On Natural Monomial Characters of $S_n$

Reinhard Knörr*

Institut für Mathematik, Universität Rostock

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Abstract

A class of natural linear characters for the centralizers of elements in the symmetric group is introduced. The character values of the corresponding monomial characters are calculated. They have a surprising combinatorial interpretation.

Key Words: group representation, ordinary character theory, symmetric groups, finite fields

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For any $0 < m \in \mathbb{N}$, let

$$f_m(x) = \sum_{d|m} \mu(d) x^{m/d},$$

where $\mu$ is the Möbius function, so $f_m$ is a monic polynomial of degree $m$ over $\mathbb{Z}$. For $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{N}_0^n$, let

$$p_\tau(x) = \prod_{m=1}^n \prod_{j=0}^{\tau_m-1} (f_m(x) + jm);$$

so if $\tau$ is a partition of $n$, i.e. $n = \sum_m m \tau_m$, then $p_\tau$ has degree $n$. Note that for $n \neq 0$ (which we assume throughout), the constant term of $p_\tau$ is 0, since this is true for every

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*Correspondence: Reinhard Knörr, Institut für Mathematik, Universität Rostock, D-18051 Rostock, Germany; E-mail: reinhard.knoerr@uni-rostock.de
We expand
\[ p_\tau(x) = \sum_{t=1}^{n} \chi_t(\tau) x^t, \]
so this produces class functions \( \chi_1, \ldots, \chi_n \) of the symmetric group \( S_n \), where of course \( \chi_t(g) = \chi_t(\tau) \) if \( g \in S_n \) is of type \( \tau \), i.e., \( g \) has exactly \( \tau \) orbits of length \( i \) (in its natural action on \( \{1, \ldots, n\} \)). The aim of this note is to show that the \( \chi_t \)'s are characters of \( S_n \). More precisely, Corollary 1 states that the \( \chi_t \)'s are sums of certain canonical monomial characters.

**Notation**

(i) For \( k \in \mathbb{N} \), let \( S_{n,k} = \{ g \in S_n \mid g \text{ has exactly } k \text{ orbits on } n \} \), so \( S_{n,k} \) is a union of conjugacy classes \( K_\beta \) of \( S_n \), namely the ones of type \( \beta \) with \( k = |\beta| := \sum_i \beta_i \). Clearly, \( S_{n,k} = \emptyset \) if \( k > n \); also, \( S_{n,0} = \emptyset \) (since \( n > 0 \)).

(ii) For \( m \in \mathbb{N} \), let
\[
\varepsilon_m = e^{\frac{2\pi i}{m}} \in \mathbb{C},
\]
so \( \varepsilon_m \) is a primitive \( m \)-th root of unity. Note that \( (\varepsilon_m)^d = \varepsilon_{m/d} \) for every \( d|m \).

**Lemma 1**

\[
\prod_{k=0}^{n-1} (x + k) = \sum_{k=0}^{n} |S_{n,k}| x^k
\]

**Proof:** Given an element \( g \in S_{n,k-1} \), we define \( \tilde{g} \in S_{n+1,k} \) by \( \tilde{g}(n+1) = n+1 \) and \( \tilde{g} = g \) on \( n \). Given \( g \in S_{n,k} \) and \( 1 \leq i \leq n \), we define \( \tilde{g}_i \in S_{n+1,k} \) by
\[
\tilde{g}_i(j) = \begin{cases} 
  n+1 & \text{if } j = i \\
  g(i) & \text{if } j = n+1 \\
  g(j) & \text{otherwise}.
\end{cases}
\]

Then \( \tilde{g} \) is a bijection between \( S_{n,k-1} \cup S_{n,k} \times n \) and \( S_{n+1,k} \); in particular \( |S_{n+1,k}| = |S_{n,k-1}| + |S_{n,k}| n \). From this, the assertion follows easily by induction.

**Remark 1** For any \( 0 < m \in \mathbb{N} \), there is a natural action of \( S_n \) on the set \( \mathbb{N}^m \) of all maps \( \mathbb{N} \to m \). Such a map \( f \) is fixed by \( g \in S_n \) if and only if \( f \) is constant on the orbits of \( g \), so the number of fixed points of \( g \) is \( n^{b(g)} \), where \( b : S_n \to \mathbb{N} \) counts the orbits. Calculating the multiplicity of the trivial character in the permutation character \( \pi_m \) gives
\[
(\pi_m, 1) = \frac{1}{n!} \sum_{g \in S_n} \pi_m(g) = \frac{1}{n!} \sum_{t} |S_{n,t}| m^t = \frac{1}{n!} \prod_{k=0}^{n-1} (m+k) = \binom{n+m-1}{n},
\]
where the third equality follows from the lemma. By Burnside’s lemma ([1], Corollary 5.15), this gives the number of orbits of \( S_n \) on \( \mathbb{N}^m \), hence the number of choices with repetitions of \( n \) objects from \( m \). So get a well known formula from basic combinatorics. A similar argument allows us to calculate the multiplicity of the sign character \( sgn \): using that \( \pi_{-m} = (-1)^n sgn \cdot \pi_m \), one gets
\[
(\pi_m, sgn) = \binom{m}{n}.
\]
Lemma 2

\[ \prod_{k=0}^{t-1} (f_m(x) + km) = \sum_{\beta \vdash t} |K_\beta| m^{t-|\beta|} f_m(x)^{|\beta|} \]

Proof:

\[ \prod_{k=0}^{t-1} (f_m(x) + km) = m^t \prod_{k=0}^{t-1} \left( \frac{1}{m} f_m(x) + k \right) \]

\[ = m^t \sum_{k=0}^{t} |S_{t,k}| \left( \frac{1}{m} f_m(x) \right)^k \quad \text{(using Lemma 1)} \]

\[ = \sum_{k=0}^{t} \left| S_{t,k} \right| m^{t-k} f_m(x)^k \]

\[ = \sum_{\beta \vdash t} |K_\beta| m^{t-|\beta|} f_m(x)^{|\beta|}. \]

Lemma 3

\[ \mu(m) = \sum_{\varepsilon \text{ primitive } m\text{-}th \ root \ of \ unity} \varepsilon \]

Proof:

\[ \sum_{d|m} \sum_{\varepsilon \text{ primitive } d\text{-}th \ root \ of \ unity} \varepsilon = \sum_{\varepsilon \text{ root of unity}} \varepsilon = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases} = \sum_{d|m} \mu(d), \]

hence the assertion.

Definition, Remark 2

(i) Let \( G \) be a finite group acting on the finite set \( X \). Fix some conjugacy class \( K \) of \( G \) and consider the set

\[ M = \{(a, B) \mid a \in K, B \text{ an orbit of } <a> \text{ on } X \}. \]

It is clear that \( M \) is a \( G \)-set by \((a, B)g = (a^g, Bg)\) and that \( \alpha : M \ni (a, B) \mapsto a \in K \) is a \( G \)-map. For every point \((a, B) \in M\), we define a linear character \((a, B)\theta\) of the stabilizer \( G_{(a,B)} \) by

\[ (a, B)\theta(g) = \varepsilon_{|B|}^j \] if \( xg = xa^j \) for some \( x \in B \).

Since \( g \in G_{(a,B)} \) commutes with \( a \), the choice of \( x \) is irrelevant. Also \( j \) is unique modulo the length \(|B|\) of the orbit. Therefore \((a, B)\theta\) is well-defined and clearly multiplicative. Obviously \((a, B)\theta^g = (a^g, Bg)\theta\), so \( \theta : (a, B) \mapsto (a, B)\theta \) is an inductible map; it follows that \( \gamma = \theta^a \) is a character of \( G \), in fact a monomial character induced from a linear character \( \lambda_a \) of \( C_G(a) \) for \( a \in K \) (compare [3] for the notation and simple facts concerning inductible maps and their induction). This character depends on \( X \) and \( K \), so \( \gamma = \gamma(X, K) \).

(ii) In the following, \( G = S_n \) and \( X = \mathbb{N} \), so it remains to specify the conjugacy class. As the classes are naturally labelled by the partitions \( \sigma \) of \( n \), we use the partitions also as labels for the \( \gamma \)'s and write \( \gamma_\sigma := \gamma(n, K_\sigma) \).
(iii) There is an alternative – and more familiar – description of the linear character \( \lambda_a \) of \( C_{S_n}(a) \) from which \( \gamma_\sigma \) is induced. As is well known, corresponding to the decomposition of \( a \) (of type \( \sigma \)) in products of cycles of equal length, there is a direct product decomposition of \( C_{S_n}(a) \). The factor \( C^{(m)} \) corresponding to the cycles \( a_1, \ldots, a_s \) (say) of length \( m \) in this direct product is in turn a semi-direct product of an abelian normal subgroup \( A = \langle a_1, \ldots, a_s \rangle \cong C_m \times \cdots \times C_m \) with a symmetric group \( S_s \) which acts by permuting the cycles. Therefore, there are \( m \) linear characters of \( A \) which are stable under \( S_s \), hence extendable to \( C_{S_n}(a) \), so we can choose a linear character \( \lambda_m \) of \( C_m \) which has order \( m \) and is trivial on \( S_s \). This character is determined only up to algebraic conjugation, but we can avoid ambiguity by specifying that \( \lambda_m(a_i) = \varepsilon_m \).

The product of these characters \( \lambda_m \) gives a character \( \lambda_a \) of \( C_{S_n}(a) \). In fact, \( \lambda_a = \varphi_a^\sigma \), as is easily seen by calculating the values of these two linear characters on a cycle of \( a \) and on an element only permuting the cycles.

Incidentally, the choice of \( \lambda_m \) is irrelevant for \( \gamma_\sigma = \lambda_{S_n}(a) \): induce first to the Young subgroup \( S_{1\sigma_1} \times \cdots \times S_{n\sigma_n} \) and use that all characters of a symmetric group are rational.

(iv) To summarize, for every \( \sigma \vdash n \), we have a monomial character \( \gamma_\sigma \) of \( S_n \) with values given by

\[
\gamma_\sigma(g) = \sum_{a \in K_\sigma} \lambda_a(g),
\]

where

\[
\lambda_a(g) = \prod_i (a, B_i) \theta(g^{e_i})
\]

for a set \( B_i \) of representatives of the orbits of \( \langle g \rangle \) on the orbits of \( \langle a \rangle \) and \( e_i = | \langle g \rangle : \langle a \rangle \rangle \).

(v) For bookkeeping, it is useful to introduce \( X^\sigma := \prod_{i=1}^n x_i^{\sigma_i} \), a monomial in \( n \) variables of total degree \( |\sigma| \), and

\[
h_g(x_1, \ldots, x_n) = \sum_{\sigma \vdash n} \gamma_\sigma(g) X^\sigma \in \mathbb{Z}[x_1, \ldots, x_n],
\]

a polynomial that collects the character values of the \( \gamma_\sigma \)'s at an element \( g \in S_n \); of course, \( h_g = h_\tau \) depends only on the conjugacy class \( K_\tau \) of \( g \).

(vi) It is clear that

\[
h_g = \sum_{\sigma \vdash n} \gamma_\sigma(g) X^\sigma = \sum_{\sigma \vdash n} \sum_{a \in K_\sigma} \lambda_a(g) X^\sigma = \sum_{a \in C} \lambda_a(g) X^{\sigma(a)},
\]

where \( C = C_{S_n}(g) \) and \( \sigma(a) \) is the type of \( a \).

Lemma 4 Let \( g_m \) be the product of all cycles of length \( m \) of \( g \in S_n \), viewed as an element of \( S_{m\tau_m} \), where \( \tau \) is the type of \( g \). Then

\[
h_g = \prod_m h_{g_m}.
\]
Proof: Let $T_m \subseteq n$ be the union of all orbits of length $m$ of $g$ and $H_m = S_{T_m}$, the symmetric group on $T_m$; also denote $C_m = C_{H_m}(g_m)$. Then

$$h_{g_m} = \sum_{a_m \in C_m} \lambda_{a_m}(g_m)X^{\sigma(a_m)},$$

so

$$\prod_{m} h_{g_m} = \sum_{(a_1, \ldots, a_n) \in m \in C_m} \lambda_{a_1}(g_1) \ldots \lambda_{a_n}(g_n)X^{\sigma(a_1)} \ldots X^{\sigma(a_n)}.$$ 

Now $C_1 \times \cdots \times C_n \ni (a_1, \ldots, a_n) \mapsto a := a_1 \cdots a_n$ is a bijection $C_1 \times \cdots \times C_n \rightarrow C := C_{S_n}(g)$; clearly, $\sigma(a) = \sigma(a_1) + \cdots + \sigma(a_n)$ and by definition $\lambda_{a}(g) = \lambda_{a_1}(g_1) \ldots \lambda_{a_n}(g_n)$. Therefore, the sum on the right simplifies to

$$\sum_{a \in C} \lambda_{a}(g)X^{\sigma(a)} = h_g$$

as claimed.

Lemma 5 Let $g$ be homocyclic, say $g$ is the product of $t$ cycles of length $m$. Then

$$h_g = \sum_{\beta \vdash t} |K_\beta| m^{-|\beta|} \prod_i \left( \sum_{d \mid m} \mu(d) x_i^{m/d} \right)^{\beta_i}. $$

Proof: Let $n = m \cdot t$ and $C = C_{S_n}(g)$. Since

$$h_g = \sum_{a \in C} \lambda_{a}(g)X^{\sigma(a)},$$

we have to calculate the contribution of $a \in C$ to this sum. Since $C$ is a semi-direct product of $S_t$ and an abelian normal subgroup $N = \langle g_1, \ldots, g_t \rangle$, where $g = g_1 \cdot \ldots \cdot g_t$ is the cycle decomposition, every element $a \in C$ can be written as $a = a_0 \cdot g_1^{e_1} \cdot \ldots \cdot g_t^{e_t}$ with $a_0 \in S_t$. We consider first the case that $a_0$ is a long cycle, so $a_0$ has order $t$. Denote $A = \langle a \rangle$ and $D := \langle g, a \rangle$; so $D$ is an abelian transitive subgroup of $S_m$. Let $b$ be the order of $a$. Then clearly $t | b$; since $a^t = g^e$, where $e = \sum_i e_i$ and since the order of $g^e$ is $d := m / \gcd(e, m)$, we find that $l = t \cdot d$. This is then the length of every orbit of $A$, so $A$ has $m \cdot t / l = m / d$ orbits. The corresponding monomial is therefore $x_i^{m/d}$. To calculate the coefficient, note that $g^{m/d}$ and $a^t$ generate the same subgroup (of order $d$), so $g^m = a^t$ for some $u$ prime to $d$. Therefore $\lambda_{a}(g) = \varepsilon_i^{t_u} = \varepsilon_i^u$ is a primitive $d$-th root of unity.

Now take $a' = a_0 \cdot g_1^{e_1} \cdot \ldots \cdot g_t^{e_t}$ and let $e' = \sum_i e_i$. For any $0 \leq s < m$, there are $m^{t-1}$ solutions $(e'_1, \ldots, e'_t)$ for $e' \equiv s \mod(m)$ with $0 \leq e'_i < s$ for all $i$. If we collect those for which $\gcd(e', m) = m / d$ for some fixed divisor $d$ of $m$, the monomial is always $x_i^{m/d}$ and each primitive $d$-th root of unity appears $m^{t-1}$ times as a coefficient. By Lemma 3, we get $m^{t-1} \mu(d) x_i^{m/d}$ for fixed $d$ and

$$\sum_{d \mid m} m^{t-1} \mu(d) x_i^{m/d} = m^{t-1} \sum_{d \mid m} \mu(d) x_i^{m/d}.$$
as contribution of \( a_0N \) to \( h_g \).

A general element \( a_0 \) of \( S_t \) will have several cycles, say \( \beta_i \) cycles of length \( i \) for some \( \beta \vdash t \). Then the above analysis can be done for each of these cycles, replacing \( t \) by \( i \). The contribution of \( a_0N \) to \( h_g \) is then

\[
\prod_i \left( m^{i-1} \sum_{d|m} \mu(d) x_i^{m/d} \right)^{\beta_i} = m^{(i-1)\beta_i} \prod_i \left( \sum_{d|m} \mu(d) x_i^{m/d} \right)^{\beta_i} = m^{t-|\beta|} \prod_i \left( \sum_{d|m} \mu(d) x_i^{m/d} \right)^{\beta_i} \]

Summing over all elements of \( S_t \) yields the result.

Combining the last two lemmas, we get

**Theorem** For \( g \in S_n \) of type \( \tau \), one has

\[
h_g = \prod_m \left[ \sum_{\beta \vdash \tau_m} |K_\beta| m^{\tau_m-|\beta|} \prod_i \left( \sum_{d|m} \mu(d) x_i^{m/d} \right)^{\beta_i} \right].
\]

**Proof:** Clear.

**Corollary 1**

\[
\chi_t = \sum_{\sigma \vdash n \atop |\sigma| = t} \gamma_\sigma
\]

**Proof:** Again for \( g \) of type \( \tau \), we get by substitution

\[
h_g(x, \ldots, x) = \sum_{\sigma \vdash n} \gamma_\sigma(g)x^{|\sigma|} = \sum_{t=1}^n \left( \sum_{\sigma \vdash n \atop |\sigma| = t} \gamma_\sigma(g) \right) x^t.
\]

On the other hand, we get from the theorem and Lemma 2 that

\[
h_g(x, \ldots, x) = \prod_m \left[ \sum_{\beta \vdash \tau_m} |K_\beta| m^{\tau_m-|\beta|} \prod_i \left( \sum_{d|m} \mu(d) x_i^{m/d} \right)^{\beta_i} \right]
\]

\[
= \prod_m \left[ \sum_{\beta \vdash \tau_m} |K_\beta| m^{\tau_m-|\beta|} f_m(x)^{|\beta|} \right]
\]

\[
= \prod_m \left[ \prod_{j=0}^{\tau_m-1} (f_m(x) + jm) \right]
\]

\[
= p_\tau(x) = \sum_{t=1}^n \chi_t(g)x^t.
\]
Now compare coefficients to get
\[ \chi_t(g) = \sum_{|\sigma|=t} \gamma_\sigma(g) ; \]
this holds for every \( g \), hence the assertion.

**Remark 3**

(i) By the theorem, the character values of the \( \gamma_\sigma \)'s can be calculated in the polynomial ring \( \mathbb{Z}[x_1, \ldots, x_n] \). This is tedious, but purely mechanical work; note that the sizes of the conjugacy classes (the only information needed from the group) are given by a straightforward formula. For instance, let \( g \in S_8 \) be of type \( \tau = (2,3,0,0,0,0,0,0) \). For \( m = 1 \), there are two partitions of \( \tau_1 = 2 \), namely \((2,0)\) and \((0,1)\); the corresponding classes have both size 1, so the first factor in \( h_g \) is
\[ 1 \cdot 1^2 - 2 \cdot \left( \mu(1) x_1^{2/1} + \mu(2) x_2^{2/2} \right)^3 \]
\[ + 3 \cdot 2^3 - 2 \cdot \left( \mu(1) x_1^{2/1} + \mu(2) x_2^{2/2} \right)^1 \cdot \left( \mu(1) x_2^{2/2} + \mu(2) x_2^{2/2} \right)^1 \]
\[ + 2 \cdot 2^3 - 1 \cdot \left( \mu(1) x_3^{2/4} + \mu(2) x_3^{2/2} \right)^1 \]
\[ = (x_1^2 - x_2)^3 + 6 (x_1^2 - x_2)(x_2^2 - x_4) + 8 (x_3^2 - x_6) \]
\[ = x_1^6 - 3 x_1^2 x_2 + 9 x_1^2 x_2 - 6 x_2^2 x_4 - 7 x_3^2 + 6 x_2 x_4 + 8 x_3^2 - 8 x_6 . \]

All other \( \tau_m = 0 \), so the corresponding factors of \( h_g \) are 1. Therefore
\[ h_g = (x_1^2 + x_2)(x_1^6 - 3 x_1^4 x_2 + 9 x_1^2 x_2^2 - 6 x_2^2 x_4 - 7 x_3^2 + 6 x_2 x_4 + 8 x_3^2 - 8 x_6) \]
\[ = x_1^8 - 2 x_1^2 x_2^2 + 6 x_1^2 x_2^2 - 6 x_1^2 x_4 + 2 x_1^2 x_2^2 + 8 x_1^2 x_2^2 - 8 x_1^2 x_4 - 7 x_2^2 \]
\[ + 6 x_2 x_4 + 8 x_2 x_3^2 - 8 x_2 x_6 . \]

From this, we can read off the character values \( \gamma_\sigma(g) \); e.g. for \( \sigma = (0,4,0,0,0,0,0,0) \), we look at the coefficient of \( x_2^4 \) to find \( \gamma_\sigma(g) = -7 \).

(ii) Similarly (but much simpler), the values of \( \chi_t \) can be calculated in \( \mathbb{Z}[x] \).

(iii) There are as many \( \gamma_\sigma \)'s as there are irreducible characters, but they do not in general span the space of class functions. Here is a list giving — for a few small \( n \) — the class number \( k \) of \( S_n \) and the dimension \( d \) of the subspace spanned by the \( \gamma_\sigma \)'s:

\[
\begin{array}{ccccccccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 10 & 15 & 20 & 25 & 30 \\
 k & 1 & 2 & 3 & 5 & 7 & 42 & 176 & 627 & 1958 & 5604 \\
 d & 1 & 2 & 3 & 4 & 6 & 38 & 161 & 577 & 1816 & 5245 \\
\end{array}
\]

From these meager data, it looks as if the quotient \( d/k \) might tend to 1 for increasing \( n \), but I have not even an argument for the existence of a limit. As is clear from the theory, the algorithm is reasonably fast, considering that some of the character values are quite large; in any case, it takes longer to calculate the dimension of the subspace than to compute the monomial characters.
(iv) There is a symmetry for the values of $\gamma_\sigma$: If $g_\tau \in K_\tau$ and $g_\sigma \in K_\sigma$, then

$$|K_\tau|\gamma_\sigma(g_\tau) = |K_\sigma|\gamma_\sigma(g_\sigma).$$

To see this, note that

$$|K_\tau|\gamma_\sigma(g_\tau) = \gamma_\sigma(\hat{K}_\tau) = \sum_{a \in K_\sigma \atop b \in K_\tau \atop ab = ba} \lambda_a(b).$$

While it is not quite true that $\lambda_a(b) = \lambda_b(a)$, both are products of the same number of algebraically conjugate roots of unity. This holds in general, i.e. for an arbitrary finite group $G$ on any finite $G$-set $X$. Here is the argument:

Denote $A = \langle a \rangle$, $B = \langle b \rangle$ and $C = \langle a, b \rangle$ for commuting elements $a, b$ and fix an orbit $xC$. If $|xC| = t$, $|xA| = r$ and $|xB| = s$ then all orbits of $A$ on $xC$ have length $r$ (since $C$ is commutative). Therefore there are $t/r$ such $A$-orbits on $xC$, transitively permuted by $B$. Hence $b^{t/r}$ is the smallest power of $b$ fixing $xA$, say $xb^{t/r} = xa^d$, so the contribution of $xC$ to $\lambda_a(b)$ is the factor $\varepsilon_r^d$. Similarly, the contribution of $xC$ to $\lambda_b(a)$ is the factor $\varepsilon_s^e$, where $xa^{t/s} = xb^e$.

The last equation shows that $b^e$ fixes $xA$, so $e$ is a multiple of $t/r$, say $e = (t/r)u$ and similarly $d = (t/s)v$. Setting $\delta = \varepsilon_t^{(t/r)(t/s)} = \varepsilon_r^{t/s}$, it follows that $\varepsilon_r^d = \varepsilon^v \in <\delta>$, hence $<\varepsilon_r^d> \leq <\delta>$. In fact, equality holds, since from

$$xa^{t/s} = xb^e = xb^{(t/r)u} = xa^{du},$$

one concludes that $t/s \equiv du \mod r$, hence $\delta = \varepsilon_r^{t/s} = \varepsilon_r^{du} \in <\varepsilon_r^d>$. Similarly, $\delta$ and $\varepsilon_s^e$ are algebraically conjugate.

For $G = S_n$, one finds by summing (essentially as in the proof of Lemma 5) separately for the orbits of $\langle a, b \rangle$, that the contributions of each such orbit to $\gamma_\sigma(\hat{K}_\tau)$ and to $\gamma_\tau(\hat{K}_\sigma)$ are equal.

(v) This symmetry can be used as a check or as a shortcut in calculations. For instance, let $\tau = (0, \ldots, 0, 1)$ be the partition corresponding to the long cycles. From the theorem, we get then simply

$$h_\tau = \sum_{d|n} \mu(d) x_d^{n/d},$$

so

$$\gamma_\sigma(g_\tau) = \begin{cases} 
\mu(d) & \text{if } \sigma \text{ is homocyclic with cycle length } d \\
0 & \text{otherwise}.
\end{cases}$$

Therefore

$$\gamma_\tau(g_\sigma) = \frac{|K_\tau|}{|K_\sigma|} \gamma_\sigma(g_\tau)$$

$$= \begin{cases} 
\mu(d) & \text{if } \sigma \text{ is homocyclic with cycle length } d \\
0 & \text{otherwise}.
\end{cases}$$

$$= \begin{cases} 
\mu(d) d^{m-1}(m-1)! & \text{if } \sigma \text{ is homocyclic with } m \text{ cycles of length } d \\
0 & \text{otherwise}.
\end{cases}$$

(It is easy to check this directly from the definition of $\gamma_\tau$.)
Here is a more theoretical application: Let $\pi_z$ be the class function defined by $\pi_z(\sigma) = z^{\sigma}$, where $z \in \mathbb{Z}$. Using the symmetry $|K_\tau| \gamma_\sigma(\tau) = |K_\sigma| \gamma_\tau(\sigma)$, one calculates the inner product

$$(\gamma_\tau, \pi_z) = \frac{1}{n!} \sum_\sigma |K_\sigma| \gamma_\tau(\sigma) z^{\sigma} = \frac{1}{n!} \sum_\sigma |K_\tau| \gamma_\sigma(\tau) z^{\sigma} = \frac{1}{n!} \sum_{t=1}^n \left( \sum_{|\sigma| = t} \gamma_\sigma(\tau) \right) z^t,$$

hence by Corollary 1

$$\frac{1}{|C_\tau|} \sum_t \chi_t(\tau) z^t = \frac{1}{|C_\tau|} p_\tau(z),$$

where $|C_\tau|$ is the order of the centralizer in $S_n$ of an element of type $\tau$. In particular, $|C_\tau|$ divides $p_\tau(z)$ for every $z \in \mathbb{Z}$, since $\pi_z$ is a generalized character.

**Corollary 2** Let $n > 1$; denote $l = \left[ \frac{n}{2} \right]$,

$$\gamma_e = \sum_{|\sigma| \equiv 0 \text{ mod}(2)} \gamma_\sigma \quad \text{and} \quad \gamma_o = \sum_{|\sigma| \equiv 1 \text{ mod}(2)} \gamma_\sigma.$$

Then

(i) $$\gamma_e(1) = \gamma_o(1) = \frac{n!}{2}$$

(ii) $$\gamma_e(g) = -\gamma_o(g) = (-1)^n 2^{l-1} l!$$

if $g$ is a 'long involution', i.e. $g$ has $l$ orbits of length 2.

(iii) $$\gamma_e(g) = \gamma_o(g) = 0$$

for all other $g \in S_n$.

**Proof:** Since

$$f_m(1) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_m(-1) = \begin{cases} -1 & \text{if } m = 1 \\ 2 & \text{if } m = 2 \\ 0 & \text{otherwise} \end{cases},$$

we can – for $g$ of type $\tau$ – calculate that

$$\gamma_e(g) + \gamma_o(g) = \sum_{|\sigma| \equiv n} \gamma_\sigma(g) \quad = \sum_k \chi_k(g) \quad \text{by Cor.1} \quad = p_\tau(1) \quad = \prod_{m=1}^n \prod_{j=0}^{\tau_m-1} (f_m(1) + jm) \quad = \begin{cases} n! & \text{if } \tau = (n, 0, \ldots, 0) \\ 0 & \text{otherwise} \end{cases},$$

9
The assertions follow.

**Remark 4**  Since $\sum_\sigma \gamma_\sigma = \rho$, the sign character is a constituent (with multiplicity 1) of exactly one $\gamma_\sigma$; Frobenius reciprocity shows that this $\sigma$ is the type of the long involutions. Alternatively, one can deduce this from Remark 3(vi), since $(-1^n)\text{sign} = \pi_{-1}$.

**Remark 5**  The argument used in the proof of Corollary 2 can be generalized. For instance, for an odd prime $r$, one has $f_r(-2) = (-2)^r - (-2) = -(2^r - 2)$, so $f_r(-2) + jr = 0$ for $j = (2^r - 2)/r$. But if $\tau_r > j$, then $f_r(x) + jr$ is a factor of $p_r(x)$. For such $\tau$ then

$$0 = p_r(-2) = \sum_t \chi_t(\tau)(-2)^t = \sum_{\sigma \vdash n} \gamma_\sigma(\tau)(-2)^{|\sigma|} .$$

It follows from Remark 3(vi) again that $(\gamma_\tau, \pi_{-2}) = 0$, still under the assumption that $\tau_r > (2^r - 2)/r$ for some odd prime $r$. Since $\pi_{-2}$ is either a character or the negative of a character, this means that $\pi_{-2}$ and $\gamma_\tau$ have no common constituents. (Instead of $-2$, any other negative integer $z$ will do; of course, the condition on $\tau$ depends on $z$.)

**Remark 6**  There is a combinatorial interpretation of the polynomials $p_r$, hence of the characters $\chi_k$: Let $F = F_q$ be the field with $q$ elements, and let $E$ be a field extension with $|E : F| = m$. Then the map $d \mapsto F(d)$, where $|F(d)| = q^d$, is a bijection between the divisors $d$ of $m$ and the intermediate fields $F \leq F(d) \leq E$. Denote $A_d := \{a \in E \mid F[a] = F(d)\}$; then $q^m = |E| = \sum_{d|m} |A_d|$ . Möbius inversion yields

$$|A_m| = \sum_{d|m} \mu(m/d) q^d = f_m(q) .$$

Now every $a \in A_m$ has minimal polynomial of degree $m$ and each of these has $m$ different zeros, all in $A_m$, so the number of monic irreducible polynomial of degree $m$ in $F[x]$ is

$$f_m^*(q) := \frac{1}{m} f_m(q) .$$

Since every polynomial is (essentially uniquely) a product of irreducibles, we can define the type $\tau$ of a polynomial $h$ by letting $\tau_m$ be the number of irreducible factors of degree
\( m \) in \( h \); so \( \tau \vdash \deg(h) \). Of course, an irreducible factor may occur with a multiplicity, so the number of monic polynomials over \( F \) of a given type \( \tau \vdash n \) is

\[
p^*_\tau(q) = \prod_{m=1}^{n} \left( \frac{f^*_m(q) + \tau_m - 1}{\tau_m} \right).
\]

Multiplication of \( p^*_\tau \) by a suitable scalar gives a monic polynomial, more precisely \(|C_\tau| p^*_\tau = p_\tau\); recall that the order of the centralizer \(|C_\tau| = \prod m^{\tau_m} \tau_m!\). Using Remark 3(vi) once again, we conclude that the number of polynomials of type \( \tau \) (over \( F_q \)) equals the inner product \((\gamma_\tau, \pi_q)\).

**References**

