Orthogonality on cosets

Reinhard Knörr*

Institut für Mathematik, Universität Rostock

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Abstract

We present a version of the orthogonality relations for group characters which appears to be new – at least in this generality. The proof given uses only Maschke's theorem and Schur's lemma. Among other consequences, Burnside's result on the existence of zeros for non–linear irreducible characters is generalized.
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Throughout, G is a finite group. The most important case $x = y$ of part (i) of the following appears as lemma in Gallagher's paper [1]; see also 3.4 in [2]. Some of the corollaries below follow already from Gallagher's result.

Theorem 1 *Orthogonality* Let V and W be $\mathbb{C}G$-modules affording the characters χ and ψ of G and let H be a subgroup.

(i) If $\chi_{|H}$ is irreducible, then

$$\frac{1}{|H|}\sum_{h\in H}\chi(hx)\overline{\chi(hy)} \;=\; \frac{\chi(xy^{-1})}{\chi(1)}$$

for all $x, y \in G$. In particular, the mean is 1 if $x = y$.

*Correspondence: Reinhard Knörr, Institut für Mathematik, Universität Rostock, D-18051 Rostock, Germany; E-mail: reinhard.knoerr@uni-rostock.de

(ii)

$$\frac{1}{|H|}\sum_{h\in H}\chi(hx)\overline{\psi(hy)} \;=\; 0 \quad\text{for all } x,y\in G$$

if and only if $V_{|H}$ and $W_{|H}$ have no common constituents.

Proof:

(i) Let $E := \mathrm{End}_{\mathbb{C}}(V)$. For any $a,b\in E$, we have a linear map $\lambda_a\rho_b : E\to E$ sending $e\in E$ to aeb. A short calculation using matrix units shows that its trace on E is $tr_E(\lambda_a\rho_b) = tr_V(a)\cdot tr_V(b)$. Therefore

$$\begin{aligned}\sum_{h\in H} tr_V(h^{-1}a)\cdot tr_V(bh) &= \sum_{h\in H} tr_E(\lambda_{h^{-1}a}\rho_{bh})\\ &= tr_E(\sum_{h\in H}\lambda_{h^{-1}a}\rho_{bh})\\ &= tr_E(\tau)\ ,\end{aligned}$$

where

$$\tau := \sum_{h\in H}\lambda_{h^{-1}a}\rho_{bh}$$

sends $e\in E$ to

$$e\tau = \sum_{h\in H} h^{-1}aebh\ .$$

Since $V_{|H}$ is (absolutely) irreducible and $e\tau$ commutes with all $h\in H$, it is a scalar matrix, say $e\tau = c\cdot\mathrm{Id}$. The scalar c can be calculated by taking the trace:

$$\begin{aligned} c\cdot\chi(1) &= tr_V(\sum_{h\in H} h^{-1}aebh)\\ &= |H|\cdot tr_V(aeb)\ .\end{aligned}$$

Therefore

$$e\tau = \frac{|H|}{\chi(1)}\cdot tr_V(aeb)\cdot\mathrm{Id}$$

for every $e\in E$. Since $\tau : E\to E$ maps all of E into the one-dimensional subspace $\mathbb{C}\cdot\mathrm{Id}$, we have

$$tr_E(\tau)\cdot\mathrm{Id} = (\mathrm{Id})\tau = \frac{|H|}{\chi(1)}\cdot tr_V(ab)\cdot\mathrm{Id}$$

or

$$\frac{1}{|H|}\sum_{h\in H} tr_V(h^{-1}a)\cdot tr_V(bh) \;=\; \frac{1}{|H|}\, tr_E(\tau) \;=\; \frac{tr_V(ab)}{\chi(1)}\ .$$

Therefore

$$\frac{1}{|H|}\sum_{h\in H}\chi(hx)\overline{\chi(hy)} \;=\; \frac{1}{|H|}\sum_{h\in H} tr_V(h^{-1}x)\cdot tr_V(y^{-1}h) \;=\; \frac{tr_V(xy^{-1})}{\chi(1)} \;=\; \frac{\chi(xy^{-1})}{\chi(1)}\ ,$$

where we have used $\overline{\chi(h^{-1}y)} = \chi(y^{-1}h)$.

(ii) The same idea works: Denote $F = \mathrm{Hom}_{\mathbb{C}}(V, W)$. For $a \in \mathrm{End}_{\mathbb{C}}(V)$ and $b \in \mathrm{End}_{\mathbb{C}}(W)$, the map $\lambda_a \rho_b : F \to F$ has trace $tr_F(\lambda_a\rho_b) = tr_V(a)tr_W(b)$ as before. But

$$\tau := \sum_{h\in H} \lambda_{h^{-1}a}\rho_{bh} : \ f \mapsto \sum_{h\in H} h^{-1}afbh \ ,$$

and this is an H-linear map $V \to W$, hence 0, if $V_{|H}$ and $W_{|H}$ have no common constituents. Therefore $\tau = 0$ and so

$$0 = tr_F(\tau) = \sum_{h\in H} tr_V(h^{-1}a)\cdot tr_W(bh) \ .$$

One direction follows.
To prove the converse, take $x = y = 1$ and use the ordinary orthogonality relations for H; note that these are a consequence of what has already been shown.

If H is normal and $x = y$, one can give a much shorter proof, using the elements of Clifford theory.

Corollary 2 Let $H \le G$ and $\chi \in \mathrm{Irr}\,(G)$. Assume that $\chi_{|H}$ is irreducible and let e be the central idempotent of $\mathbb{C}H$ corresponding to $\chi_{|H}$. Then

$$\varphi(x) = \frac{\chi(1)}{|H|}\sum_{h\in H}\chi(xh^{-1})h$$

for $x \in \mathbb{C}G$ defines an algebra homomorphism $\varphi : \mathbb{C}G \to \mathbb{C}H$ with $\chi(\varphi(g)) = \chi(g)$ for all $g \in G$, $\varphi(h) = he$ for all $h \in H$ and $\varphi(g^y) = \varphi(g)^y$ for all $g \in G$ and $y \in N_G(H)$.

Proof: Take $x, y \in G$ and look at the coefficient of some fixed $a \in H$ for $\varphi(xy)$ and for $\varphi(x)\varphi(y)$ respectively. We get

$$\frac{\chi(1)}{|H|}\chi(xya^{-1}) \quad \text{and}$$

$$\sum_{\substack{h,k\in H\\ hk=a}} \frac{\chi(1)}{|H|}\chi(xh^{-1})\frac{\chi(1)}{|H|}\chi(yk^{-1}) = \left(\frac{\chi(1)}{|H|}\right)^2 \sum_{h\in H}\chi(xh^{-1})\chi(ya^{-1}h)$$

respectively. By part (i) of the theorem, the last sum ist just

$$\frac{|H|}{\chi(1)}\chi(xya^{-1}) \ .$$

This shows that φ is multiplicative. The other properties are obvious.

Under the assumptions of the corollary, let $K = \mathbb{Q}(\chi)$. By its very definition, $\varphi : KG \to KH$, so for any field $E \ge K$, there exists an E-representation affording χ if and only if there exists an E-representation affording $\chi_{|H}$.

It is possible to prove the corollary first and then deduce part (i) of the theorem. To do so, note first that the representation of G on V induces an algebra homomorphism $\mathbb{C}G \to \mathrm{End}_{\mathbb{C}}V$. It is well known that $\mathrm{End}_{\mathbb{C}}V \cong e\mathbb{C}H \leq \mathbb{C}H$. This shows the existence of an algebra homomorphism $\varphi : \mathbb{C}G \to \mathbb{C}H$ with $\chi(\varphi(g)) = \chi(g)$ for all $g \in G$. One can then use the regular character of H to deduce the above formula for φ. Since φ is multiplicative, part (i) of the theorem is a consequence. This approach follows closely the one in [3], 2.13.

Corollary 3 Let $H \leq G$ and $\chi \in \mathrm{Irr}\,(G)$ with $\chi_{|H}$ irreducible.

(i) Given $a_1, \ldots, a_n \in G$ for some $n \geq 1$, one has

$$\chi(a_1 a_2 \ldots a_n) = \left(\frac{\chi(1)}{|H|}\right)^{n-1} \cdot \sum_{\substack{(h_0,\ldots,h_n)\in H^{n+1} \\ h_0 = h_n = 1}} \prod_{i=1}^{n} \chi(\, h_{i-1}^{-1}\, a_i\, h_i \,) \quad .$$

(ii) If the values of χ on some cosets $Hg_1, \ldots, Hg_s,\ s > 0$ are known, one can calculate the character values on all elements in the subgroup generated by these cosets. In particular, if the values on Hg are known, the values on $\langle H, g\rangle$ can be deduced. They all belong to the field $\mathbb{Q}(\chi(x)\,|\,x \in Hg)$.

Proof:

(i) Use induction on n. Replacing y by y^{-1}, on gets from part (i) of the theorem that

$$\frac{\chi(1)}{|H|}\sum_{h\in H}\chi(xh)\chi(h^{-1}y) \;=\; \chi(xy) \quad .$$

This is the statement for $n = 2$ (the case $n = 1$ being trivial) and provides the induction step.

(ii) By the first part, we can calculate (rationally) the character value on any product of elements from $\bigcup_i Hg_i$ once we know $\frac{\chi(1)}{|H|}$. For this, choose $a \in Hg$ with $\chi(a) \neq 0$. Since

$$\chi(a) = \chi(a^{-1}aa) = \left(\frac{\chi(1)}{|H|}\right)^2 \sum_{h,k\in H}\chi(a^{-1}h)\chi(h^{-1}ak)\chi(k^{-1}a)$$

and $\chi(a^{-1}h) = \overline{\chi(h^{-1}a)}$, $\chi(h^{-1}ak) = \chi(kh^{-1}a)$ and $\chi(k^{-1}a)$ are all known, the sum can be calculated, and therefore $\left(\frac{\chi(1)}{|H|}\right)^2$ and $\frac{\chi(1)}{|H|}$ as well.

Taking $H = G$ and $g = 1$ in the next corollary gives Burnside's result mentioned above. Another example illustrating the point nicely is $G = SL(2,3)$ and $H = Q_8$. The irreducible characters of degree 2 have no zero outside H.

Corollary 4 Let $H \leq G$, $g \in G$ of order k and $\chi \in \mathrm{Irr}\,(G)$. If $\chi_{|H}$ is irreducible and χ has no zero on the coset Hg, then $\chi(g) = \pm\,\varepsilon$, where ε is a k–th root of unity.
Proof: (Compare [3], 3.14, 3.15) Let $\chi = \chi_1, \ldots, \chi_t$ be all the algebraic conjugates of χ.

Note that they are all irreducible under restriction to H, so it follows from part (i) of the theorem that

$$\frac{1}{t|H|}\sum_{i,\ h}|\chi_i(hg)|^2 = 1\ .$$

No χ_i has a zero on Hg, so

$$0\ <\ \prod_{i,\ h}|\chi_i(hg)|^2\ \ .$$

Since the product is a rational algebraic integer, it is at least 1. Now use the fact that the inequality

$$\frac{1}{n}\sum_{i=1}^{n}r_i\ \geq\ (\prod_{i=1}^{n}r_i)^{1/n}$$

between arithmetic and geometric mean (for $0 \leq r_i \in \mathbb{R}$) is *proper* unless all r_i's are equal. Therefore $|\chi_i(hg)| = 1$ for all i and h and in particular $|\chi(g)| = 1$. It is then actually a root of unity by exercise 3.2 in [3]. Since $\chi(g)$ belongs to the cyclotomic field $\mathbb{Q}_k$, the assertion follows from [6], Satz 11.13.

In some cases, the sign of the character value in the last result can be determined: Let $k = o(g)$ be a power of some prime p. If ζ is a primitive k–th root of unity and I any maximal ideal of $\mathbb{Z}[\zeta]$ containing p, then $\zeta^s \equiv 1 \bmod I$ for every s, hence $\chi(g) \equiv \chi(1) \bmod I$. So if $\chi(g) = \varepsilon$, a k–th root of unity, then $\chi(1) \equiv 1 \bmod p$ and if $\chi(g) = -\varepsilon$ then $\chi(1) \equiv -1 \bmod p$. In particular, if Hg contains a p–element, then χ has a zero on Hg unless $\chi(1) \equiv \pm 1 \bmod p$ (still assuming $\chi_{|H}$ irreducible).

The next result is independent of the theorem, but it fits in with the context, because it is proved by the same arguments as the last corollary and because one can again deduce Burnside's result, this time taking $H = 1$.

Proposition 5 Let $H \leq G$ and $\chi \in \mathrm{Irr}\,(G)$. If $\chi_{|H}$ is reducible then χ has a zero on $G \setminus H$.

Proof: Let again $\chi = \chi_1, \ldots, \chi_t$ be the algebraic conjugates of χ. If $\chi(g) \neq 0$, then $\sum_i |\chi_i(g)|^2 \geq t$ by the arithmetic-geometric inequality. So if $\chi(g) \neq 0$ for all $g \in G \setminus H$, we get

$$t|G| = \sum_{i,\,g\in G}|\chi_i(g)|^2 \geq \sum_{i,\,h\in H}|\chi_i(h)|^2 + \sum_{g\in G\setminus H} t\ \ ,$$

hence

$$t|H| \geq \sum_{i,h\in H}|\chi_i(h)|^2 = t|H|(\chi,\chi)_H\ \ ,$$

contradicting the assumption on $\chi_{|H}$.

It is of course not possible to guaranty a zero of χ on a given coset of H in the last result. Examples of non–linear χ's with no zero on $G \setminus H$ are given below.

The following is a slight generalization of a well known result (see [3], Theorem 3.7 for instance).

Proposition 6 *Central characters* Let $H \leq G$, $g \in G$ and $L = g^H$ be the H–conjugacy class of g. Let χ be a character of G and let $e = e_\sigma$ be the central idempotent of $\mathbb{C}H$ corresponding to some $\sigma \in \mathrm{Irr}\,(H)$. Finally, denote $\omega_{\chi,\sigma}(\widehat{L}) := \frac{\chi(e\widehat{L})}{\sigma(1)}$. Then

(i)
$$\omega_{\chi,\sigma}(\widehat{L}) = \frac{|L|\chi(eg)}{\sigma(1)} = \frac{|L|}{|H|}\sum_{h\in H}\chi(hg)\overline{\sigma(h)}$$

(ii) If $(\chi, \sigma)_H = 1$, then $\omega_{\chi,\sigma}$ defines an algebra homomorphism $A := C_{\mathbb{C}G}(\mathbb{C}H) \to \mathbb{C}$; moreover, $\omega_{\chi,\sigma}(\widehat{L})$ is an algebraic integer.

(iii) If $\chi_{|H} = \sigma$ is irreducible, then $\omega_{\chi,\sigma}(\widehat{L}) = \frac{|L|\chi(g)}{\chi(1)}$ is an algebraic integer.

Proof:

(i) Since e commutes with any $h \in H$, we have $eg^h = (eg)^h$, so $\chi(ex) = \chi(eg)$ for all $x \in L$ and therefore $\chi(e\widehat{L}) = |L|\chi(eg)$. Now apply χ to $eg = \frac{\sigma(1)}{|H|}\sum_{h\in H}\overline{\sigma(h)}hg$.

(ii) Let $L = L_1, \ldots, L_t$ be all the H–conjugacy classes in G. The class sums $\widehat{L_i}$ form a basis of A. Note that any product $\widehat{L_i}\cdot\widehat{L_j}$ is a $\mathbb{Z}$–linear combination of the $\widehat{L_k}$'s, so the $\mathbb{Z}$–span of the $\widehat{L_k}$'s is a $\mathbb{Z}$–subalgera $B \leq A$.
Now let V be a G–module affording χ and S an H–module affording σ. Multiplication with $a \in A$ induces an H–endomorphism of V and therefore also of its S-homogeneous part Ve.
By assumption, $Ve \cong S$ is irreducible, so Schur's Lemma tells us that any element of A acts by scalar multiplication on the module Ve. This defines then an algebra homomorphism $\omega : A \to \mathbb{C}$. Since e is the projection on Ve, we have $\chi(e\widehat{L}) = \chi(e\cdot\omega(\widehat{L})) = \chi(e)\cdot\omega(\widehat{L}) = \sigma(1)\cdot\omega(\widehat{L})$, i.e. $\omega = \omega_{\chi,\sigma}$. In particular, $\omega_{\chi,\sigma}(B)$ is a unitary subring of $\mathbb{C}$ and finitely generated as a $\mathbb{Z}$–module. All elements in $\omega_{\chi,\sigma}(B)$ are then algebraic integers by [3], Theorem 3.4.

(iii) Clear from (ii), since e_σ acts as identity on V.

Even under the assumptions of (ii), $\chi(eg)$ is in general not integral, so the usefulness of the statement is not clear. For (iii), the situation is better:

Corollary 7 Let $\chi \in \mathrm{Irr}\,(G)$ and $K = g^G$ be a conjugacy class. If there is a subgroup $H \leq G$ such that $\chi_{|H}$ is irreducible and $1 = gcd\,\{\chi(1), |L_1|, \ldots, |L_t|\}$, where $L_1, \ldots, L_t$ are all the H– conjugacy classes contained in K, then either $g \in Z(\chi)$ or else $\chi(g) = 0$.

Proof: Mimic the proof for Burnside's theorem as given in [3], Theorem 3.8. The only change is that we choose integers $u, v_1, \ldots, v_t$ such that $1 = u\chi(1) + v_1|L_1| + \ldots + v_t|L_t|$.

Corollary 8 Let $g \in G$ and $\chi \in \mathrm{Irr}\,(G)$ such that $\chi(g)$ is a root of unity. If $\chi_{|H}$ is irreducible for some $H \leq G$, then $\chi(1)$ divides $|H : C_H(g)|$.

Proof: Clear since $|g^H| = |H : C_H(g)|$ and $\frac{|g^H|}{\chi(1)}$ is integral by Proposition 5.

Example Let X be a doubly transitive H–set, say $|X| = n$, and let $S \leq H$ with $|H : S|$ not divisible by $n-1$. Further let π be the permutation character of the symmetric group $G = S_X \geq H$. Then $\chi = \pi - \mathbf{1}$ is irreducible, even when restricted to H (by 2-transitivity). If $g \in C = C_G(S)$, then $S \leq C_H(g)$, so $\chi(1) = n-1$ does not divide $|H : C_H(g)|$. By the last corollary, $\chi(g) \neq \pm 1$, so $\pi(g) \neq 0, 2$. In particular, every element of C fixes some point of X.
A closer look shows that there is even some $x \in X$ such that $S_x \neq S_y$ for all $x \neq y \in X$. Such an x is clearly fixed by all of C.

Corollary 9 Let $H, U \leq G$ (not necessarily distinct), $g \in G$ and $\chi \in \mathrm{Irr}\,(G)$. If both $\chi_{|H}$ and $\chi_{|U}$ are irreducible and if χ has no zero on the coset Hg, then $\chi(1)$ divides $|U : C_U(hg)|$ for all $h \in H$.

Proof: $\chi(hg)$ is a root of unity by Corollary 4, so the assertion follows from Corollary 8, used for U.

Following Exercise 3.15 (attributed to Thompson) in Isaacs' book [3], we can use a result of Siegel to improve Corollary 4.

Corollary 10 Let $H \leq G$, $g \in G$, $\chi \in \mathrm{Irr}\,(G)$ and assume that $\chi_{|H}$ is irreducible. Denote $z = |\{x \in Hg \,|\, \chi(x) = 0\}|$ and $u = |\{x \in Hg \,|\, \chi(x) \text{ is a root of unity }\}|$. Then

$$u \leq |H| \leq u + 3z.$$

Proof: Let $\chi = \chi_1, \ldots, \chi_t$ be all the algebraic conjugates of χ. If $x \in Hg$ is a zero for χ, then $\chi_i(x) = 0$ for any i and similarly for the roots of unity. If $x \in Hg$ is one of the $r = |H| - z - u$ remaining elements, then $|\chi_i(x)|^2 \neq 1, 0$ for all i, so $\frac{3t}{2} \leq \sum_i |\chi_i(x)|^2$ by [5], Theorem III. Therefore

$$(z + u + r) \cdot t \;=\; |H| \cdot t \;=\; \sum_{\substack{x \in Hg \\ i}} |\chi_i(x)|^2 \;\geq\; u \cdot t \;+\; r \cdot \frac{3t}{2} \quad .$$

Hence $z \geq \frac{r}{2}$, so $|H| = u + z + r \leq u + 3z$.

Remark

(i) If $z = 0$ in the last result, then clearly $u = |H|$. This is essentially the statement of Corollary 4. It is a trivial consequence of the theorem that $r = 0$ also implies $u = |H|$.

(ii) It is clear from the above proof that we have $|H| = u + 3z$ if and only if $\frac{3t}{2} = \sum_i |\chi_i(x)|^2$ for every x with $|\chi_i(x)| \neq 1, 0$. As Siegel has shown, this happens only for $\xi := |\chi(x)|^2 = \frac{1}{2}(3 \pm \sqrt{5})$. Since $\xi \in \mathbb{Q}_5$ and $\mathbb{Q}_n \cap \mathbb{Q}_m = \mathbb{Q}$ if n and m are relatively prim ([6], Satz 11.14), we must then have that the order of such an x is divisible by 5.

(iii) If χ is not linear and we consider the coset $Hg = H$, then $x = 1$ is one of these 'remaining elements' and has order not divisible by 5. Therefore we get the stronger inequality $|H| < u + 3z$ in this case. For other cosets, however, we may have equality. Take $G = A_5$ and $H = A_4$, for instance; then for the irreducible characters of degree 3, one finds $u = z = 3$ on any coset $\neq H$.

In the following proof (and again at the end of the paper), we use the fact that for a character χ of G and a subgroup H with $(\chi, \mathbf{1})_{|H} = 0$, all coset sums $\sum_{x \in Hg} \chi(x)$ are 0, since $\widehat{H}$ and then also $\widehat{Hg}$ act as 0 on the module affording χ, as the referee pointed out.

Corollary 11 Let $H \leq G$, $g \in G$ and $\chi \in \mathrm{Irr}\,(G)$ real-valued and not linear. If $\chi_{|H}$ is irreducible and if χ has no zero on the coset Hg, then χ is rational on $\langle H, g \rangle$. Moreover $\chi(x) = 1$ for exactly half of the elements in Hg and $\chi(x) = -1$ for the other half. In particular, H has even order, and in fact $4\chi(1)$ divides $|H|$.

Proof: By Corollary 4, the only values χ can take on Hg are ± 1, and the preceding remark shows that both values are taken equally often, say k times, so $|H| = 2k$ is even. It is clear from Corollary 2 that χ is rational on $\langle H, g \rangle$.
Now let

$$\alpha = \sum_{\substack{h \in H \\ \chi(hg)=1}} \chi(h)$$

and

$$\beta = \sum_{\substack{h \in H \\ \chi(hg)=-1}} \chi(h) \quad ,$$

so these are rational integers. Then, by Theorem 1

$$\frac{|H|}{\chi(1)} \chi(g) = \sum_{h \in H} \chi(hg)\chi(h) = \alpha - \beta = 2\alpha,$$

since $\alpha + \beta = 0$, again by the above remark.
Now $\chi(g) = \pm 1$, so $4\chi(1)$ divides $|H|$ if and only if α is even.
Assume α is odd; then $\chi(a)$ is odd for some $a \in H$. For $c \in H$ arbitrary but fixed and $\epsilon, \delta \in \{\pm 1\}$, let $r_{\epsilon,\delta}$ be the number of elements $h \in H$ such that $\chi(hg) = \epsilon$ and $\chi(chg) = \delta$. Then $r_{1,1} + r_{1,-1} = k = r_{1,1} + r_{-1,1}$, so $r_{1,-1} = r_{-1,1} =: r$. Therefore $\chi(hg) \neq \chi(chg)$ for exactly $2r$ elements and consequently $\chi(hg) = \chi(chg)$ for $2k - 2r$ elements. Again using Theorem 1, we get

$$\frac{|H|}{\chi(1)} \chi(c) = \sum_{h \in H} \chi(chg)\chi(hg) = (2k - 2r) - 2r \equiv |H| \bmod 4$$

independent of the choice of c. Choosing c such that $\chi(c) = 0$ (recall that χ is not linear) shows that $0 \equiv |H| \bmod 4$, hence also

$$\frac{|H|}{\chi(1)} \chi(a) \equiv 0 \bmod 4 \quad .$$

The assertion (and a contradiction) follow, since $\chi(a)$ is odd and $\chi(1)$ divides $|H|$.

The next corollary shows that a coset of a doubly transitive subgroup usually contains an element with exactly one fixed point.

Corollary 12 Let G be a permutation group of degree $n > 2$ and H a 2–transitive subgroup. Then any coset Hg either contains an element with exactly one fixed point or else half of the elements in Hg have no fixed point and the other half have exactly two fixed points. In this exceptional case, $4(n-1)$ divides $|H|$.

Proof: If $\pi = \chi + \mathbf{1}$ denotes the permutation character of G, then χ is rational valued, not linear and, by 2–transitivity, irreducible under restriction to H. If χ has a zero on Hg, then some element in this coset fixes exactly one point. If not, then χ satisfies the conditions of the previous corollary. Noting that $\chi(1) = n-1$ and $\chi(x)+1 = \pi(x)$, the assertions are clear.

Of course, the exception cannot occur for 3–transitive H, because then any coset contains an element fixing at least 3 points.
An earlier version of the paper did not contain Corollary 11; instead, Corollary 12 was proved directly. The referee proposed the use of real-valued characters and made helpful suggestions for the proof.

Examples

(i) Take $g \in G = GL_2(q)$ and $H = SL_2(q)$ for some prime power q. Which characters χ of G with $\chi_{|H}$ irreducible have no zero on Hg?
If $\det(g) = s^2$ is a square in F_q (this is always the case if q is even), then Hg contains the central element sI, so $\chi(1) = 1$ by Corollary 9. Conversely of course, the linear characters have no zeros (on Hg).
Otherwise, the characteristic polynomial of an element $x \in Hg$ is either irreducible over F_q or has two different zeros (and both cases occur). Accordingly, $|G : C_G(x)|$ is either $q(q-1)$ or $q(q+1)$. By Corollary 9, the degree of χ has to divide both. This is satisfied by only two degrees, namely 1 (again, as expected) and q; the list of character degrees (indeed the character table) can be found for instance in [4], Theorem 28.5. Denote by B the subgroup of upper triangular matrices. Then H acts doubly transitive on the cosets $\{By \,|\, y \in G\}$ and the permutation character π (of degree $\pi(1) = |G : B| = q+1$) takes on $x \in Hg$ the value 0 (if the characteristic polynomial of x is irreducible) or 2 (otherwise). So the character $\chi = \pi - \mathbf{1}$ (known as the Steinberg character) has degree q and no zero on Hg. Multiplying χ with linear characters gives the other solutions.

(ii) Let F be a finite field of characteristic 2 and let G be the group of all semi-linear transformations $F \ni x \mapsto x^\sigma a + b,\ \sigma \in \mathrm{Aut}(F),\ a, b \in F,\ a \neq 0$. The subgroup H of all linear transformations $x \mapsto xa + b$ is doubly transitive. Let $\phi : x \mapsto x^2$ be the Frobenius automorphism. If x is a fixed point of some $h\phi = (a, b)\phi \in H\phi$, then $x = (ax+b)^2$, so x is a zero of a separable polynomial of degree 2. Therefore, $h\phi$ has no fixed point (if the polynomial is irreducible in $F[x]$) or exactly 2 fixed points. This conclusion holds also for other cosets $H\gamma$, provided γ is a generator of $\mathrm{Aut}(F)$:

If $\gamma^s = \phi$, choose t prime to $|G|$ such that $t \equiv s \bmod |\mathrm{Aut}(F)|$. Then $h\gamma$ and $(h\gamma)^t$ generate the same group, so have the same fixed points. Since $(h\gamma)^t \in H\phi$, their number is 0 or 2. In the smallest case $|F| = 2$, we have $H = G = S_2$. If $|F| = 4$, one gets $H = A_4$ and $G = S_4$; this is the only faithful example with $|H| = 4\chi(1)$. If $|F| = 2^p$ for some prime p, any $g \in G \setminus H$ has either 0 or 2 fixed points, since any $\gamma \neq \mathrm{id}$ generates $\mathrm{Aut}(F)$, so $\chi(g) = \pm 1$ on $G \setminus H$.

(iii) The following question was asked in a previous version of this paper:
Is there an example of a faithful character, irreducible when restricted to some subgroup H and non–zero on all elements of a coset Hg, where $g \notin N_G(H)$?
The answer is "yes"; in fact both readers of the paper came up with such examples.
The first is due to Frieder Ladisch and is actually a rational character:
With F and H as in the last example, consider the permutation σ on F defined by $0\sigma = 0$ and $x\sigma = x^{-1}$ for $0 \neq x \in F$. It is easily seen that σ does not normalize H except for very small F. If x is fixed by $\sigma h = \sigma(a, b)$, then $x = x\sigma a + b$. If $x = 0$, this means $b = 0$. In this case, the unique solution of $x^2 = a$ gives the second fixed point. If $b \neq 0$, we must have $x = x^{-1}a + b$ or $x^2 + bx + a = 0$. Since this polynomial is separable, it has either two zeros or none (in F).
The second example was given by the referee:
Let K be a p'–group which acts faithfully on an extra-special p–group N of order p^{1+2n}, fixing its center Z. Let $G = KN$ be the semi-direct product and assume in addition that $\overline{G} = G/Z$ is a Frobenius group with kernel $\overline{N}$, so $|K|$ divides $p^{2n} - 1$. Therefore all elements in $\overline{G} \setminus \overline{N}$ are conjugate to some element in $\overline{K}$, hence all elements in $G \setminus N$ are conjugate to some element kz with $1 \neq k \in K$ and $z \in Z$. The non-linear irreducible characters $\theta_1, \ldots, \theta_{p-1}$ of N are G-stable (since they vanish off Z), hence extendible to G, because $|G : N|$ and $|N|$ are relatively prime (see [3], Corollary 8.16). The θ_i's are Galois-conjugate, so we may assume that their extensions χ_i are as well. These χ_i's have no zero on $G \setminus N$:
By the above, it is enough to consider an element $k \in K$. If $\chi_1(k) = 0$, then $\chi_i(k) = 0$ for all i. But one gets all the faithful irreducible characters of G by multiplying the χ_i's with the irreducible characters of G/N, so they all vanish on k. The second orthogonality relation then gives

$$|C_G(k)| = \sum_{\chi \in \mathrm{Irr}(G)} |\chi(k)|^2 = \sum_{\psi \in \mathrm{Irr}(\overline{G})} |\psi(\overline{k})|^2 = |C_{\overline{G}}(\overline{k})|$$

which is obviously false (by a factor p).
Now one gets examples by taking $H = SN$, where S is any non–normal subgroup of K. Specifically, one can take N of order 5^3 and exponent 5 and K the normalizer of a Sylow 3–subgroup in $SL_2(5)$, so $|K| = 12$. Then a Sylow 2–subgroup S of K is self-normalizing.
Two comments:
(a) Any prime divisor of $|K|$ divides $p^{2n} - 1 = (\chi_i(1) + 1)(\chi_i(1) - 1)$, since $\chi_i(1) = p^n$. This illustrates the remark following Corollary 4.
(b) It is known and not hard to prove that $|\chi_i(k)|^2 = |C_{\overline{N}}(k)|$ even if $\overline{G}$ is not a Frobenius group.

Question Are there examples of a G–set of odd cardinality $n = \pi(1)$ and a 2–transitive

subgroup H, such that $\pi(x) \neq 1$ for all x in some coset Hg?
This looks unlikely in view of the fact – observed by Ladisch – that any transitive subgroup $U \leq H$ must be of even order. To see this, note that $(\mathbf{1}, \pi)_U = 1$ by transitivity, so $(\mathbf{1}, \chi)_U = 0$, hence $0 = \sum_{x \in Ug} \chi(x)$ by an earlier remark. Since all summands are ± 1, their number must be even.
As a consequence, if n is odd, it cannot be a prime power. Also there can be no solvable transitive subgroup in H, because its $2'$–Hall group would still be transitive. By 2–transitivity, Hg contains elements of even order, but it contains no 2–elements, because $\pi(x) \equiv n \bmod 2$ for such elements (or use the remark following Corollary 4).
Note, however, that the weaker conditions of Corollary 11 do not imply $\chi(1)$ odd, as the rational irreducible character of degree 2 for $SL(2,3)$ shows. Here $|H| = |Q_8| = 4\chi(1)$.

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