A note on permutation groups

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Abstract

A k–transitive group of degree $n > k$ is generated by all the the elements fixing precisely i points if $i \leq k-3$. The cases $i = k-2$, $i = k-1$ and $i = k$ are also considered.
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Throughout, G is a finite group and X a finite G–set, say $|X| = n$. The permutation character of G on X is denoted by π. For $0 \leq i \leq n$, we set

$$N_i = N_i(G,X) = \langle\, g \in G \,|\, \pi(g) = i \,\rangle \quad .$$

It is clear that N_i is normal in G, since its generating set is closed under conjugation. There are good general reasons why normal subgroups of multiply transitive groups tend to be large. Here, only the N_i's are considered .

Theorem

(i) Assume G transitive. Then N_0 is transitive and $N_i \leq N_0$ for every $i \neq 1$; also $N_0 = G$ or $N_1 = G$.

(ii) Assume G 2–transitive. Then $1 + |G : N_0| = (\pi, \pi)_{N_0}$; also N_0 is 2–transitive if and only if $N_0 = G$.

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(iii) Let G be k–transitive for some $0 < k < n$. Then N_i is $(i+1)$–transitive for every $i < k$.

(iv) Let G be k–transitive for some $1 < k < n$. Then $N_0 = N_1 = \cdots = N_{k-3} = G$; also, $N_{k-1} = G$ or $N_{k-2} = N_k = G$.

Proof:

(i) Since G is transitive, we have $|G| = |G|(\pi, \mathbf{1})_G = \sum_{g \in G} \pi(g)$ (see [1], Proposition 16.9). Clearly, $|N_0| \le |N_0|(\pi, \mathbf{1})_{N_0} = \sum_{n \in N_0} \pi(n)$ with equality if and only if N_0 is transitive. All elements outside N_0 fix at least one point by construction of N_0, so

$$|G \setminus N_0| \le \sum_{g \in G \setminus N_0} \pi(g)$$

with equality if and only if $\pi(g) = 1$ for all such g. Combining the inequalities, we have

$$|G| = |N_0| + |G \setminus N_0| \le \sum_{n \in N_0} \pi(n) + \sum_{g \in G \setminus N} \pi(g) = \sum_{g \in G} \pi(g) = |G| \quad ,$$

so we must have equality throughout. This proves transitivity of N_0; also all elements outside N_0 fix exactly one point, so belong to N_1. Therefore $G = N_0 \cup N_1$, but no group is a union of two proper subgroups. For $i \neq 1$, the generating elements of N_i belong to N_0, so $N_i \le N_0$.

(ii) $2 = (\pi, \pi)_G$ since G is 2–transitive (see again [1], Proposition 16.9). Therefore

$$2|G| = \sum_{g \in G} \pi(g)^2 = \sum_{n \in N_0} \pi(n)^2 + \sum_{g \in G \setminus N_0} \pi(g)^2 = |N_0|(\pi, \pi)_{N_0} + |G \setminus N_0|$$

using (i), so

$$(\pi, \pi)_{N_0} = \frac{2|G| - |G \setminus N_0|}{|N_0|} = |G : N_0| + 1 \quad .$$

By the result just quoted, N_0 is 2–transitive if and only if this value is 2, i.e. if and only if $G = N_0$.

(iii) By (i), we may assume $i > 0$, so $k > 1$. Pick i different points $x_1, \dots, x_i \in X$; since G is (at least) $(i+1)$–transitive by assumption, their stabilizer $H := G_{x_1, \dots, x_i}$ acts transitively on $X' := X \setminus \{x_1, \dots, x_i\}$. Let $M_r = N_r(H, X')$, so

$$M_r = \langle\, h \in H \,|\, \pi(h) = i + r \,\rangle \le N_{i+r} \quad .$$

By (i), M_0 acts transitively on X'. Since $|X'| = n - i \ge n - (k-1) > 1$, this action cannot be trivial, so N_i acts non–trivially on X. Since G is 2–transitive, N_i is transitive by a well-known result (see e.g. [1], Propositions 3.8 and 4.4). To show $(i+1)$–transitivity of N_i, it is therefore enough to show that a point stabilizer $(N_i)_x = N_i \cap G_x$ is i–transitive on $X^* := X \setminus x$. But $N_i \cap G_x \ge N_{i-1}(G_x, X^*)$ and this last group is i–transitive by induction.

(iv) The group H above is clearly $(k-i)$–transitive on X', so if $i \le k-3$, then H is 3–transitive, so by (iii) M_2 is 3–transitive on X' as well. Since $M_2 \le M_0$ by (i), we find that M_0 is 2–transitive, so $M_0 = H$ by (ii) and a fortiori $N_i \ge H$. But by i–transitivity of N_i, we have $G = HN_i = N_i$.

Finally, assume $N_{k-1} \ne G$; we must show $N_{k-2} = N_k = G$. First, let H and X' be as before but with $i = k-2$. Note that $G = HN_{k-2} = HN_{k-1}$ since both N_{k-2} and N_{k-1} are $(k-2)$–transitive by (iii). If $H = M_1 \le N_{k-1}$, we find $G = N_{k-1}$ contrary to the assumption. Since H is transitive (even 2–transitive) on X', we get from (i) that $H = M_0 \le N_{k-2}$, so $G = N_{k-2}$.

To show $G = N_k$, we repeat the argument, this time with $i = k-1$. Since $N_{k-1} \ne G$, we conclude now that $H \ne M_0$, so $H = M_1 \le N_k$. We are done if N_k is $(k-1)$–transitive, since then $G = HN_k = N_k$. Clearly, N_k is transitive, since it is a normal subgroup acting non-trivially (since M_1 acts non trivially on X'). Suppose we know already that N_k acts t–transitively for some $t < k-1$ and consider $L := N_k \cap G_{x_1,\dots,x_t}$. Since $L \lhd G_{x_1,\dots,x_t}$ and $G_{x_1,\dots,x_t}$ acts $(k-t)$–, so at least 2–transitively on the remaining elements, and since L acts non–trivially (because $M_1 \le L$), we find that L acts transitively on $X \setminus \{x_1, \dots, x_t\}$, so N_k is $(t+1)$–transitive. By induction, N_k is $(k-1)$–transitive, as claimed.

Examples For the first three examples, assume $n \ge 4$.

(1) Let $G = S_n$ acting on n letters and let $k = n-1$. Since no permutation fixes exactly $n-1$ letters, we find $N_k = 1$, so $G = N_{k-1}$ by (iv) above; this is the well known fact that S_n is generated by involutions. The same conclusion follows from the observation that N_{k-2} is generated by 3–cycles, hence contained in A_n, so in particular $N_{k-2} \ne G$.

(2) Let $G = A_n$ acting on n letters and let $k = n-2$. Then A_n is k–transitive. Since no involution belongs to A_n, we find again $N_k = 1$, so $G = N_{k-1}$ by (iv) above, that is A_n is generated by 3–cycles.

(3) Take again $G = A_n$ and $k = n-2$. Then N_{k-2} is generated by all even permutations moving exactly 4 points, so $N_{k-2} = V_4 < A_4$ if $n = 4$, while $N_{k-2} = A_n$ if $n > 4$. Of course, since A_n is then a simple group, $N_i = A_n$ for all $i < n-2$.

(4) If G is a Frobenius group, N_0 is the Frobenius kernel.

(5) Every sharply 2–transitive group provides an example with N_0 *not* 2–transitive.

(6) It is well known that $G = PGL(2,q)$ is sharply 3–transitive on the projective line, so $n = q+1$. Clearly $N_i = 1$ if $i \ge 3$.

If $q = 2$, then $G = S_3$, $N_0 = A_3$, $N_1 = G$ and $N_2 = 1$.

If $q > 2$, then N_2 is 3–transitive by (iii), hence $N_2 = G$; also $N_0 = G$ by (iv) (or by (i)). Since every $g \in G$ with $\pi(g) = 1$ belongs to $PSL(2,q)$ (its pre-image in $GL(2,q)$ has determinant a square), it follows that $N_1 \le PSL(2,q)$. Since N_1 is 2–transitive, certainly $N_1 \ne 1$. Therefore $N_1 = PSL(2,q)$, if this group is simple. It is easy to see that this holds in the exceptional case $q = 3$ as well. Therefore $N_1 < G$ if and only if q is odd.

References

[1] D.S. Passman, Permutation groups, W.A.Benjamin, New York 1968