Mandatory Representation Designs MRD$(4, k; v)$ with $k \equiv 2 \bmod 3$

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Abstract

A mandatory representation design MRD$(K; v)$ is a pairwise balanced design on v points with block sizes from the set K in which for each $k \in K$ there is at least one block in the design of size k. In this paper, we show that the necessary criteria for an MRD$(K; v)$ to exist are asymptotically sufficient for finite K. Furthermore, we consider MRDs with $K = \{4, k\}$, where $k \equiv 2 \bmod 3, k \geq 5$. Here, we prove that the necessary conditions for existence are sufficient if $v \equiv 2 \bmod 3$ and $v \geq 18k^2$, or $v \equiv 0 \bmod 3$ and $v \geq 12k^3$, or $v \equiv 1 \bmod 3$ and $v \geq 8k^4$.

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1 Introduction

A pairwise balanced design PBD is a pair $(X, \mathcal{B})$, where X is a set of *points* and $\mathcal{B}$ is a collection of subsets of X called *blocks*, such that each pair of distinct points from X occurs in a unique block. A PBD$(K; v)$ is a pairwise balanced design on v points in which each block has size an integer in the set K. A *mandatory representation design* MRD$(K; v)$ is a PBD$(K; v)$ in which for each $k \in K$ there is at least one block in the design of size k. Necessary conditions for the existence of a PBD$(K; v)$ are

$$(v-1) \equiv 0 \bmod \alpha(K) \quad \text{and} \quad v(v-1) \equiv 0 \bmod \beta(K), \qquad (1)$$

where $\alpha(K) = \gcd\{k-1 \,|\, k \in K\}$ and $\beta(K) = \gcd\{k(k-1) \,|\, k \in K\}$. In a series of three papers R.M. Wilson [27, 28, 29] developed an existence theory for PBDs and proved that the necessary conditions are asymptotically sufficient, that is, there exists a constant $v_0(K)$ such that a PBD(v, K) exists for all $v \geq v_0(K)$ which satisfy the congruences in (1). The problem is that we can not conclude that every block size occurs in such a PBD. So using a result of Lamken and Wilson [18] we will prove in Section 2 that the necessary conditions (1) for the existence of an MRD are asymptotically sufficient for finite K.

Although the existence proof of Lamken and Wilson is somehow constructive, the estimate of the constant is very large. Therefore, one attempts to determine the spectrum $B(K) = \{v : \exists MRD(K; v)\}$ for given K as accurately as possible. Mandatory representation designs have been extensively studied by Mendelsohn and Rees [20], Rees [22, 23], Grüttmüller [13], Grüttmüller and Rees [15, 17, 16], and Ge [10]. In particular, in the case $K = \{4, k\}$ with $k \equiv 1 \bmod 3$ we have the following result which is the culmination of the contributions of several authors [6, 7, 10, 17, 24, 25, 26]. Note that the MRDs in part (i) are equivalent to the embedding of a $(k, 4, 1)$-BIBD into a $(v, 4, 1)$-BIBD.

Theorem 1.1 *Let $k \equiv 1 \bmod 3$. There exists a mandatory representation design MRD$(\{4, k\}, v)$*

(i) if $k \equiv 1, 4 \bmod 12$, $v \equiv 1, 4 \bmod 12$ and $v \geq 3k + 1$; or

(ii) if $k \equiv 7, 10 \bmod 12$, $v \equiv 7, 10 \bmod 12$ and $v \geq 3k + 1$; or

(iii) if $k \equiv 7, 10 \bmod 12$, $v \equiv 1, 4 \bmod 12$ and $v \geq 4k - 3$, except possibly when $(k, v) \in \{(10, 52), (22, 121), (22, 124), (22, 133), (22, 136), (22, 145), (22, 148), (22, 244), (34, 229), (34, 232)\}$.

In this paper, we continue to investigate the spectrum for MRDs with $K = \{4, k\}$ now with $k \equiv 2 \bmod 3$. The necessary conditions for such MRDs are as follows (we use the notation $\lceil x \rceil_{a;b}$ to mean the smallest integer not less than x which is congruent to a modulo b and define $p(t) = \min\{n > 0 :$ the complete graph K_n contains t edge-disjoint $K_4s\})$.

Theorem 1.2 *([16, Theorem 1.5]) Let $k \equiv 2$ mod 3, and suppose that there exists a mandatory representation design MRD$(4, k; v)$. Then the following conditions hold*

(i) If $k \equiv 2$ mod 3 and $v \equiv 1$ mod 3, then either $k \equiv 2$ or 11 mod 12, or $k \equiv 5$ or 8 mod 12 and $v \equiv 1$ or 4 mod 12; in either case $v \geq \frac{1}{3}k(2k + 2)$.

(ii) If $k \equiv 2$ mod 3 and $v \equiv 2$ mod 3, then either

(a) $k \equiv 5$ or 8 mod $12, v \equiv 5$ or 8 mod 12 and $v \geq kp(t) - 3t$, where $t = \lfloor \frac{kq-v}{3} \rfloor$ and $q = \lceil \frac{v}{k} \rceil_{1;3}$, or

(b) $k \equiv 2$ or 11 mod 12 and $v \geq kp(t) - 3t$, where $t = \lfloor \frac{kq-v}{3} \rfloor$, and $q = \lceil \frac{v}{k} \rceil_{1;6}$ when $v \equiv 2$ or 11 mod 12 while $q = \lceil \frac{v}{k} \rceil_{4;6}$ when $v \equiv 5$ or 8 mod 12, with the possible exceptions $(k, v) = (11, 113)$ and $(14, 161)$.

(iii) If $k \equiv 2$ mod 3 and $v \equiv 0$ mod 3, then either $k \equiv 2$ or 11 mod 12, or $k \equiv 5$ or 8 mod 12 and $v \equiv 0$ or 9 mod 12; furthermore,

$$v \geq \begin{cases} \frac{1}{2}k(k+1) & \text{if } k \equiv 2, 8, 17, 23 \bmod 24, \\ \frac{1}{2}k(k+4) - \frac{3}{2}\lfloor \frac{k+4}{5} \rfloor & \text{if } k \equiv 5, 11, 14, 20 \bmod 24 \text{ and } 5 | (k+4), \\ \frac{1}{2}k(k+4) - \frac{3}{2}\lfloor \frac{k}{5} \rfloor & \text{if } k \equiv 5, 11, 14, 20 \bmod 24 \text{ and } 5 \not| (k+4). \end{cases}$$

In Section 3 we will show that the necessary conditions for existence are sufficient whenever $v \equiv 2$ mod 3 and $v \geq 18k^2$, or $v \equiv 0$ mod 3 and $v \geq 12k^3$, or $v \equiv 1$ mod 3 and $v \geq 8k^4$.

In the rest of the introduction, we give some definition and notations as well as some preliminary results which will be used in the sequel. We refer the reader to [4] and [9] for undefined terms as well as a general overview of design theory.

Fundamental to our constructions are a number of designs which we define now. A *group-divisible design* (GDD) is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a set of *points*, $\mathcal{G}$ is a partition of V into *groups* and $\mathcal{B}$ is a collection of subsets of V (called *blocks*) such that any pair of distinct points in V occurs together either in some group or in exactly one block, but not both. A K-GDD of type $g_1^{t_1} g_2^{t_2} \dots g_r^{t_r}$ is a GDD in which each block has size from the set K and in which there are t_i groups of size $g_i, i = 1, 2, \dots, r$. We will denote a $\{k\}$-GDD as a k-GDD.

The following families of 4-GDDs will be very useful for our constructions.

Lemma 1.3 *([8]) Let t and u be positive integers. Then there exists a 4-GDD of type t^u if and only if the conditions in the following table are satisfied.*

Existence of $4-$GDDs of type t^u		
t	u	Condition
$1, 5\ mod\ 6$	$1, 4\ mod\ 12$	
$2, 4\ mod\ 6$	$1\ mod\ 3$	$(t, u) \neq (2, 4)$
$3\ mod\ 6$	$0, 1\ mod\ 4$	
$0\ mod\ 6$	none	$u = 1\ or\ u \geq 4, (t, u) \neq (6, 4)$

Lemma 1.4 *([5, 6, 7, 3, 25, 26, 24]) Let t and u be positive integers. Then there exists a 4-GDD of type $t^1 1^u$ if and only if the conditions in the following table are satisfied.*

Existence of $4-$GDDs of type $t^1 1^u$		
t	u	Condition
$1, 7\ mod\ 12$	$0, 3\ mod\ 12$	$u \geq 2t + 1$
$4, 10\ mod\ 12$	$0, 9\ mod\ 12$	$u \geq 2t + 1$

Lemma 1.5 *([11, Theorem 5.2(ii)-(iv)]) Let g,u and m be positive integers. Then there exists a 4-GDD of type $g^u m^1$ if the conditions in the following table are satisfied.*

Existence of $4-$GDDs of type $\mathrm{g^u m^1}$			
g	u	m	Condition
$1, 5\ mod\ 6$	$0\ mod\ 12$	$g\ mod\ 3$	$g \neq 11, 17, u \neq 12, 24, 72, 120, 168,$ $u \geq 2m + 3/g + 1$
$2, 4\ mod\ 6$	$0\ mod\ 3$	$g\ mod\ 3$	$g \neq 2, u \geq 192, u \neq 231, 234, 237,$ $u \geq 2m/g + 1$
$3\ mod\ 6$	$0\ mod\ 4$	$0\ mod\ 3$	$u \neq 8, 12,\ u \geq (2m + 3)/g + 1$

We proceed with the definition of a type of design called modified group divisible design (also known as grid design or as a particular class of double group divisible designs) which serves as an essential tool in our constructions. Let k, g, u be positive integers. A *modified group divisible design* k-MGDD of type g^u is a quadruple $(V, \mathcal{G}, \mathcal{H}, \mathcal{B})$, where V is a finite set of cardinality gu, $\mathcal{G}$ and $\mathcal{H}$ are two partitions of V into parts (*groups* and *holes*) and $\mathcal{B}$ is a family of subsets (*blocks*) of V which satisfy the properties:

(i) if $G \in \mathcal{G}$, then $|G| = g$;

(ii) if $B \in \mathcal{B}$, then $|B| = k$;

(iii) if $G \in \mathcal{G}$ and $H \in \mathcal{H}$, then $|G \cap H| = 1$;

(iv) every pair of distinct elements of V occurs either in exactly one block, or exactly one group or one hole, but not both.

Assaf and Wei [1], Ling and Colbourn [19], and Ge, Wang and Wei [12] have completely determined the spectrum of 4-MGDDs as recorded in Lemma 1.6.

Lemma 1.6 *A modified group divisible design* 4*-MGDD of type* g^u *exists if and only if* $(g-1)(u-1) \equiv 0$ *mod* 3 *and* $g, u \geq 4$*, except for* $(g, u) = (6, 4)$.

2 Asymptotic Sufficiency of the Necessary Conditions

In this section, we show that the necessary conditions (1) are also asymptotically sufficient for the existence of an MRD. We want to use a result of Lamken and Wilson concerning decompositions of edge-colored complete digraphs. As we only need one color and no direction on the edges we state here a simplified version of Theorem 1.2 from [18]. We will require the following notation. Given a family $\mathcal{G}$ of simple graphs, a family $\mathcal{F}$ of subgraphs of K_n (the complete graph on n vertices) is called a $\mathcal{G}$-decompositions of K_n if every edge $e \in E(K_n)$ belongs to exactly one member of $\mathcal{F}$ and every $F \in \mathcal{F}$ is isomorphic to some graph $G \in \mathcal{G}$. For a vertex x of a graph G let $\tau(x)$ denote the degree of x and denote by $\alpha(\mathcal{G})$ the greatest common divisor of $\tau(x)$ as x ranges over all vertices of all graphs in $\mathcal{G}$. Let $\mu(G)$ be the number of edges in G and define $\beta(\mathcal{G})$ to be two times the greatest common divisor of $\mu(G), G \in \mathcal{G}$.

Theorem 2.1 *([18, Theorem 1.2]) Let* $\mathcal{G}$ *be a family of simple graphs. Then there exists a constant* $n_0 = n_0(\mathcal{G})$ *such that* $\mathcal{G}$*-decompositions of* K_n *exist for all* $n \geq n_0$ *satisfying the congruences*

$$n - 1 \equiv 0 \ mod\ \alpha(\mathcal{G}) \quad and \quad n(n-1) \equiv 0 \ mod\ \beta(\mathcal{G}). \tag{2}$$

If we define $\mathcal{G} = \{K_k : k \in K\}$, then a $\mathcal{G}$-decomposition $\mathcal{F}$ of K_n is equivalent to a PBD(K, n) but not necessarily equivalent to an MRD(K, n) as we can not assume that for every $k \in K$ there is a graph $F \in \mathcal{F}$ such that $F \simeq K_k$. But with a different choice of the graphs in $\mathcal{G}$ we can prove the following result.

Theorem 2.2 *Let K be a finite set of positive integers. Then there exists a constant $v_0 = v_0(K)$ such that MRD(K, v) exist for all $v \geq v_0$ satisfying the congruences*

$$v - 1 \equiv 0 \ mod\ \alpha(K) \quad and \quad v(v-1) \equiv 0 \ mod\ \beta(K). \tag{3}$$

Proof. Let U be the disjoint union of all K_k with $k \in K$, define G_k to be the disjoint union $U \cup K_k$ and let $\mathcal{G} = \{U\} \cup \{G_k : k \in K\}$. It follows immediately from the definition that $\alpha(\mathcal{G}) = \alpha(K)$. Moreover, $\beta(\mathcal{G}) = \gcd(2e(U), 2e(U) + k(k-1) : k \in K)$, where $e(U)$ denotes the sum of all $k(k-1)$ with $k \in K$. That implies that $\beta(K)$ is a divisor of $\beta(\mathcal{G})$ and vice versa. Hence, $\beta(K) = \beta(\mathcal{G})$ and the claim follows from Theorem 2.1. □

3 Constructions and Results

In this section, we develop the constructions for MRD$(\{4, k\}, v)$s required to prove the main result Theorem 3.28. In order to facilitate this, we state an additional necessary condition which does not influence the asymptotic existence question but is important when considering small orders of v and useful to structure the paper. Let x be an arbitrary point and let γ_k denote the number of blocks of size k which contain x. Then counting pairs containing x gives $3\gamma_4 + (k-1)\gamma_k = v - 1$, which reduces for $k \equiv 2 \bmod 3$ to $\gamma_k \equiv v - 1 \bmod 3$. It will be convenient to consider these cases in separate subsections where we will first investigate MRDs with $\gamma_k \equiv 1 \bmod 3$, i.e. $v \equiv 2 \bmod 3$. Then, these MRDs will be used to construct MRDs with $\gamma_k \equiv 2 \bmod 3$, i.e. $v \equiv 0 \bmod 3$. And finally both types of MRDs form the basis for the construction of MRDs with $\gamma_k \equiv 0 \bmod 3$, i.e. $v \equiv 1 \bmod 3$.

3.1 $v \equiv 2$ mod 3, $\gamma_k \equiv 1$ mod 3

We start with constructing some basic MRDs with $v \equiv 2 \bmod 3$ from 4-GDDs which will serve as ingredient designs in further constructions.

Lemma 3.1 *Let $k \equiv 2$ mod $3, k \geq 5$. There is a mandatory representation design MRD$(\{4, k\}; ku)$*

(i) if $k \equiv 2, 8$ mod 12 and $u \equiv 1$ mod $3, u \geq 4$; or

(ii) if $k \equiv 5, 11$ mod 12 and $u \equiv 1, 4$ mod $12, u \geq 4$.

Moreover, there is a mandatory representation design MRD$(\{4, k\}; (k-1)u+1)$

(i) if $k \equiv 2, 8$ *mod* 12 *and* $u \equiv 1, 4$ *mod* $12, u \geq 4$*; or*

(ii) if $k \equiv 5, 11$ *mod* 12 *and* $u \equiv 1$ *mod* $3, u \geq 4$.

Proof. Take a 4-GDD of type k^u from Lemma 1.3 and consider the groups to be blocks of size k to obtain the desired MRD($\{4,k\}; ku$). Furthermore, adjoin a new point at infinity to a 4-GDD of type $(k-1)^u$ and replace each group and the infinity point by a block of size k to produce an MRD($\{4,k\}; (k-1)u+1$). $\square$

Note, that in the designs constructed $v \equiv 2$ mod 3 and each point lies on either 1 or $u \equiv 1$ mod 3 blocks of size k, so the condition $\gamma_k \equiv v-1$ mod 3 is satisfied.

The two constructions following next allow us to construct an infinite sequence of mandatory representation designs from just one ingredient design with the property that if for all points in the ingredient design $\gamma_k \equiv 1$ mod 3, then also in the resulting MRD holds $\gamma_k \equiv 1$ mod 3 for each point.

Construction 3.2 *Let* $k \equiv 2$ *mod* $6, k \geq 8$ *and suppose there is a PBD*($\{4,k\}; m$) *with* $m \equiv 2$ *mod* 3. *Then there is a mandatory representation design MRD*($\{4,k\}; v$) *for all* $v \geq 3m+k$ *with* $v \equiv m$ *mod* $3k, v \geq 192k+m, v \neq 231k+m, 234k+m, 237k+m$.

Proof. Use a 4-GDD of type $k^u m^1$ which exists by Lemma 1.5 for all $u \equiv 0$ mod $3, u \geq 192, u \neq 231, 234, 237, u \geq 2m/k+1$, consider groups of size k to be blocks and fill the group of size m by the PBD($\{4,k\}; m$) to produce a PBD($\{4,k\}; v = uk+m$). Clearly, $v \equiv m$ mod $3k$ and we get a PBD for each such v with $v \geq \lceil 2m/k+1 \rceil_{0;3} k + m \geq 3m+k$ with the three exceptions listed. Since there is more than one group in the 4-GDD there are blocks of size 4 and k. So the resulting PBD is indeed an MRD($\{4,k\}; v$) as desired. $\square$

Construction 3.3 *Let* $k \equiv 5$ *mod* 6 *and suppose there is a PBD*($\{4,k\}; m$) *with* $m \equiv 2$ *mod* 3. *Then there is a mandatory representation design MRD*($\{4,k\}; v$) *for all* $v \geq 3m+k-3$ *with* $v \equiv m$ *mod* $3(k-1), v \geq 192(k-1)+m, v \neq 231(k-1)+m, 234(k-1)+m, 237(k-1)+m$.

Proof. Take a 4-GDD of type $(k-1)^u(m-1)^1$ which exists by Lemma 1.5 for all $u \equiv 0$ mod $3, u \geq 192, u \neq 231, 234, 237, u \geq 2(m-1)/(k-1)+1$, adjoin one infinite point and fill in the groups together with the infinite point by blocks of size k or the PBD($\{4,k\}; m$) to obtain a PBD($\{4,k\}; v = u(k-1)+m$). Obviously, $v \equiv m$ mod $3(k-1)$ and $v \geq \lceil 2(m-1)/(k-1) + 1 \rceil_{0;3}(k-1) + m \geq 3m+k-3$. Again, if m is relatively small we can have three possible exceptions. Note, that the construction method ensures that there are blocks of size 4 and k and, therefore, the PBD constructed is an MRD. $\square$

In the following we want apply Constructions 3.2 and 3.3. If we are able to provide a representative PBD in each possible residue class modulo $3k$ or $3(k-1)$, then we have established the existence of an MRD$(\{4,k\};v)$ for all $v \geq 3m_{\max}+k$ or $v \geq 3m_{\max}+k-3$ where $m_{\max}$ is the number of points in the largest representative PBD. To be more precise, we need according to the necessary conditions representative PBD$(\{4,k\},m_t)$ with $m_t \equiv 3t+2 \bmod 3k$ if $k \equiv 2 \bmod 12$ for each $t=0,1,\ldots,k-1$, or $m_t \equiv 3t+2 \bmod 3(k-1)$ if $k \equiv 11 \bmod 12$ for each $t=0,1,\ldots,k-2$, or $m_t \equiv 12t+5, 12t+8 \bmod 3(k-1)$ if $k \equiv 5 \bmod 12$ for each $t=0,1,\ldots,(k-1)/4-1$, or $m_t \equiv 12t+5, 12t+8 \bmod 3k$ if $k \equiv 8 \bmod 12$ for each $t=0,1,\ldots,k/4-1$. In the next lemmata we will provide these representative PBDs and compute the corresponding bounds for v.

Lemma 3.4 *Let $k \equiv 2$ mod 6 and $v \equiv 2$ mod 3. There exists a mandatory representation design MRD$(\{4,k\};v)$*

(i) if $k \equiv 2$ mod 12, $k \geq 26$, $v \equiv 2$ mod 3 and $v \geq 18k^2-41k+27$; or

(ii) if $k \equiv 8$ mod 12, $k \geq 44$, $v \equiv 5, 8$ mod 12 and $v \geq 9k^2-32k+27$.

Proof. We take as representative designs MRD$(\{4,k\};m_s=(k-1)u_s+1)$ which exist by Lemma 3.1 for all $u_s=12s+a$ where $s \in \mathbb{N}, a \in \{1,4\}$ and $(s,a) \neq (0,1)$. In the latter case we use as representative PBD just a block of size k. Then $m_s \equiv -12s+k-a+1 \bmod 3k$. If $k \equiv 2 \bmod 12$, then $\gcd(12,3k)=6$ and thus with $a=1$ and $s=0,1,\ldots,k/2-1$ we get all residues $6t+2$ modulo $3k$. Moreover, with $a=4$ and $s=0,1,\ldots,k/2-1$ we get all residues $6t+5$ modulo $3k$. Therefore, we obtained a representative design congruent $3t+2$ modulo $3k$ for each $t=0,1,\ldots,k-1$. The largest representative design has order $m_{\max}=(k-1)(12s_{\max}+4)+1=6k^2-14k+9$. Hence using Construction 3.2 establishes the bound $v \geq 3m_{\max}+k$ in Case (i). Note, that the exceptional cases $v \neq 231k+m, 234k+m, 237k+m$ do not affect the bound in general as $2m_{\max}+k > 237k$ if $k \geq 26$.

Similarly, for $k \equiv 8 \bmod 12$ with $a=1$ or 4 and $s=0,1,\ldots,k/4-1$ we get all residues $12t+5$ or $12t+8$ modulo $3k$. Again using these representative MRD$(\{4,k\},m_s)$ with $m_{\max}=3k^2-11k+9$ in Construction 3.2 yields the bound in Case (ii). It is easily checked that $2m_{\max}+k > 237k$ if $k \geq 44$, so the exceptional cases listed in Construction 3.2 do not apply. $\square$

Lemma 3.5 *Let $k \equiv 5$ mod 6 and $v \equiv 2$ mod 3. There exists a mandatory representation design MRD$(\{4,k\};v)$*

(i) if $k \equiv 5$ mod 12, $k \geq 53$, $v \equiv 5, 8$ mod 12 and $v \geq 9k^2 - 32k - 3$; or

(ii) if $k \equiv 11$ mod 12, $k \geq 23$, $v \equiv 2$ mod 3 and $v \geq 18k^2 - 41k - 3$.

Proof. Here, we use representative designs MRD$(\{4, k\}; m_s = ku_s)$ which exist by Lemma 3.1 for all $u_s = 12s + a$ where $s \in \mathbb{N}, a \in \{1, 4\}$ and $(s, a) \neq (0, 1)$ or a representative PBD$(\{4, k\}; k)$. Then $m_s \equiv 12s + ak \bmod 3(k-1)$. If $k \equiv 11 \bmod 12$, then $\gcd(12, 3(k-1)) = 6$ and thus with $a = 1$ and $s = 0, 1, \dots, (k-1)/2-1$ we get all residues $6t+5$ modulo $3(k-1)$. Moreover, with $a = 4$ and $s = 0, 1, \dots, (k-1)/2-1$ we get all residues $6t+2$ modulo $3(k-1)$. Therefore, we obtained a representative design congruent to $3t+2$ modulo $3(k-1)$ for each $t = 0, 1, \dots, k-2$. The largest representative design has order $m_{\max} = k(12s_{\max} + 4) = 6k^2 - 14k$. Hence using Construction 3.3 establishes the bound $v \geq 3m_{\max} + k - 3$ in Case (i). Note, that the exceptional cases $v \neq 231k + m, 234k + m, 237k + m$ do not affect the bound in general as $2m_{\max} + k - 3 > 237k$ if $k \geq 23$.

Similarly, for $k \equiv 5 \bmod 12$ with $a = 1$ or 4 and $s = 0, 1, \dots, (k-1)/4-1$ we get all residues $12t+5$ or $12t+8$ modulo $3(k-1)$. Again using these representative MRD$(\{4, k\}, m_s)$ with $m_{\max} = 3k^2 - 11k$ in Construction 3.3 yields the bound in Case (ii). Again $k \geq 53$ implies that $2m_{\max} + k - 3 > 237k$, so the exceptional cases listed in the construction do not need to be considered. □

In view of the lemmata above it remains to investigate $k = 5, 8, 11, 14, 17, 20, 29, 32, 41$. The closure of $K = \{4, 5\}$ and $K = \{4, 8\}$ are almost completely know, see [4, 2, 21], so we just need to trace back the constructions and see which of them ensure that the designs constructed contain both blocks of size 4 and 5 or 8, respectively.

Lemma 3.6 *If $v \equiv 5, 8$ mod $12, v \geq 17$, then there is a mandatory representation design MRD$(\{4, 5\}, v)$.*

Proof. $B(\{4, 5\}) = \mathbb{N}_{0,1 \bmod 4} \setminus \{8, 9, 12\}$, thus if $v \equiv 5, 8 \bmod 12, v \geq 17$, then $v \in B(\{4, 5\})$. Moreover, if $v \not\equiv 5, 41 \bmod 60$, then the necessary conditions imply that $v \notin B(\{4\}), B(\{5\})$ and, therefore, there is an MRD$(\{4, 5\}, v)$. Now, it is easily seen that each $v \equiv 5, 41 \bmod 60, v \geq 65$ has a representation $v = 4g + a$ with $g \equiv 0, 1 \bmod 4, g \geq 16$, $a \in \{1, 5, 13\}$. Take a transversal design TD$(5, g)$ which exists for all $g \geq 11$ (see [9]), delete all but a points from the last group and fill in groups by a PBD$(\{4, 5\}, g)$ or PBD$(\{4, 5\}, a)$ to obtain an MRD$(\{4, 5\}, v = 4g+a)$. Noting that by Lemma 3.1 there exists an MRD$(\{4, 5\}, 41)$ completes the proof. □

Lemma 3.7 *If $v \equiv 5, 8$ mod $12, v \geq 176$, then there is a mandatory representation design MRD$(\{4, 8\}, v)$.*

Proof. $B(\{4,8\}) \supseteq \mathbb{N}_{0,1 \bmod 4} \setminus \{5,9,12,17,20,21,24,33,41,44,45,48,53,60,65,69,77,89,101,161,164,173\}$, thus if $v \equiv 5,8 \bmod 12, v \geq 176$, $v \not\equiv 8,113 \bmod 168$ then $v \in B(\{4,8\})$ but $v \notin B(\{4\}), B(\{8\})$ and, therefore, there is an MRD$(\{4,8\},v)$. Each $v \equiv 8 \bmod 168, v \geq 176$ has a representation $v = 8u$ where $u \equiv 1 \bmod 3$ and each $v \equiv 113 \bmod 168$ has a representation $v = 7u+1$ where $u \equiv 4 \bmod 12$, so Lemma 3.1 provides in each case an MRD$(\{4,8\},v)$. $\square$

So far we did not use 4-GDDs of type $g^u m^1$ with $g \equiv 1,5 \bmod 6$ from Lemma 1.5 as these give in general worse bounds compared to the bounds we already have. But there are less possible exceptions, so these GDDs are useful for small k.

Construction 3.8 *Let $k \equiv 5 \bmod 6, k \neq 11,17$ and suppose there is a PBD$(\{4,k\};m)$ with $m \equiv 2 \bmod 3$. Then there is a mandatory representation design MRD$(\{4,k\};v)$ for all $v \geq 3m+k+3$ with $v \equiv m \bmod 12k, v \neq 12k+m, 24k+m, 72k+m, 120k+m, 168k+m$.*

Proof. Use a 4-GDD of type $k^u m^1$ which exists by Lemma 1.5 for all $u \equiv 0 \bmod 12, u \neq 12,24,72,120,168, u \geq (2m+3)/k+1$, consider groups of size k to be blocks and fill the group of size m by the PBD$(\{4,k\};m)$ to produce a PBD$(\{4,k\}; v = uk+m)$. Clearly, $v \equiv m \bmod 12k$ and we get a PBD for each such v with $v \geq \lceil (2m+3)/k+1 \rceil_{0;12} k + m \geq 3m+k+3$ with the five exceptions listed. Since there is more than one group in the 4-GDD there are blocks of size 4 and k. So the resulting PBD is indeed an MRD$(\{4,k\};v)$ as desired. $\square$

Construction 3.9 *Let $k \equiv 2 \bmod 6, k \geq 8$ and suppose there is a PBD$(\{4,k\};m)$ with $m \equiv 2 \bmod 3$. Then there is a mandatory representation design MRD$(\{4,k\};v)$ for all $v \geq 3m+k$ with $v \equiv m \bmod 12(k-1), v \neq 12(k-1)+m, 24(k-1)+m, 72(k-1)+m, 120(k-1)+m, 168(k-1)+m$.*

Proof. Take a 4-GDD of type $(k-1)^u (m-1)^1$ which exists by Lemma 1.5 for all $u \equiv 0 \bmod 12, u \neq 12,24,72,120,168, u \geq (2(m-1)+3)/(k-1)+1$, adjoin one infinite point and fill in the groups together with the infinite point by blocks of size k or the PBD$(\{4,k\};m)$ to obtain a PBD$(\{4,k\}; v = u(k-1)+m)$. Obviously, $v \equiv m \bmod 12(k-1)$ and $v \geq \lceil (2(m-1)+3)/(k-1)+1 \rceil_{0;12}(k-1)+m \geq 3m+k$. Again, if m is relatively small we can have five possible exceptions. Note, that the construction method ensures that there are blocks of size 4 and k and, therefore, the PBD constructed is an MRD. $\square$

Lemma 3.10 *If* $v \equiv 2 \ mod\ 3, v > 2492, v \neq 2513, 2516, 2546, 2585, 2615, 2618, 2645, 2648, 2678, 2717, 2747, 2750, 2777, 2780, 2810, 2849, 2879, 2882, 2909, 2912, 2942$*, then there is a mandatory representation design MRD*$(\{4, 11\}, v)$.

Proof. Let $M = \{11, 44, 143, 176, 275, 308, 407, 440, 539, 572\}$. Lemma 3.1 provides a PBD$(\{4, 11\}, m)$ for all $m \in M$ which represent each residue class $3t + 2$ modulo 30. Hence, Construction 3.3 yields an MRD$(\{4, 11\}, v)$ for all $v \equiv 2$ mod 3 with $v \geq 3m_{\max} + 11 - 3 = 1724$, $v > 192 \cdot 10 + m_{\max} = 2492$ and $v \neq 231 \cdot 10 + M, 234 \cdot 10 + M, 237 \cdot 10 + M$. $\square$

Lemma 3.11 *If* $v \equiv 5, 11 \ mod\ 12, v > 3677, v \neq 3680, 3719, 3758, 3836, 3875, 3914, 3992, 4031, 4070$*, or if* $v \equiv 2, 8 \ mod\ 12, v \geq 3206, v \neq 3206, 3248$ *then there is a mandatory representation design MRD*$(\{4, 14\}, v)$.

Proof. Let $M = \{156t + 53 : t = 0, \ldots, 6\}$. Lemma 3.1 provides a PBD$(\{4, 14\}, m)$ for all $m \in M$ which represent each residue class $6t + 5$ modulo 42. Hence, Construction 3.2 yields an MRD$(\{4, 14\}, v)$ for all $v \equiv 5, 11$ mod 12 with $v \geq 3m_{\max} + 14 = 2981$, $v > 192 \cdot 14 + m_{\max} = 3677$ and $v \neq 231 \cdot 14 + M, 234 \cdot 14 + M, 237 \cdot 14 + M$. Now let $R = \{42t + 14 : t = 0, \ldots, 25\}$. Lemma 3.1 provides a PBD$(\{4, 14\}, r)$ for all $r \in R$ which represent each residue class $6t+2$ modulo 156. Hence, Construction 3.9 yields an MRD$(\{4, 14\}, v)$ for all $v \equiv 2, 8$ mod 12 with $v \geq 3r_{\max} + 14 = 3206$ and $v \neq 169 \cdot 13 + R$. $\square$

Lemma 3.12 *If* $v \equiv 5, 8 \ mod\ 12, v > 3752, v \neq 3761, 3764, 3809, 3812, 3860, 3917, 3965, 3968, 4013, 4016, 4064, 4121, 4169, 4172, 4217, 4220, 4268, 4325, 4373, 4376, 4421, 4424, 4472$*, then there is a mandatory representation design MRD*$(\{4, 17\}, v)$.

Proof. Let $M = \{17, 68, 221, 272, 425, 476, 629, 680\}$. Lemma 3.1 provides a PBD$(\{4, 17\}, m)$ for all $m \in M$ which represent each residue class $12t+5$ or $12t+8$ modulo 48. Hence, Construction 3.3 yields an MRD$(\{4, 17\}, v)$ for all $v \equiv 5, 8$ mod 12 with $v \geq 3m_{\max} + 17 - 3 = 2054$, $v > 192 \cdot 16 + m_{\max} = 3752$ and $v \neq 231 \cdot 16 + M, 234 \cdot 16 + M, 237 \cdot 16 + M$. $\square$

Lemma 3.13 *If* $v \equiv 8 \ mod\ 12, v > 4772, v \neq 4868, 4928, 4988, 5096, 5156, 5216, 5324, 5384, 5444, 5552, 5612, 5672$*, or if* $v \equiv 5 \ mod\ 12, v \geq 3320, v \neq 3320, 3332, 3380, 3392, 3452, 3512, 3572, 3632, 3692, 3752, 3812, 3872, 3932, 3992, 4052, 4112, 4172, 4232, 4292$*, then there is a mandatory representation design MRD*$(\{4, 20\}, v)$.

Proof. Let $M = \{228t + 20 : t = 0, \ldots, 4\}$. Lemma 3.1 provides a PBD$(\{4, 20\}, m)$ for all $m \in M$ which represent each residue class $12t + 8$ modulo 60. Hence, Construction 3.2 yields an MRD$(\{4, 20\}, v)$ for all $v \equiv 8 \bmod 12$ with $v \geq 3m_{\max} + 20 = 2816$, $v > 192 \cdot 20 + m_{\max} = 4772$ and $v \neq 4620+M, 4680+M, 4740+M$. Now let $R = \{60t+20 : t = 0, \ldots, 18\}$. Lemma 3.1 provides a PBD$(\{4, 20\}, r)$ for all $r \in R$ which represent each residue class $12t + 5$ modulo 228. Hence, Construction 3.9 yields an MRD$(\{4, 20\}, v)$ for all $v \equiv 5 \bmod 12$ with $v \geq 3r_{\max}+20 = 3320$ and $v \neq 120\cdot 19+R, 169\cdot 19+R$. $\square$

Lemma 3.14 *If* $v \equiv 8 \ mod\ 12, v > 7580, v \neq 7628, 7712, 7796, 7976, 8060, 8144, 8324, 8408, 8492, 8672, 8756, 8840$*, or if* $v \equiv 5 \ mod\ 12, v \geq 7175, v \neq 7253$*, then there is a mandatory representation design MRD*$(\{4, 29\}, v)$.

Proof. Let $M = \{348t + 116 : t = 0, \ldots, 6\}$. Lemma 3.1 provides a PBD$(\{4, 29\}, m)$ for all $m \in M$ which represent each residue class $12t + 8$ modulo 84. Hence, Construction 3.3 yields an MRD$(\{4, 29\}, v)$ for all $v \equiv 8 \bmod 12$ with $v \geq 3m_{\max} + 29 - 3 = 6638$, $v > 192 \cdot 28 + m_{\max} = 7580$ and $v \neq 231 \cdot 28 + M, 234 \cdot 28 + M, 237 \cdot 28 + M$. Now let $R = \{84t + 29 : t = 0, \ldots, 28\}$. Lemma 3.1 provides a PBD$(\{4, 29\}, r)$ for all $r \in R$ which represent each residue class $12t + 5$ modulo 348. Hence, Construction 3.8 yields an MRD$(\{4, 29\}, v)$ for all $v \equiv 5 \bmod 12$ with $v \geq 3r_{\max} + 29 + 3 = 7175$, and $v \neq 168 \cdot 29 + R$. $\square$

Lemma 3.15 *If* $v \equiv 8 \ mod\ 12, v > 8780, v \neq 8912, 9008, 9104, 9284, 9380, 9476, 9656, 9752, 9848, 10028, 10124, 10220$*, or if* $v \equiv 5 \ mod\ 12, v \geq 8768$*, then there is a mandatory representation design MRD*$(\{4, 32\}, v)$.

Proof. Let $M = \{372t + 32 : t = 0, \ldots, 7\}$. Lemma 3.1 provides a PBD$(\{4, 32\}, m)$ for all $m \in M$ which represent each residue class $12t + 8$ modulo 96. Hence, Construction 3.2 yields an MRD$(\{4, 32\}, v)$ for all $v \equiv 8 \bmod 12$ with $v \geq 3m_{\max} + 32 = 7940$, $v > 192 \cdot 32 + m_{\max} = 8780$ and $v \neq 231 \cdot 32 + M, 234 \cdot 32 + M, 237 \cdot 32 + M$. Now let $R = \{96t + 32 : t = 0, \ldots, 30\}$. Lemma 3.1 provides a PBD$(\{4, 32\}, r)$ for all $r \in R$ which represent each residue class $12t + 5$ modulo 372. Hence, Construction 3.9 yields an MRD$(\{4, 32\}, v)$ for all $v \equiv 5 \bmod 12$ with $v \geq 3r_{\max} + 32 = 8768$. $\square$

Lemma 3.16 *If* $v \equiv 5, 8 \ mod\ 12, v \geq 13814, v \neq 13832, 13952, 14072$*, then there is a mandatory representation design MRD*$(\{4, 41\}, v)$.

Proof. Let $M = \{492t + 41, 492t + 164 : t = 0, \ldots, 9\}$. Lemma 3.1 provides a PBD$(\{4, 41\}, m)$ for all $m \in M$ which represent each residue class $12t+5$ or $12t+8$ modulo 120. Hence, Construction 3.3 yields an MRD$(\{4, 41\}, v)$ for all $v \equiv 5, 8$ mod 12 with $v \geq 3m_{\max}+41-3 = 13814$, $v > 192 \cdot 40 + m_{\max} = 12272$ and $v \neq 231 \cdot 40 + M, 234 \cdot 40 + M, 237 \cdot 40 + M$. The latter inequality gives a list of five possible exceptions with $v \geq 13814$. We can delete $v = 13829, 12949$ from that list as there are a 4-GDD of type $41^{228}4481^1$, a 4-GDD of type $41^{228}4601^1$ (Lemma 1.5) and an MRD$(\{4, 41\}, 4481)$ and an MRD$(\{4, 41\}, 4601)$ (Lemma 3.1). Thus filling in groups yields the desired MRDs leaving three possible exceptions $v = 13832, 13952, 14072$. $\square$

3.2 $v \equiv 0$ mod 3, $\gamma_k \equiv 2$ mod 3

Now, we turn our attention to MRDs on $v \equiv 0$ mod 3 points where γ_k needs to be congruent to 2 mod 3. The basic idea is to take a modified group divisible design and to construct on each group and on each hole an MRD with $\gamma_k \equiv 1$ mod 3 which provides, as every point occurs in exactly one group and exactly one hole, an MRD with $\gamma_k \equiv 2$ mod 3. But, first we state a more general construction using modified group divisible designs.

Construction 3.17 *Let $k \equiv 2$ mod $3, k \geq 5$. If there is a PBD$(\{4, k\}; v)$, then there is a mandatory representation design MRD$(\{4, k\}; (k-1)v + 1)$. If there is a PBD$(\{4, k\}; v)$ with $v \equiv 2$ mod 3, then there is a mandatory representation design MRD$(\{4, k\}; k(v-1) + 1)$.*

Proof. Clearly, $((k-1) - 1)(v-1) \equiv 0$ mod 3 and therefore Lemma 1.6 implies that there is a 4-MGDD of type $(k-1)^v$. So take that 4-MGDD and fill each hole by the PBD$(\{4, k\}; v)$. Furthermore, adjoin a new point to the point set and replace each group and the new point by a k-block to produce an MRD with blocks of size 4 and k on $(k-1)v + 1$ points. Similarly, if in addition $v \equiv 2$ mod 3, then $(k-1)((v-1) - 1) \equiv 0$ mod 3 which implies that there exists a 4-MGDD of type k^{v-1}. Again, we adjoin a new point and replace now each hole and this new point by the PBD$(\{4, k\}; v)$. If we consider all groups to be blocks of size k, then we get the desired MRD$(\{4, k\}; k(v-1) + 1)$. $\square$

Using this construction together with designs from Lemma 3.1 we obtain the following three corollaries.

Corollary 3.18 *Let $k \equiv 2$ mod $3, k \geq 5$. There is a mandatory representation design MRD$(\{4, k\}; k(k-1)u + 1)$ for all $u \equiv 1$ mod 3.*

Proof. For $u \geq 4$ take an MRD($\{4,k\}; ku$) from Lemma 3.1 as ingredient PBD in Construction 3.17 to obtain an MRD($\{4,k\}, k(k-1)u+1$). For $u = 1$ just use a single block of size k as a (trivial) PBD in Construction 3.17. This provides all required MRDs except for $k \equiv 5, 11 \bmod 12$ and $u \equiv 7, 10 \bmod 12$. Here take an MRD($\{4,k\}, k(k-1)+1$) just constructed and fill in the groups of a 4-GDD of type $(k(k-1))^u$ with a point at infinity adjoint which exists for all $u \equiv 1 \bmod 3$ by Lemma 1.3. □

Corollary 3.19 *Let $k \equiv 2 \bmod 3, k \geq 5$. There is a mandatory representation design MRD($\{4,k\}; k(ku-1)+1$)*

(i) if $k \equiv 2, 8 \bmod 12$ and $u \equiv 1 \bmod 3$; or

(ii) if $k \equiv 5, 11 \bmod 12$ and $u \equiv 1, 4 \bmod 12$.

Proof. An MRD with $k(k-1)+1$ points is already constructed in Corollary 3.18 for all $k \equiv 2 \bmod 3, k \geq 5$, so we only need to consider $u \geq 4$. For that purpose take an MRD with $v = ku$ from Lemma 3.1 for which clearly $v \equiv 2 \bmod 3$ holds. Thus applying Construction 3.17 yields the desired MRD($\{4,k\}; k(ku-1)+1$). □

Corollary 3.20 *Let $k \equiv 2 \bmod 3, k \geq 5$. There is a mandatory representation design MRD($\{4,k\}; (k-1)^2u+k$)*

(i) if $k \equiv 2, 8 \bmod 12$ and $u \equiv 1, 4 \bmod 12$; or

(ii) if $k \equiv 5, 11 \bmod 12$ and $u \equiv 1 \bmod 3$.

Proof. For $u = 1$ take an MRD with $v = k(k-1)+1$ from Corollary 3.18. For $u \geq 4$ apply Construction 3.17 with an MRD($\{4,k\}; (k-1)u+1$) constructed in Lemma 3.1 to obtain an MRD on $(k-1)((k-1)u+1)+1 = (k-1)^2u+k$ points. □

We remark, that for the number of points $v = k(k-1)u+1$, $v = k(ku-1)+1$ or $v = (k-1)^2u+k$ of the MRDs constructed above holds $v \equiv 0 \bmod 3$ and that every point is contained in either $2, k, u+1, 2u, ku$ or $(k-1)u+1$ blocks of size k. Hence, $\gamma_k \equiv 2 \bmod 3$ as desired.

Similar as in the case $v \equiv 2 \bmod 3$ one can construct an infinite sequence of mandatory representation designs from just one ingredient MRD with the property that if $\gamma_k \equiv 2 \bmod 3$ for each point in the ingredient MRD, then also in the resulting MRD $\gamma_k \equiv 2 \bmod 3$ for all points.

Construction 3.21 *Let $k \equiv 2 \bmod 3$ and suppose there is an MRD($\{4,k\}; m$) with $m \equiv 0 \bmod 3$. Then there is a mandatory representation design MRD($\{4,k\}; v$) for all $v \geq 3m+k(k-1)+4$ with $v \equiv m \bmod 4(k(k-1)+1), v \neq 8(k(k-1)+1)+m, 12(k(k-1)+1)+m$.*

Proof. If $k \equiv 2 \bmod 3$, then $k(k-1)+1 \equiv 3 \bmod 6$ and so by Lemma 1.5 there is a 4-GDD of type $(k(k-1)+1)^u m^1$ for all $u \equiv 0 \bmod 4, u \neq 8, 12, u \geq (2m+3)/(k(k-1)+1)+1$. Replacing groups of size $k(k-1)+1$ by an MRD($\{4,k\}; k(k-1)+1$) which exists by Corollary 3.18 and the group of size m by the MRD($\{4,k\}; m$) produces an MRD($\{4,k\}; v = u(k(k-1)+1)+m$). Thus we get an MRD for all $v \equiv m \bmod 4(k(k-1)+1)$ where $v \geq \lceil (2m+3)/(k(k-1)+1)+1 \rceil_{0;4}(k(k-1)+1)+m \geq 3m+k(k-1)+4$ with the exception of $v = 8(k(k-1)+1)+m$ or $v = 12(k(k-1)+1)+m$. $\square$

In what follows we want apply Construction 3.21. If we are able to provide a representative MRD($\{4,k\}, m_t$) with $m_t \equiv 3t \bmod 4(k(k-1)+1)$ for each $t = 0, 1, \ldots, 4(k(k-1)+1)/3-1$ if $k \equiv 2, 11 \bmod 12$; or with $m_t \equiv 6t+3 \bmod 4(k(k-1)+1)$ for each $t = 0, 1, \ldots, 4(k(k-1)+1)/6-1$ if $k \equiv 5, 8 \bmod 12$, then we have established the existence of an MRD($\{4,k\}; v$) for all $v \geq 3m_{\max}+g+3$ where $m_{\max} = \max\{m_t\}$. We remark that the designs resulting from Construction 3.21 lie in the same residue class modulo 12 as the second ingredient MRD($\{4,k\}; m$). So we will need to consider different types of ingredient MRDs to obtain the desired designs in each residue class modulo 12. This will be done in the next three lemmata.

Lemma 3.22 *Let $k \equiv 2 \bmod 3$ and $v \equiv 3, 9 \bmod 12$. There exists a mandatory representation design MRD($\{4,k\}; v$)*

(i) if $k \equiv 2, 11 \bmod 12$, $v \equiv 3, 9 \bmod 12$ and $v \geq 6k^4-12k^3+25k^2-19k+7$; or

(ii) if $k \equiv 5, 8 \bmod 12$, $v \equiv 9 \bmod 12$ and $v \geq 3k^4-6k^3+10k^2-7k+7$.

Proof. For $k \equiv 2, 11 \bmod 12$ it suffices to provide representative MRD($\{4,k\}, m_t$) with $m_t \equiv 12t+3 \bmod 4(k(k-1)+1)$ and $m_t \equiv 12t+9 \bmod 4(k(k-1)+1)$ for each $t = 0, 1, \ldots, (k(k-1)+1)/3-1$ to obtain the desired bounds. These representative MRDs are taken from Corollary 3.18: an MRD($\{4,k\}; m_s$) with $m_s = k(k-1)u_s+1$ exists for all $u_s = 12s+a$ where $s \in \mathbb{N}, a \in \{1, 4, 7, 10\}$. Then $m_s = 3s(4(k(k-1)+1)) - 12s + ak(k-1)+1$ and thus $m_s \equiv -12s + ak(k-1)+1 \bmod 4(k(k-1)+1)$. Since $\gcd(12, 4(k(k-1)+1)) = 12$ it is easy to check that with $a = 1$ or $a = 7$ and $s = 0, 1, \ldots, (k(k-1)-2)/6$ we get all residues $12t+3$ modulo $4(k(k-1)+1)$. Moreover, with $a = 4$ or $a = 10$ and $s = 0, 1, \ldots, (k(k-1)-2)/6$ we get all residues $12t+9$ modulo $4(k(k-1)+1)$. The largest representative MRD has order $m_{\max} = k(k-1)(12s_{\max}+10)+1 = 2k^2(k-1)^2+6k(k-1)+1$. Hence using Construction 3.21 establishes the bound $v \geq 3m_{\max}+g+3$ in

Case (i). Note, that the exceptional case $v \neq 8g+m, 12g+m$ does not affect the bound as $2m_{\max} > 12g$.

Similarly, for $k \equiv 5, 8 \bmod 12$ it suffices to present representative $\mathrm{MRD}(\{4,k\}, m_t)$ with $m_t \equiv 12t + 9 \bmod 4(k(k-1)+1)$ for each $t = 0, 1, \ldots, (k(k-1)+1)/3 - 1$ to obtain the desired bounds. With $u = 12s + a$, $a = 1, 4, 7$ or 10 and $s = 0, 1, \ldots, (k(k-1)-8)/12$ we get all residues $12t+9$ modulo $4(k(k-1)+1)$. Again using these representative $\mathrm{MRD}(\{4,k\}, m_s)$ with $m_{\max} = k^2(k-1)^2 + 2k(k-1) + 1$ in Construction 3.21 yields the bound in Case (ii). $\square$

Lemma 3.23 *Let $k \equiv 5, 11$ mod 12 and $v \equiv 0, 6$ mod 12. There exists a mandatory representation design MRD$(\{4,k\}; v)$,*

(i) if $k \equiv 5$ mod 12, $v \equiv 0$ mod 12 and $v \geq 4k^4 - 5k^3 + 43k^2 - 12k + 14$; or

(ii) if $k \equiv 11$ mod 12, $v \equiv 0, 6$ mod 12 and $v \geq 8k^4 - 10k^3 + 49k^2 - 14k + 15$.

Proof. First, let $k \equiv 5 \bmod 12$. It suffices to provide representative $\mathrm{MRD}(\{4,k\}, m_t)$ with $m_t \equiv 12t \bmod 4(k(k-1)+1)$ for each $t = 0, 1, \ldots, (k(k-1)+1)/3 - 1$. There is an $\mathrm{MRD}(\{4,k\}; k(4k-1)+1)$ by Corollary 3.19 with $k(4k-1)+1 \equiv 0 \bmod 12$. Lemma 1.3 implies that there exists a 4-GDD of type $(k(4k-1)+1)^s$ for all $s \geq 4$ and, therefore, an $\mathrm{MRD}(\{4,k\}, m_s = (k(4k-1)+1)s)$. It is easy to check that $m_s = s(4(k(k-1)+1)) + 3(k-1)t$ and thus $m_s \equiv 3(k-1)s \bmod 4(k(k-1)+1)$. Clearly $\gcd(3(k-1), 4(k(k-1)+1)) = 12$, so it follows immediately that with $s = 4, 5, \ldots, (k(k-1)+10)/3$ we get all residues $12t$ modulo $4(k(k-1)+1)$. The largest representative MRD has order $m_{\max} = (k(4k-1)+1)s_{\max} = \frac{1}{3}(4k^4 - 5k^3 + 42k^2 - 11k + 10)$. Hence using Construction 3.21 establishes the bound in Case (i).

Now, let $k \equiv 11 \bmod 12$. It suffices to provide representative $\mathrm{MRD}(\{4,k\}, m_t)$ with $m_t \equiv 12t$, or $12t + 6 \bmod 4(k(k-1)+1)$ for each $t = 0, 1, \ldots, (k(k-1)+1)/3 - 1$. In a similar way as in the discussion above one shows that there is an $\mathrm{MRD}(\{4,k\}, m_s = (k(4k-1)+1)s)$ for all $s \geq 4$. Note, that $k(4k-1)+1 \equiv 6 \bmod 12$ and hence if s is even and $s = 0, 2, \ldots, 2(k(k-1)+4)/3$, then we get all residues $12t$ modulo $4(k(k-1)+1)$. While, if s is odd and $s = 1, 3, \ldots, 2(k(k-1)+4)/3 + 1$, then we get all residues $12t+6$ modulo $4(k(k-1)+1)$. Again using these representative $\mathrm{MRD}(\{4,k\}, m_s)$ with $m_{\max} = \frac{1}{3}(8k^4 - 10k^3 + 48k^2 - 13k + 11)$ in Construction 3.21 yields the bound in Case (ii). $\square$

Lemma 3.24 *Let $k \equiv 2, 8$ mod 12 and $v \equiv 0, 6$ mod 12. There exists a mandatory representation design MRD$(\{4,k\}; v)$,* *(i) if* $k \equiv 2 \mod 12$*,* $v \equiv 0, 6 \mod 12$ *and* $v \geq 8k^4 - 22k^3 + 67k^2 - 86k + 48$*; or*

(iii) if $k \equiv 8 \mod 12$*,* $v \equiv 0 \mod 12$ *and* $v \geq 4k^4 - 11k^3 + 52k^2 - 75k + 44$*.*

Proof. First, let $k \equiv 8 \bmod 12$. It suffices to provide representative MRD$(\{4, k\}, m_t)$ with $m_t \equiv 12t \bmod 4(k(k-1)+1)$ for each $t = 0, 1, \ldots, (k(k-1)+1)/3 - 1$. There is an MRD$(\{4, k\}; 4(k-1)^2 + k)$ by Corollary 3.20 with $4(k-1)^2 + k \equiv 0 \bmod 12$. Lemma 1.3 implies that there exists a 4-GDD of type $(4(k-1)^2 + k)^s$ for all $s \geq 4$ and, therefore, an MRD$(\{4, k\}, m_s = (4(k-1)^2 + k)s)$. It is easy to check that $m_s = s(4(k(k-1)+1)) - 3ks$ and thus $m_s \equiv -3ks \bmod 4(k(k-1)+1)$. Clearly $\gcd(3k, 4(k(k-1)+1)) = 12$, so it follows immediately that with $s = 4, 5, \ldots, (k(k-1)+10)/3$ we get all residues $12t$ modulo $4(k(k-1)+1)$. The largest representative MRD has order $m_{\max} = (4(k-1)^2 + k)s_{\max} = \frac{1}{3}(4k^4 - 11k^3 + 51k^2 - 74k + 40)$. Hence using Construction 3.21 establishes the bound in Case (ii).

Now, let $k \equiv 2 \bmod 12$. It suffices to provide representative MRD$(\{4, k\}, m_t)$ with $m_t \equiv 12t$, or $12t + 6 \bmod 4(k(k-1)+1)$ for each $t = 0, 1, \ldots, (k(k-1)+1)/3 - 1$. As described above there is an MRD$(\{4, k\}, m_s = (4(k-1)^2 + k)s)$ for all $s \geq 4$. Note, that $4(k-1)^2 + k \equiv 6 \bmod 12$ and hence if s is even and $s = 0, 2, \ldots, 2(k(k-1)+4)/3$, then we get all residues $12t$ modulo $4(k(k-1)+1)$. While, if s is odd and $s = 1, 3, \ldots, 2(k(k-1)+4)/3+1$, then we get all residues $12t + 6$ modulo $4(k(k-1)+1)$. Again using these representative MRD$(\{4, k\}, m_s)$ with $m_{\max} = \frac{1}{3}(8k^4 - 22k^3 + 66k^2 - 85k + 44)$ in Construction 3.21 yields the bound in Case (i). $\square$

3.3 $v \equiv 1$ mod 3, $\gamma_k \equiv 0$ mod 3

Using MRDs with $\gamma_k \equiv 2 \bmod 3$ from the previous subsection we are now able in conjunction with Construction 3.17 to establish the existence of some MRDs with $\gamma_k \equiv 0 \bmod 3$ for each $k \equiv 2 \bmod 3$. These MRDs are then used to fill groups of appropriate 4-GDDs.

Corollary 3.25 *Let* $k \equiv 2 \mod 3, k \geq 5$*. There is a mandatory representation design MRD*$(\{4, k\}; k(k-1)^2 u + k)$

(i) if $k \equiv 2, 8 \mod 12$ *and* $u \equiv 1 \mod 3, u \geq 4$*; or*

(ii) if $k \equiv 5, 11 \mod 12$ *and* $u \equiv 1, 4 \mod 12, u \geq 4$*.*

Proof. Use an MRD$(\{4, k\}; k(k-1)u + 1)$ constructed in Lemma 3.18 as ingredient PBD in Construction 3.17 to obtain an MRD$(\{4, k\}; (k-1)(k(k-1)u + 1) + 1)$. $\square$

Construction 3.26 *Let* $t \equiv 1 \mod 3$ *and suppose there is an MRD*$(\{4,k\};t)$*. Then there is a mandatory representation design MRD*$(\{4,k\};v)$ *for all* $v \geq 3t+1$

(i) with $v \equiv 1,4 \mod 12$ *if* $t \equiv 1,4 \mod 12$*; or*

(ii) with $v \equiv 7,10 \mod 12$ *if* $t \equiv 0,9 \mod 12$*.*

Proof. Use a 4-GDD of type $t^1 1^{v-t}$ from Lemma 1.4 and replace the group of size t by the MRD$(\{4,k\};t)$. $\square$

Lemma 3.27 *Let* $k \equiv 2 \mod 3$ *and* $v \equiv 1 \mod 3$*. There exists a mandatory representation design MRD*$(\{4,k\};v)$

(i) if $k \equiv 2,5,8 \mod 12$*,* $v \equiv 1,4 \mod 12$ *and* $v \geq 3k(k-1)^2+3k+1$*; or*

(ii) if $k \equiv 11 \mod 12$*,* $v \equiv 1,4 \mod 12$ *and* $v \geq 12k(k-1)^2+12k-8$*; or*

(iii) if $k \equiv 2 \mod 12$*,* $v \equiv 7,10 \mod 12$ *and* $v \geq 12k(k-1)^2+3k+1$*; or*

(iv) if $k \equiv 11 \mod 12$*,* $v \equiv 7,10 \mod 12$ *and* $v \geq 3k(k-1)^2+3k+1$*.*

Proof. Start with an MRD$(\{4,k\};k(k-1)^2+k)$ from 3.25 and use it as ingredient in Construction 3.26. If $k \equiv 2,5,8 \mod 12$ the number of points $t = k(k-1)^2+k$ is congruent 1 or 4 modulo 12, so an MRD for all $v \geq 3(k(k-1)^2+k)+1, v \equiv 1,4 \mod 12$ is produced which gives the bound for Case (i). While if $k \equiv 11 \mod 12$ we have $t \equiv 7 \mod 12$ and, therefore, $v \equiv 7,10 \mod 12$ (Case (iv)).

For $k \equiv 11 \mod 12$ we continue by filling in the MRD$(\{4,k\};k(k-1)^2+k)$ into the groups of a 4-GDD of type $(k(k-1)^2+k)^4$ and get an MRD on $t = 4(k(k-1)^2+k)$ where $t \equiv 4 \mod 12$. Thus, if used with Construction 3.26 MRD$(\{4,k\};v)$s for all $v \geq 12(k(k-1)^2+k)+1, v \equiv 1,4 \mod 12$ are obtained (Case (ii)).

Finally, take for $k \equiv 2 \mod 12$ an MRD$(\{4,k\};4k(k-1)^2+k)$ which is obtained from 3.25 by setting $u = 4$. Here, $t = 4k(k-1)^2+k \equiv 10 \mod 12$ so using again Construction 3.26 yields an MRD$(\{4,k\};v)$s for all $v \geq 12k(k-1)^2+3k+1, v \equiv 1,4 \mod 12$ and establishes the bound in Case (iii). $\square$

3.4 Main Result

We summarize the main result of the section which is a combination of Lemmata 3.4–3.7, 3.10–3.16, 3.22–3.24 and 3.27.

Theorem 3.28 *Let $k \equiv 2 \bmod 3, k \geq 5$. There exists a mandatory representation design MRD$(\{4,k\};v)$*

(i) with $v \equiv 1 \bmod 3$, if

(a) $k \equiv 2,5,8 \bmod 12, v \equiv 1,4 \bmod 12$, or $k \equiv 11 \bmod 12, v \equiv 7,10 \bmod 12$, and $v \geq 3k(k-1)^2+3k+1$, or

(b) $k \equiv 2 \bmod 12, v \equiv 7,10 \bmod 12$, or $k \equiv 11 \bmod 12, v \equiv 1,4 \bmod 1$, and $v \geq 12k(k-1)^2+12k-8$;

(ii) with $v \equiv 2 \bmod 3$, if

(a) $k \equiv 5,8 \bmod 12, k = 5,8$ or $k \geq 44, v \equiv 5,8 \bmod 12$ and $v \geq 9k^2-32k+27$, or

(b) $k \equiv 2,11 \bmod 12, k \geq 23$ and $v \geq 18k^2-41k+27$, or

(c) $k = 17,20,29,32,41$ and $v > 192k+3k^2-11k+9$, or

(d) $k = 11,14$ and $v > 192k+6k^2-14k+9$;

(iii) with $v \equiv 0 \bmod 3$, if

(a) $k \equiv 5,8 \bmod 12, v \equiv 9 \bmod 12$ and $v \geq 3k^4-6k^3+10k^2-7k+7$, or

(b) $k \equiv 2,11 \bmod 12, v \equiv 3,9 \bmod 12$ and $v \geq 6k^4-12k^3+25k^2-19k+7$, or

(c) $k \equiv 5,8 \bmod 12, v \equiv 0 \bmod 12$ and $v \geq 4k^4-5k^3+43k^2-12k+14$, or

(d) $k \equiv 2,11 \bmod 12, v \equiv 0,6 \bmod 12$ and $v \geq 8k^4-10k^3+49k^2-14k+15$.

4 Conclusion

After having established upper bounds for the case $k \equiv 2 \bmod 3$, it remains to close the gap between lower and upper bounds. We remark that the analogue problem for $K = \{3,k\}$ is difficult and far from being completely solved, see [14] for recent advances.

Also, the determination of the MRD-closure in the case $k \equiv 0 \bmod 3$ is open. Here, the difficulties arise out of the fact that each point lies on either $\gamma_k \equiv 0 \bmod 3$ or $\gamma_k \equiv 1 \bmod 3$ blocks of size k. So in order to construct MRDs with $\gamma_k \equiv 0 \bmod 3$ we can not use modified group divisible designs together with MRDs with $\gamma_k \equiv 1$ or $2 \bmod 3$ since the latter simply do not exist. It would be of considerable interest to establish something like a *further* modified group divisible design, i.e. a design with three parallel classes of equal sized holes, as this would allow to fill each parallel class just with MRDs with $\gamma_k \equiv 1 \bmod 3$.

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