On Natural Monomial Characters of S_n

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Abstract

A class of natural linear characters for the centralizers of elements in the symmetric group is introduced. The character values of the corresponding monomial characters are calculated. They have a surprising combinatorial interpretation.
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For any $0 < m \in \mathbb{N}$, let

$$f_m(x) = \sum_{d|m} \mu(d)\, x^{m/d} \ ,$$

where μ is the Möbius function, so f_m is a monic polynomial of degree m over $\mathbb{Z}$. For $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{N}_0^n$, let

$$p_\tau(x) = \prod_{m=1}^{n} \prod_{j=0}^{\tau_m - 1} (f_m(x) + jm) \ ;$$

so if τ is a partition of n, i.e. $n = \sum_m m\, \tau_m$, then p_τ has degree n. Note that for $n \neq 0$ (which we assume throughout), the constant term of p_τ is 0, since this is true for every

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f_m. We expand

$$p_\tau(x) = \sum_{t=1}^{n} \chi_t(\tau)\, x^t \ ,$$

so this produces class functions $\chi_1, \ldots, \chi_n$ of the symmetric group S_n, where of course $\chi_t(g) = \chi_t(\tau)$ if $g \in S_n$ is of type τ, i.e. g has exactly τ_i orbits of length i (in its natural action on $\underline{n} = \{1, \ldots, n\}$). The aim of this note is to show that the χ_t's are characters of S_n . More precisely, Corollary 1 states that the χ_t's are sums of certain canonical monomial characters.

Notation

(i) For $k \in \mathbb{N}$, let $S_{n,k} = \{g \in S_n \,|\, g$ has exactly k orbits on $\underline{n}\}$, so $S_{n,k}$ is a union of conjugacy classes K_β of S_n , namely the ones of type β with $k = |\beta| := \sum_i \beta_i$. Clearly, $S_{n,k} = \emptyset$ if $k > n$; also, $S_{n,0} = \emptyset$ (since $n > 0$).

(ii) For $m \in \mathbb{N}$, let

$$\varepsilon_m = e^{\frac{2\pi i}{m}} \in \mathbb{C} \ ,$$

so ε_m is a primitive m-th root of unity. Note that $(\varepsilon_m)^d = \varepsilon_{m/d}$ for every $d|m$.

Lemma 1

$$\prod_{k=0}^{n-1}(x+k) = \sum_{k=0}^{n} |S_{n,k}|\, x^k$$

Proof: Given an element $g \in S_{n,k-1}$, we define $\widetilde{g} \in S_{n+1,k}$ by $\widetilde{g}(n+1) = n+1$ and $\widetilde{g} = g$ on $\underline{n}$. Given $g \in S_{n,k}$ and $1 \le i \le n$, we define $\widetilde{g}_i \in S_{n+1,k}$ by

$$\widetilde{g}_i(j) = \begin{cases} n+1 & \text{if } j = i \\ g(i) & \text{if } j = n+1 \\ g(j) & \text{otherwise.} \end{cases}$$

Then $\widetilde{\ }$ is a bijection between $S_{n,k-1} \cup S_{n,k} \times \underline{n}$ and $S_{n+1,k}$; in particular $|S_{n+1,k}| = |S_{n,k-1}| + |S_{n,k}|n$. From this, the assertion follows easily by induction.

Remark 1 For any $0 < m \in \mathbb{N}$, there is a natural action of S_n on the set $\underline{m}^{\underline{n}}$ of all maps $\underline{n} \to \underline{m}$. Such a map f is fixed by $g \in S_n$ if and only if f is constant on the orbits of g, so the number of fixed points of g is $m^{b(g)}$, where $b : S_n \to \mathbb{N}$ counts the orbits. Calculating the multiplicity of the trivial character in the permutation character π_m gives

$$(\pi_m, \mathbf{1}) = \frac{1}{n!}\sum_{g \in S_n} \pi_m(g) = \frac{1}{n!}\sum_t |S_{n,t}| m^t = \frac{1}{n!}\prod_{k=0}^{n-1}(m+k) = \binom{n+m-1}{n} \ ,$$

where the third equality follows from the lemma. By Burnside's lemma ([1], Corollary 5.15), this gives the number of orbits of S_n on $\underline{m}^{\underline{n}}$, hence the number of choices with repetitions of n objects from m. So get a well known formula from basic combinatorics. A similar argument allows us to calculate the multiplicity of the sign character sgn: using that $\pi_{-m} = (-1)^n sgn \cdot \pi_m$, one gets

$$(\pi_m, sgn) = \binom{m}{n} \ .$$

Lemma 2

$$\prod_{k=0}^{t-1}(f_m(x)+km) = \sum_{\beta\vdash t} |K_\beta|\, m^{t-|\beta|} f_m(x)^{|\beta|}$$

Proof:

$$\begin{aligned}
\prod_{k=0}^{t-1}(f_m(x)+km) &= m^t \prod_{k=0}^{t-1}(\tfrac{1}{m}f_m(x)+k) \\
&= m^t \sum_{k=0}^{t} |S_{t,k}|\,(\tfrac{1}{m}f_m(x))^k \qquad \text{(using Lemma 1)} \\
&= \sum_{k=0}^{t} |S_{t,k}|\, m^{t-k} f_m(x)^k \\
&= \sum_{\beta\vdash t} |K_\beta|\, m^{t-|\beta|} f_m(x)^{|\beta|} \quad .
\end{aligned}$$

Lemma 3

$$\mu(m) = \sum_{\substack{\varepsilon \text{ primitive} \\ m\text{-th root} \\ \text{of unity}}} \varepsilon$$

Proof:

$$\sum_{d|m} \sum_{\substack{\varepsilon \text{ primitive} \\ d\text{-th root} \\ \text{of unity}}} \varepsilon = \sum_{\substack{\varepsilon\ m\text{-th} \\ \text{root of} \\ \text{unity}}} \varepsilon = \begin{cases} 1 & \text{if } m=1 \\ 0 & \text{otherwise} \end{cases} = \sum_{d|m} \mu(d)\ ,$$

hence the assertion.

Definition, Remark 2

(i) Let G be a finite group acting on the finite set X. Fix some conjugacy class K of G and consider the set

$$M = \{(a,\mathcal{B})\,|\, a\in K,\ \mathcal{B} \text{ an orbit of } <a> \text{ on } X\}\ .$$

It is clear that M is a G-set by $(a,\mathcal{B})g = (a^g,\mathcal{B}g)$ and that $\alpha : M \ni (a,\mathcal{B}) \mapsto a \in K$ is a G-map. For every point $(a,\mathcal{B})\in M$, we define a linear character $(a,\mathcal{B})\theta$ of the stabilizer $G_{(a,\mathcal{B})}$ by

$$(a,\mathcal{B})\theta(g) = \varepsilon_{|\mathcal{B}|}^{j} \ \text{ if } \ xg = xa^j \ \text{ for some } \ x\in\mathcal{B}\ .$$

Since $g\in G_{(a,\mathcal{B})}$ commutes with a, the choice of x is irrelevant. Also j is unique modulo the length $|\mathcal{B}|$ of the orbit. Therefore $(a,\mathcal{B})\theta$ is well–defined and clearly multiplicative. Obviously $(a,\mathcal{B})\theta^g = (a^g,\mathcal{B}g)\theta$, so $\theta : (a,\mathcal{B}) \mapsto (a,\mathcal{B})\theta$ is an inductible map; it follows that $\gamma = \theta^\alpha$ is a character of G, in fact a monomial character induced from a linear character λ_a of $C_G(a)$ for $a\in K$ (compare [3] for the notation and simple facts concerning inductible maps and their induction). This character depends on X and K, so $\gamma = \gamma(X,K)$.

(ii) In the following, $G = S_n$ and $X = \underline{n}$, so it remains to specify the conjugacy class. As the classes are naturally labelled by the partitions σ of n, we use the partitions also as labels for the γ's and write $\gamma_\sigma := \gamma(\underline{n}, K_\sigma)$.

(iii) There is an alternative – and more familiar – description of the linear character λ_a of $C_{S_n}(a)$ from which γ_σ is induced. As is well known, corresponding to the decomposition of a (of type σ) in products of cycles of equal length, there is a direct product decomposition of $C_{S_n}(a)$. The factor $C^{(m)}$ corresponding to the cycles $a_1, \ldots, a_s$ (say) of length m in this direct product is in turn a semi-direct product of an abelian normal subgroup $A = <a_1, \ldots, a_s> \cong C_m \times \cdots \times C_m$ with a symmetric group S_s which acts by permuting the cycles. Therefore, there are m linear characters of A which are stable under S_s, hence extendable to $C^{(m)}$, so we can choose a linear character λ_m of $C^{(m)}$ which has order m and is trivial on S_s. This character is determined only up to algebraic conjugation, but we can avoid ambiguity by specifying that $\lambda_m(a_i) = \varepsilon_m$.
The product of these characters λ_m gives a character λ_a of $C_{S_n}(a)$. In fact, $\lambda_a = \vartheta_a^\alpha$, as is easily seen by calculating the values of these two linear characters on a cycle of a and on an element only permuting the cycles.
Incidentally, the choice of λ_m is irrelevant for $\gamma_\sigma = \lambda_a^{S_n}$: induce first to the Young subgroup $S_{1\sigma_1} \times \ldots \times S_{n\sigma_n}$ and use that all characters of a symmetric group are rational.

(iv) To summarize, for every $\sigma \vdash n$, we have a monomial character γ_σ of S_n with values given by

$$\gamma_\sigma(g) = \sum_{\substack{a \in K_\sigma \\ ga = ag}} \lambda_a(g) \ ,$$

where

$$\lambda_a(g) = \prod_i (a, \mathcal{B}_i) \theta(g^{e_i})$$

for a set $\mathcal{B}_i$ of representatives of the orbits of $<g>$ on the orbits of $<a>$ and $e_i = | <g> : <g>_{\mathcal{B}_i} |$.

(v) For bookkeeping, it is useful to introduce $X^\sigma := \prod_{i=1}^{n} x_i^{\sigma_i}$, a monomial in n variables of total degree $|\sigma|$, and

$$h_g(x_1, \ldots, x_n) = \sum_{\sigma \vdash n} \gamma_\sigma(g)\, X^\sigma \ \in \mathbb{Z}[x_1, \ldots, x_n] \ ,$$

a polynomial that collects the character values of the γ_σ's at an element $g \in S_n$; of course, $h_g = h_\tau$ depends only on the conjugacy class K_τ of g.

(vi) It is clear that

$$h_g = \sum_{\sigma \vdash n} \gamma_\sigma(g) X^\sigma = \sum_{\sigma \vdash n} \sum_{\substack{a \in K_\sigma \\ ag = ga}} \lambda_a(g) X^\sigma = \sum_{a \in C} \lambda_a(g) X^{\sigma(a)} \ ,$$

where $C = C_{S_n}(g)$ and $\sigma(a)$ is the type of a.

Lemma 4 Let g_m be the product of all cycles of length m of $g \in S_n$, viewed as an element of $S_{m\tau_m}$, where τ is the type of g. Then

$$h_g = \prod_m h_{g_m} \ .$$

Proof: Let $T_m \subseteq \underline{n}$ be the union of all orbits of length m of g and $H_m = S_{T_m}$, the symmetric group on T_m; also denote $C_m = C_{H_m}(g_m)$. Then

$$h_{g_m} = \sum_{a_m \in C_m} \lambda_{a_m}(g_m) X^{\sigma(a_m)} \quad ,$$

so

$$\prod_m h_{g_m} = \sum_{\substack{(a_1,\ldots,a_n) \\ a_m \in C_m}} \lambda_{a_1}(g_1)\ldots\lambda_{a_n}(g_n) X^{\sigma(a_1)} \ldots X^{\sigma(a_n)} \quad .$$

Now $C_1 \times \cdots \times C_n \ni (a_1,\ldots,a_n) \mapsto a := a_1 \cdots a_n$ is a bijection $C_1 \times \cdots \times C_n \to C := C_{S_n}(g)$; clearly, $\sigma(a) = \sigma(a_1) + \cdots + \sigma(a_n)$ and by definition $\lambda_a(g) = \lambda_{a_1}(g_1)\ldots\lambda_{a_n}(g_n)$. Therefore, the sum on the right simplifies to

$$\sum_{a \in C} \lambda_a(g) X^{\sigma(a)} = h_g$$

as claimed.

Lemma 5 Let g be homocyclic, say g is the product of t cycles of length m. Then

$$h_g = \sum_{\beta \vdash t} |K_\beta|\, m^{t-|\beta|} \prod_i \left(\sum_{d|m} \mu(d)\, x_{i\cdot d}^{\;m/d} \right)^{\beta_i} \quad .$$

Proof: Let $n = m \cdot t$ and $C = C_{S_n}(g)$. Since

$$h_g = \sum_{a \in C} \lambda_a(g) X^{\sigma(a)} \quad ,$$

we have to calculate the contribution of $a \in C$ to this sum.
Since C is a semi-direct product of S_t and an abelian normal subgroup $N = <g_1,\ldots,g_t>$, where $g = g_1 \cdot \ldots \cdot g_t$ is the cycle decomposition, every element $a \in C$ can be written as $a = a_0 \cdot g_1^{e_1} \cdot \ldots \cdot g_t^{e_t}$ with $a_0 \in S_t$. We consider first the case that a_0 is a long cycle, so a_0 has order t. Denote $A = <a>$ and $D := <g, a>$; so D is an abelian transitive subgroup of S_n. Let l be the order of a. Then clearly $t|l$; since $a^t = g^e$, where $e = \sum_i e_i$ and since the order of g^e is $d := m/\gcd(e,m)$, we find that $l = t \cdot d$. This is then the length of every orbit of A, so A has $m \cdot t/l = m/d$ orbits. The corresponding monomial is therefore $x_{t\cdot d}^{m/d}$. To calculate the coefficient, note that $g^{m/d}$ and a^t generate the same subgroup (of order d), so $g^{m/d} = a^{t\cdot u}$ for some u prime to d. Therefore $\lambda_a(g) = \varepsilon_l^{t\cdot u} = \varepsilon_d^u$ is a primitive d-th root of unity.
Now take $a' = a_0 \cdot g_1^{e'_1} \cdot \ldots \cdot g_t^{e'_t}$ and let $e' = \sum_i e'_i$. For any $0 \le s < m$, there are m^{t-1} solutions $(e'_1, \ldots e'_t)$ for $e' \equiv s \bmod(m)$ with $0 \le e'_i < m$ for all i. If we collect those for which $\gcd(e', m) = m/d$ for some fixed divisor d of m, the monomial is always $x_{t\cdot d}^{m/d}$ and each primitive d-th root of unity appears m^{t-1} times as a coefficient. By Lemma 3, we get $m^{t-1}\mu(d)\, x_{t\cdot d}^{m/d}$ for fixed d and

$$\sum_{d|m} m^{t-1}\mu(d)\, x_{t\cdot d}^{m/d} = m^{t-1} \sum_{d|m} \mu(d)\, x_{t\cdot d}^{m/d}$$

as contribution of $a_0 N$ to h_g.
A general element a_0 of S_t will have several cycles, say β_i cycles of length i for some $\beta \vdash t$. Then the above analysis can be done for each of these cycles, replacing t by i. The contribution of $a_0 N$ to h_g is then

$$\begin{aligned}\prod_i \left(m^{i-1} \sum_{d|m} \mu(d)\, x_{i\cdot d}^{m/d} \right)^{\beta_i} &= m^{\sum_i (i-1)\beta_i} \prod_i \left(\sum_{d|m} \mu(d)\, x_{i\cdot d}^{m/d} \right)^{\beta_i} \\ &= m^{t-|\beta|} \prod_i \left(\sum_{d|m} \mu(d)\, x_{i\cdot d}^{m/d} \right)^{\beta_i} \quad .\end{aligned}$$

Summing over all elements of S_t yields the result.

Combining the last two lemmas, we get

Theorem For $g \in S_n$ of type τ, one has

$$h_g = \prod_m \left[\sum_{\beta \vdash \tau_m} |K_\beta|\, m^{\tau_m - |\beta|} \prod_i \left(\sum_{d|m} \mu(d)\, x_{i\cdot d}^{m/d} \right)^{\beta_i} \right] \quad .$$

Proof: Clear.

Corollary 1

$$\chi_t = \sum_{\substack{\sigma \vdash n \\ |\sigma| = t}} \gamma_\sigma$$

Proof: Again for g of type τ, we get by substitution

$$h_g(x, \ldots, x) = \sum_{\sigma \vdash n} \gamma_\sigma(g) x^{|\sigma|} = \sum_{t=1}^{n} \left(\sum_{\substack{\sigma \vdash n \\ |\sigma| = t}} \gamma_\sigma(g) \right) x^t \quad .$$

On the other hand, we get from the theorem and Lemma 2 that

$$\begin{aligned} h_g(x, \ldots, x) &= \prod_m \left[\sum_{\beta \vdash \tau_m} |K_\beta|\, m^{\tau_m - |\beta|} \prod_i \left(\sum_{d|m} \mu(d)\, x^{m/d} \right)^{\beta_i} \right] \\ &= \prod_m \left[\sum_{\beta \vdash \tau_m} |K_\beta|\, m^{\tau_m - |\beta|} f_m(x)^{|\beta|} \right] \\ &= \prod_m \prod_{j=0}^{\tau_m - 1} (f_m(x) + jm) \\ &= p_\tau(x) = \sum_{t=1}^{n} \chi_t(g) x^t \quad . \end{aligned}$$

Now compare coefficients to get

$$\chi_t(g) = \sum_{\substack{\sigma \vdash n \\ |\sigma| = t}} \gamma_\sigma(g) \quad ;$$

this holds for every g, hence the assertion.

Remark 3

(i) By the theorem, the character values of the γ_σ's can be calculated in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$. This is tedious, but purely mechanical work; note that the sizes of the conjugacy classes (the only information needed from the group) are given by a straightforward formula. For instance, let $g \in S_8$ be of type $\tau = (2, 3, 0, 0, 0, 0, 0, 0)$. For $m = 1$, there are two partitions of $\tau_1 = 2$, namely $(2, 0)$ and $(0, 1)$; the corresponding classes have both size 1, so the first factor in h_g is

$$1 \cdot 1^{2-2} \cdot \left(\mu(1)\, x_{1\cdot 1}^{1/1}\right)^2 + 1 \cdot 1^{2-1} \cdot \left(\mu(1)\, x_{2\cdot 1}^{1/1}\right)^1 = x_1^2 + x_2 \quad .$$

For $m = 2$, there are three partitions of $\tau_2 = 3$, namely $(3, 0, 0)$, $(1, 1, 0)$ and $(0, 0, 1)$; the corresponding classes have size 1, 3 and 2 respectively, so the second factor in h_g is

$$\begin{aligned}
&1 \cdot 2^{3-3} \cdot \left(\mu(1)\, x_{1\cdot 1}^{2/1} + \mu(2)\, x_{1\cdot 2}^{2/2}\right)^3 \\
&+ 3 \cdot 2^{3-2} \cdot \left(\mu(1)\, x_{1\cdot 1}^{2/1} + \mu(2)\, x_{1\cdot 2}^{2/2}\right)^1 \cdot \left(\mu(1)\, x_{2\cdot 1}^{2/1} + \mu(2)\, x_{2\cdot 2}^{2/2}\right)^1 \\
&+ 2 \cdot 2^{3-1} \cdot \left(\mu(1)\, x_{3\cdot 1}^{2/1} + \mu(2)\, x_{3\cdot 2}^{2/2}\right)^1 \\
&\qquad = (x_1^2 - x_2)^3 + 6\,(x_1^2 - x_2)(x_2^2 - x_4) + 8\,(x_3^2 - x_6) \\
&\qquad = x_1^6 - 3\,x_1^4 x_2 + 9\,x_1^2 x_2^2 - 6\,x_1^2 x_4 - 7\,x_2^3 + 6\,x_2 x_4 + 8\,x_3^2 - 8\,x_6 \quad .
\end{aligned}$$

All other $\tau_m = 0$, so the corresponding factors of h_g are 1. Therefore

$$\begin{aligned}
h_g &= (x_1^2 + x_2)(x_1^6 - 3\,x_1^4 x_2 + 9\,x_1^2 x_2^2 - 6\,x_1^2 x_4 - 7\,x_2^3 + 6\,x_2 x_4 + 8\,x_3^2 - 8\,x_6) \\
&= x_1^8 - 2\,x_1^6 x_2 + 6\,x_1^4 x_2^2 - 6\,x_1^4 x_4 + 2\,x_1^2 x_2^3 + 8\,x_1^2 x_3^2 - 8\,x_1^2 x_6 - 7\,x_2^4 \\
&\qquad + 6\,x_2^2 x_4 + 8\,x_2 x_3^2 - 8\,x_2 x_6 \quad .
\end{aligned}$$

From this, we can read off the character values $\gamma_\sigma(g)$; e.g. for $\sigma = (0, 4, 0, 0, 0, 0, 0, 0)$, we look at the coefficient of x_2^4 to find $\gamma_\sigma(g) = -7$.

(ii) Similarly (but much simpler), the values of χ_t can be calculated in $\mathbb{Z}[x]$.

(iii) There are as many γ_σ's as there are irreducible characters, but they do not in general span the space of class functions. Here is a list giving – for a few small n – the class number k of S_n and the dimension d of the subspace spanned by the γ_σ's :

n	1	2	3	4	5	10	15	20	25	30
k	1	2	3	5	7	42	176	627	1958	5604
d	1	2	3	4	6	38	161	577	1816	5245

From these meager data, it looks as if the quotient d/k might tend to 1 for increasing n, but I have not even an argument for the existence of a limit. As is clear from the theory, the algorithm is reasonably fast, considering that some of the character values are quite large; in any case, it takes longer to calculate the dimension of the subspace than to compute the monomial characters.

(iv) There is a symmetry for the values of γ_σ: If $g_\tau \in K_\tau$ and $g_\sigma \in K_\sigma$, then

$$|K_\tau|\gamma_\sigma(g_\tau) = |K_\sigma|\gamma_\tau(g_\sigma) \quad .$$

To see this, note that

$$|K_\tau|\gamma_\sigma(g_\tau) = \gamma_\sigma(\widehat{K}_\tau) = \sum_{\substack{a\in K_\sigma\\ b\in K_\tau\\ ab=ba}} \lambda_a(b) \quad .$$

While it is not quite true that $\lambda_a(b) = \lambda_b(a)$, both are products of the same number of algebraically conjugate roots of unity. This holds in general, i.e. for an arbitrary finite group G on any finite G-set X. Here is the argument:
Denote $A = <a>$, $B = <b>$ and $C = <a, b>$ for commuting elements a, b and fix an orbit xC. If $|xC| = t$, $|xA| = r$ and $|xB| = s$ then all orbits of A on xC have length r (since C is commutative). Therefore there are t/r such A-orbits on xC, transitively permuted by B. Hence $b^{t/r}$ is the smallest power of b fixing xA, say $xb^{t/r} = xa^d$, so the contribution of xC to $\lambda_a(b)$ is the factor ε_r^d. Similarly, the contribution of xC to $\lambda_b(a)$ is the factor ε_s^e, where $xa^{t/s} = xb^e$.
The last equation shows that b^e fixes xA, so e is a multiple of t/r, say $e = (t/r)u$ and similarly $d = (t/s)v$. Setting $\delta = \varepsilon_t^{(t/r)(t/s)} = \varepsilon_r^{t/s}$, it follows that $\varepsilon_r^d = \delta^v \in <\delta>$, hence $<\varepsilon_r^d> \leq <\delta>$. In fact, equality holds, since from

$$xa^{t/s} = xb^e = xb^{(t/r)u} = xa^{du} \ ,$$

one concludes that $t/s \equiv du \bmod r$, hence $\delta = \varepsilon_r^{t/s} = \varepsilon_r^{du} \in <\varepsilon_r^d>$. Similarly, δ and ε_s^e are algebraically conjugate.
For $G = S_n$, one finds by summing (essentially as in the proof of Lemma 5) separately for the orbits of $<a, b>$, that the contributions of each such orbit to $\gamma_\sigma(\widehat{K}_\tau)$ and to $\gamma_\tau(\widehat{K}_\sigma)$ are equal.

(v) This symmetry can be used as a check or as a shortcut in calculations. For instance, let $\tau = (0, \ldots, 0, 1)$ be the partition corresponding to the long cycles. From the theorem, we get then simply

$$h_\tau = \sum_{d|n} \mu(d)\, x_d^{n/d} \quad ,$$

so

$$\gamma_\sigma(g_\tau) = \begin{cases} \mu(d) & \text{if } \sigma \text{ is homocyclic with cycle length } d \\ 0 & \text{otherwise} \quad . \end{cases}$$

Therefore

$$\begin{aligned} \gamma_\tau(g_\sigma) &= \tfrac{|K_\tau|}{|K_\sigma|}\,\gamma_\sigma(g_\tau) \\ &= \begin{cases} \mu(d)\tfrac{|K_\tau|}{|K_\sigma|} & \text{if } \sigma \text{ is homocyclic with cycle length } d \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu(d)\, d^{m-1}(m-1)! & \text{if } \sigma \text{ is homocyclic with } m \text{ cycles of length } d \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(It is easy to check this directly from the definition of γ_τ.)

(vi) Here is a more theoretical application: Let π_z be the class function defined by $\pi_z(\sigma) = z^{|\sigma|}$, where $z \in \mathbb{Z}$. Using the symmetry $|K_\tau|\gamma_\sigma(\tau) = |K_\sigma|\gamma_\tau(\sigma)$, one calculates the inner product

$$(\gamma_\tau, \pi_z) = \frac{1}{n!}\sum_\sigma |K_\sigma|\gamma_\tau(\sigma)z^{|\sigma|} = \frac{1}{n!}\sum_\sigma |K_\tau|\gamma_\sigma(\tau)z^{|\sigma|} = \frac{|K_\tau|}{n!}\sum_{t=1}^{n}\left(\sum_{\substack{\sigma\vdash n\\|\sigma|=t}}\gamma_\sigma(\tau)\right)z^t \quad ,$$

hence by Corollary 1

$$(\gamma_\tau, \pi_z) = \frac{1}{|C_\tau|}\sum_t \chi_t(\tau)\, z^t = \frac{1}{|C_\tau|}\, p_\tau(z) \quad ,$$

where $|C_\tau|$ is the order of the centralizer in S_n of an element of type τ. In particular, $|C_\tau|$ divides $p_\tau(z)$ for every $z \in \mathbb{Z}$, since π_z is a generalized character.

Corollary 2 Let $n > 1$; denote $l = [\frac{n}{2}]$,

$$\gamma_e \; = \sum_{\substack{\sigma\vdash n\\|\sigma|\equiv 0 \bmod(2)}} \gamma_\sigma \quad \text{and} \quad \gamma_o \; = \sum_{\substack{\sigma\vdash n\\|\sigma|\equiv 1 \bmod(2)}} \gamma_\sigma \quad .$$

Then

(i)

$$\gamma_e(1) = \gamma_o(1) = \frac{n!}{2}$$

(ii)

$$\gamma_e(g) = -\gamma_o(g) = (-1)^n\, 2^{l-1}\, l!$$

if g is a 'long involution', i.e. g has l orbits of length 2.

(iii)

$$\gamma_e(g) = \gamma_o(g) = 0$$

for all other $g \in S_n$.

Proof: Since

$$f_m(1) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \quad f_m(-1) = \begin{cases} -1 & \text{if } m = 1 \\ 2 & \text{if } m = 2 \\ 0 & \text{otherwise,} \end{cases}$$

we can – for g of type τ – calculate that

$$\begin{aligned} \gamma_e(g) + \gamma_o(g) &= \sum_{\sigma\vdash n}\gamma_\sigma(g) \\ &= \sum_k \chi_k(g) \quad \text{by Cor.1} \\ &= p_\tau(1) \\ &= \prod_{m=1}^{n}\prod_{j=0}^{\tau_m-1}(f_m(1) + jm) \\ &= \begin{cases} n! & \text{if } \tau = (n, 0, \ldots, 0) \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

so $\gamma_e + \gamma_o = \rho$, the regular character of S_n . Similarly,

$$\begin{aligned}
\gamma_e(g) - \gamma_o(g) &= \sum_k (-1)^k \chi_k(g) \\
&= p_\tau(-1) \\
&= \prod_{m=1}^{n} \prod_{j=0}^{\tau_m - 1} (f_m(-1) + jm) \\
&= \begin{cases} \prod_{j=0}^{l-1} (2 + j2) & \text{if } \tau = (0, l, 0, \ldots, 0) \ \ (n \text{ even}) \\ -\prod_{j=0}^{l-1} (2 + j2) & \text{if } \tau = (1, l, 0, \ldots, 0) \ \ (n \text{ odd}) \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} (-1)^n \, 2^l \, l! & \text{if } g \text{ is a long involution} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The assertions follow.

Remark 4 Since $\sum_\sigma \gamma_\sigma = \rho$, the sign character is a constituent (with multiplicity 1) of exactly one γ_σ ; Frobenius reciprocity shows that this σ is the type of the long involutions. Alternatively, one can deduce this from Remark 3(vi), since $(-1^n) sign = \pi_{-1}$.

Remark 5 The argument used in the proof of Corollary 2 can be generalized. For instance, for an odd prime r, one has $f_r(-2) = (-2)^r - (-2) = -(2^r - 2)$, so $f_r(-2) + jr = 0$ for $j = (2^r - 2)/r$. But if $\tau_r > j$, then $f_r(x) + jr$ is a factor of $p_\tau(x)$. For such τ then

$$0 = p_\tau(-2) = \sum_t \chi_t(\tau)(-2)^t = \sum_{\sigma \vdash n} \gamma_\sigma(\tau)(-2)^{|\sigma|} \ .$$

It follows from Remark 3(vi) again that $(\gamma_\tau, \pi_{-2}) = 0$, still under the assumption that $\tau_r > (2^r - 2)/r$ for some odd prime r. Since π_{-2} is either a character or the negative of a character, this means that π_{-2} and γ_τ have no common constituents. (Instead of -2, any other negative integer z will do; of course, the condition on τ depends on z.)

Remark 6 There is a combinatorial interpretation of the polynomials p_τ, hence of the characters χ_k: Let $F = F_q$ be the field with q elements, and let E be a field extension with $|E : F| = m$. Then the map $d \mapsto F(d)$, where $|F(d)| = q^d$, is a bijection between the divisors d of m and the intermediate fields $F \leq F(d) \leq E$. Denote $A_d := \{a \in E \,|\, F[a] = F(d)\}$; then $q^m = |E| = \sum_{d|m} |A_d|$. Möbius inversion yields

$$|A_m| = \sum_{d|m} \mu(\frac{m}{d}) \, q^d = f_m(q) \ .$$

Now every $a \in A_m$ has minimal polynomial of degree m and each of these has m different zeros, all in A_m, so the number of monic irreducible polynomial of degree m in $F[x]$ is

$$f_m^*(q) := \frac{1}{m} f_m(q) \ .$$

Since every polynomial is (essentially uniquely) a product of irreducibles, we can define the *type* τ of a polynomial h by letting τ_m be the number of irreducible factors of degree

m in h; so $\tau \vdash \deg(h)$. Of course, an irreducible factor may occur with a multiplicity, so the number of monic polynomials over F of a given type $\tau \vdash n$ is

$$p^*_\tau(q) = \prod_{m=1}^{n} \binom{f^*_m(q) + \tau_m - 1}{\tau_m} \ .$$

Multiplication of p^*_τ by a suitable scalar gives a monic polynomial, more precisely $|C_\tau|\, p^*_\tau = p_\tau$; recall that the order of the centralizer $|C_\tau| = \prod\limits_m m^{\tau_m}\tau_m!$. Using Remark 3(vi) once again, we conclude that the number of polynomials of type τ (over F_q) equals the inner product (γ_τ, π_q).

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