Multileaf collimator field segmentation without tongue–and–groove effect

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Abstract

We present an algorithm for optimal step–and–shoot intensity modulated radiation therapy without interleaf collision and with elimination of tongue–and–groove effects. Adapting the concepts of [6] we characterize the minimal number of monitor units as the maximal weight of a path in a properly constructed weighted digraph. We also show that this number of monitor units can be realized by an unidirectional plan, thus proving that the algorithm of Kamath *et al.* [10] is monitor unit optimal in general and not only for unidirectional leaf movement. Our characterization of the minimal number of monitor units has the advantage that it can be used to derive a heuristic for the reduction of the number of segments following the ideas of [7].

Key words: leaf sequencing, radiation therapy optimization, intensity modulation, multileaf collimator, IMRT

2000 MSC: 92C50, 90C90

1 Introduction

In intensity modulated radiation therapy (IMRT), the desired fluence distribution can be described by an intensity matrix, that is a nonnegative integer $m \times n$–matrix A, where the irradiated region is discretized into $m \times n$ *bixels* and the entry $a_{i,j}$ represents the required fluence at bixel (i, j). A modern way to realize such intensity maps is the usage of a multileaf collimator (MLC).

An MLC consists of metal leaves which can block the radiation. To each row of the intensity matrix there is associated a pair of leaves, a left leaf and a right leaf that can be moved in the direction of the row. When the MLC is used in the so called step–and–shoot mode the given fluence distribution is realized by superimposing a number of differently shaped homogeneous fields coming from different combinations of the leaf positions. For example, Figure 1 shows a sequence of leaf positions for the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & 2 & 1 & 0 \end{pmatrix},$$

where the shading indicates the region which is covered by the leaves.

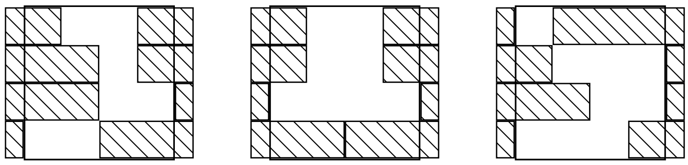

Figure 1: A realization of the intensity matrix A using an MLC.

The problem of realizing a given intensity matrix A leads to the problem of representing A as a positive integer combination of certain $(0, 1)$–matrices, called segments, which represent the possible leaf positions. The main objectives in constructing such a *segmentation* are to minimize the total number of monitor units (TNMU) and the number of segments (NS). An important restriction for the possible segments in some of the MLCs used in clinical practice is the interleaf collision constraint (ICC) which forbids the overlapping of opposite leaves in adjacent rows. Due to the tongue–and–groove design of the MLCs there is a narrow strip in the border region between two adjacent rows that is covered by both leaves and this may lead to underdosage effects in these regions. In order to minimize these effects we require that $a_{i,j} \leq a_{i+1,j}$ implies that bixel $(i + 1, j)$ is exposed whenever bixel (i, j) is exposed (similarly for $i - 1$ instead of $i + 1$). Thus we assure that the overlap region of two bixels always receives the smaller one of the relevant doses. We say that a segmentation of A satisfies the tongue–and–groove constraint (TGC) if this condition is fulfilled for all segments. Starting with [3] and [5] there were proposed several algorithms for the segmentation problem [1, 2, 4, 9, 13, 14]. Only recently there were presented two algorithms eliminating the tongue–and–groove effect [10, 12]. The algorithm from [10] is TNMU–optimal, as is shown for unidirectional plans in [10] and will be proved without restriction on the leaf movement direction in the present paper. Adapting the approach of [4], in [6] we characterized the minimal TNMU for the segmentation with ICC as the maximal weight of a path in a certain digraph. In this paper we further modify this approach such that the TGC is included, thus solving the TNMU–minimization problem for the two most relevant restrictions completely. In addition, we derive a greedy heuristic for the reduction of the number of segments and present some numerical test results.

2 Mathematical formulation of the TNMU–segmentation problem with ICC and TGC

Throughout the rest of the paper, for a natural number n, $[n]$ denotes the set $\{1, 2, \ldots, n\}$ and for natural numbers $m < n$, $[m, n]$ denotes the set $\{m, m+1, \ldots, n\}$. In this section we formulate an LP–relaxation of the segmentation problem that is very similar to the one used in [6]. We start with a formal characterization of the $(0, 1)$–matrices that are allowed as segments for a given intensity matrix A.

Definition 1. Let A be an intensity matrix. An *A–segment* is an $m \times n$-matrix $S = (s_{i,j})$ with entries from $\{0, 1\}$, such that there exist integers $l_i,\ r_i$ $(i \in [m])$ with the following properties:

$$l_i \leq r_i + 1 \qquad (i \in [m]), \tag{1}$$

$$s_{i,j} = \begin{cases} 1 & \text{if } l_i \leq j \leq r_i \\ 0 & \text{otherwise} \end{cases} \qquad (i \in [m],\ j \in [n]), \tag{2}$$

$$\text{ICC:} \quad l_i \leq r_{i+1} + 1,\ r_i \geq l_{i+1} - 1 \qquad (i \in [m-1]), \tag{3}$$

and we have

$$\text{TGC:} \begin{cases} a_{i,j} \leq a_{i+1,j} \wedge s_{i,j} = 1 & \Rightarrow \quad s_{i+1,j} = 1 \ (i \in [m-1],\ j \in [n]), \\ a_{i,j} \leq a_{i-1,j} \wedge s_{i,j} = 1 & \Rightarrow \quad s_{i-1,j} = 1 \ (i \in [2, m],\ j \in [n]). \end{cases} \tag{4}$$

Now a *segmentation* of an intensity matrix A is a representation

$$A = \sum_{i=1}^{t} u_i S^{(i)}$$

with positive integers u_i and A–segments $S^{(i)}$ $(i \in [t])$, the TNMU of this segmentation is $\sum_{i=1}^{t} u_i$ and the TNMU–segmentation problem is to find, for given A, a segmentation with minimal TNMU. Let A be a fixed intensity matrix. We denote by $\mathcal{F}$ the family of subsets of $[m] \times [n]$ that correspond to $A-$segments, precisely

$$\mathcal{F} = \{T \subseteq [m] \times [n] : \text{ There exists an } A - \text{segment } S \\ \text{with } (i,j) \in T \iff s_{i,j} = 1\}.$$

Now the formulations of the linear program and its dual are exactly the same as in [6]. We associate with a segmentation $A = \sum_{i=1}^{k} u_i S_i$ a function $f : \mathcal{F} \to \mathbb{N}$: for $1 \leq i \leq k$ we put $f(T) = u_i$ for the $T \subseteq [m] \times [n]$ corresponding to the segment S_i, and for the remaining T we put $f(T) = 0$. Now the LP–relaxation of the TNMU–segmentation problem is:

$$(P) \begin{cases} \text{minimize} \quad \sum\limits_{T \in \mathcal{F}} f(T) \quad \text{subject to} \\ \\ f(T) \geq 0 \qquad \forall T \in \mathcal{F}, \\ \\ \sum\limits_{T \in \mathcal{F} : (i,j) \in T} f(T) = a_{i,j} \qquad \forall (i,j) \in [m] \times [n]. \end{cases}$$

The dual variables (one variable for each $(i, j) \in [m] \times [n]$) can be considered as a function $g : [m] \times [n] \to \mathbb{R}$ and in this formulation the dual program is

$$(D) \begin{cases} \text{maximize} \quad \sum\limits_{(i,j) \in [m] \times [n]} a_{i,j} g(i,j) \quad \text{subject to} \\ \\ \sum\limits_{(i,j) \in T} g(i,j) \leq 1 \qquad \forall T \in \mathcal{F}. \end{cases}$$

We construct a digraph $G = (V, E)$ as follows.

$$V = \{0, 1\} \cup ([m] \times [0, n+1]), \; E = E_1 \cup E_2 \cup E_3 \cup E_4 \text{ where}$$

$$\begin{aligned} E_1 &= \{(0, (i, 0)) \; : \; i \in [m]\} \cup \{((i, n+1), 1) \; : \; i \in [m]\}, \\ E_2 &= \{((i,j), (i+1, j)) \; : \; i \in [m-1], \; j \in [n-1]\}, \\ E_3 &= \{((i,j), (i-1, j)) \; : \; i \in [2, m], \; j \in [n-1]\}, \\ E_4 &= \{((i, j-1), (i, j)) \; : \; i \in [m], \; j \in [n+1]\}. \end{aligned}$$

Here 0 and 1 serve as starting and end point, respectively, and the vertices in $[m] \times [n]$ correspond to the entries of A. The two extra columns $[m] \times \{0\}$ and $[m] \times \{n+1\}$ have the purpose to simplify the notation: they assure that for every $(i, j) \in [m] \times [n]$ there are vertices $(i, j-1)$ and $(i, j+1)$. Without this, in several of the arguments below, it would be necessary to treat the first and the last column seperately (then 0 and 1 would have to play the role of $(i,0)$ and $(i,n+1)$, respectively). To be able to treat the first and the n-th column exactly as the remaining columns, we also put $a_{i,0} = a_{i,n+1} = 0$ $(i \in [m])$. Observe that we omit the vertical arcs $((i,n),(i \pm 1,n))$ in the n-th column: this is to make the vertex (i,n) of any $(0,1)$–path unique. This is no loss of generality, because we will see that we are interested only in $(0,1)$–paths of maximal weight, the weights of vertical arcs are nonpositive and the arcs right of column n have weight 0. Now we define the weight function $w : E \rightarrow \mathbb{Z}$:

$$\begin{aligned}
w(0,(i,0)) &= w((i,n+1),1) = 0 \quad (i \in [m]),\\
w((i,j),(i+1,j)) &= \min\{0, a_{i+1,j} - a_{i,j}\} \quad (i \in [m-1],\ j \in [n]),\\
w((i,j),(i-1,j)) &= \min\{0, a_{i-1,j} - a_{i,j}\} \quad (i \in [2,m],\ j \in [n]),\\
w((i,j-1),(i,j)) &= \max\{0, a_{i,j} - a_{i,j-1}\} \quad (i \in [m],\ j \in [n+1]).
\end{aligned}$$

Example 1. Figure 2 shows G corresponding to the matrix

$$A = \begin{pmatrix} 4 & 5 & 0 & 1 & 4 & 5 \\ 2 & 4 & 1 & 3 & 1 & 4 \\ 2 & 3 & 2 & 1 & 2 & 4 \\ 5 & 3 & 3 & 2 & 5 & 3 \end{pmatrix}.$$

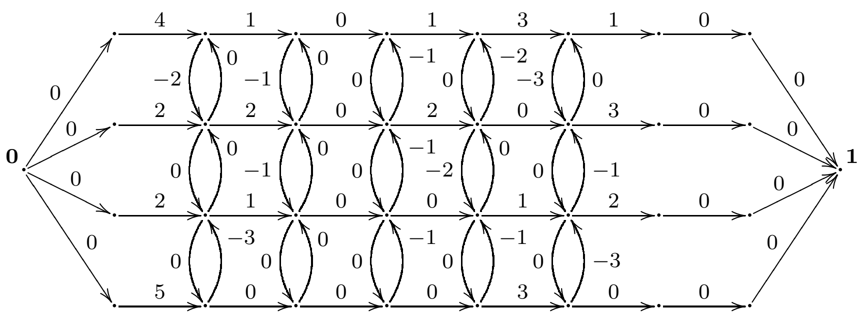

Figure 2: The digraph G corresponding to matrix A.

The following theorem, whose proof is the content the next two sections, is the main result of this paper.

Theorem 1. *The minimal TNMU of a segmentation of a nonnegative matrix A equals the maximal weight of a $(0,1)-$path in G.*

3 The lower bound

In this section we associate with a $(0,1)$–path P a dual feasible solution and prove the lower bound of the theorem. For an intensity matrix A let

$$c(A) = \max\{w(P) \ : \ P \text{ is a } (0,1) - \text{path in } G\}.$$

Let A be an intensity matrix and let P be a $(0,1)-$path in G. We define a function $g : [m] \times [n] \to \{0,1\}$ as follows.

$$g(i,j) = \begin{cases} 1 & \text{if } (i,j-1),(i,j),(i,j+1) \in P,\ a_{i,j} \geq a_{i,j-1},\ a_{i,j} > a_{i,j+1} \\ 1 & \text{if } (i-1,j),(i,j),(i,j+1) \in P,\ a_{i,j} < a_{i-1,j},\ a_{i,j} > a_{i,j+1} \\ 1 & \text{if } (i+1,j),(i,j),(i,j+1) \in P,\ a_{i,j} < a_{i+1,j},\ a_{i,j} > a_{i,j+1} \\ 1 & \text{if } (i-1,j),(i,j),(i+1,j) \in P,\ a_{i,j} < a_{i-1,j},\ a_{i,j} \leq a_{i+1,j} \\ 1 & \text{if } (i+1,j),(i,j),(i-1,j) \in P,\ a_{i,j} < a_{i+1,j},\ a_{i,j} \leq a_{i-1,j} \\ 1 & \text{if } (i,j-1),(i,j),(i+1,j) \in P,\ a_{i,j} \geq a_{i,j-1},\ a_{i,j} \leq a_{i+1,j} \\ 1 & \text{if } (i,j-1),(i,j),(i-1,j) \in P,\ a_{i,j} \geq a_{i,j-1},\ a_{i,j} \leq a_{i-1,j} \\ -1 & \text{if } (i,j-1),(i,j),(i,j+1) \in P,\ a_{i,j} < a_{i,j-1},\ a_{i,j} \leq a_{i,j+1} \\ -1 & \text{if } (i-1,j),(i,j),(i,j+1) \in P,\ a_{i,j} \geq a_{i-1,j},\ a_{i,j} \leq a_{i,j+1} \\ -1 & \text{if } (i+1,j),(i,j),(i,j+1) \in P,\ a_{i,j} \geq a_{i+1,j},\ a_{i,j} \leq a_{i,j+1} \\ -1 & \text{if } (i-1,j),(i,j),(i+1,j) \in P,\ a_{i,j} \geq a_{i-1,j},\ a_{i,j} > a_{i+1,j} \\ -1 & \text{if } (i+1,j),(i,j),(i-1,j) \in P,\ a_{i,j} \geq a_{i+1,j},\ a_{i,j} > a_{i-1,j} \\ -1 & \text{if } (i,j-1),(i,j),(i+1,j) \in P,\ a_{i,j} < a_{i,j-1},\ a_{i,j} > a_{i+1,j} \\ -1 & \text{if } (i,j-1),(i,j),(i-1,j) \in P,\ a_{i,j} < a_{i,j-1},\ a_{i,j} > a_{i-1,j} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Here by $x, y, z \in P$ we mean that P runs through these vertices in the given order, i.e. (x,y) and (y,z) are arcs of P. Observe that $g(i,j) \neq 0$ is possible only if (i,j) lies on P. The definition of g is illustrated in Figure 3.

Lemma 1. *Let P be a $(0,1)-$path in G and let g be defined according to (5). Then g is feasible for the dual program (D).*

Proof. Denote the vertices (i,j) with $g(i,j) \neq 0$ by $(i_1,j_1),\dots,(i_t,j_t)$ in the order in which they occur on P. Using the definition of g, it is easy to check that $g(i_1,j_1) = 1$ and

$$g(i_p,j_p)g(i_{p+1},j_{p+1}) = -1 \qquad (p \in [t-1]). \tag{6}$$

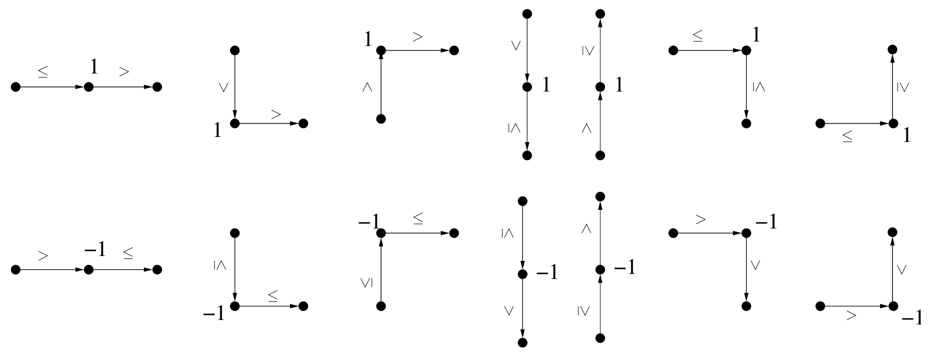

Figure 3: Illustration of the definition of the dual solution g. We depicted the possibilities for the path P to pass through a vertex (i,j) (always the middle vertex) which lead to a nonzero value of $g(i,j)$. The labels of the arcs indicate the relation of $a_{i,j}$ to its neighbours, and the label of the middle vertex is the resulting value of $g(i,j)$.

The definition of g also implies, for $p \in [t-1]$, that for arcs $((i,j),(i,j+1))$ on the $((i_p,j_p),(i_{p+1},j_{p+1}))$–subpath of P, we have

$$a_{i,j+1} \geq a_{i,j} \quad \text{if } g(i_p,j_p) = -1, \tag{7}$$

$$a_{i,j+1} < a_{i,j} \quad \text{if } g(i_p,j_p) = 1, \tag{8}$$

and for arcs $((i,j),(i\pm 1,j))$ on the $((i_p,j_p),(i_{p+1},j_{p+1}))$–subpath of P, we have

$$a_{i\pm 1,j} < a_{i,j} \quad \text{if } g(i_p,j_p) = -1, \tag{9}$$

$$a_{i\pm 1,j} \geq a_{i,j} \quad \text{if } g(i_p,j_p) = 1. \tag{10}$$

Let S be an $A-$segment with parameters l_i, r_i $(i \in [m])$. We have to show that

$$\sum_{(i,j)\in[m]\times[n]} g(i,j)s_{i,j} \leq 1.$$

Let $p,q \in [t]$ such that $p < q$,

$$g(i_p,j_p)s_{i_p,j_p} = g(i_q,j_q)s_{i_q,j_q} = 1$$

and $g(i_v,j_v)s_{i_v,j_v} \leq 0$ for $p < v < q$. Hence,

$$s_{i_v,j_v} = 0 \quad \text{if } p < v < q \text{ and } g(i_v,j_v) = 1.$$

A path from (i_p,j_p) to (i_q,j_q) with a possible leaf setting is shown in Figure 4. From (6) we obtain $q \geq p+2$. We claim that

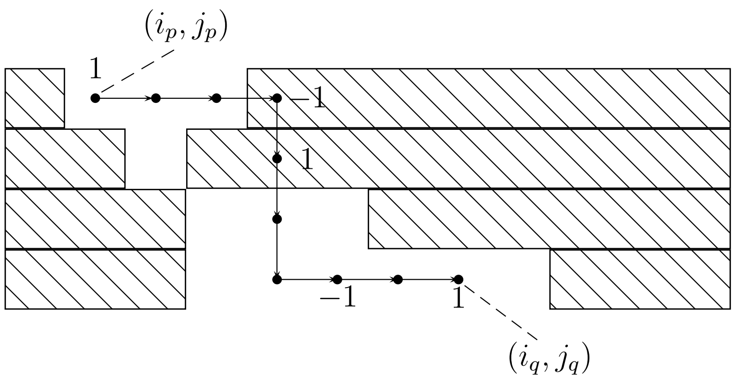

Figure 4: A possible path from (i_p, j_p) to (i_q, j_q). The labels at the vertices are the nonzero values of $g(i, j)$.

$$\sum_{v=p+1}^{q-1} g(i_v, j_v) s_{i_v, j_v} \le -1,$$

and clearly this proves the lemma. Assume

$$\sum_{v=p+1}^{q-1} g(i_v, j_v) s_{i_v, j_v} = 0,$$

or equivalently

$$s_{i_v, j_v} = 0 \text{ for } p < v < q.$$

We need the following claims.

Claim 1. If $g(i_v, j_v) = -1$ and $l_{i_v} > j_v$, then $l_{i_{v-1}} > j_{v-1}$ $(p < v < q)$.

Claim 2. If $g(i_{v-1}, j_{v-1}) = -1$ and $r_{i_{v-1}} < j_{v-1}$, then $r_{i_v} < j_v$ $(p < v \le q)$.

Proof of the claims. If $i_v = i_{v-1}$ the claims are obvious. So suppose that $i_v \ne i_{v-1}$. Let j' be the first column $j' \ge j_{v-1}$ where P leaves row i_{v-1}, and let j'' be the last column $j'' \le j_v$ where P enters row i_v, i.e. the $((i_{v-1}, j_{v-1}), (i_v, j_v))$–subpath of P has the form

$$(i_{v-1}, j_{v-1}), (i_{v-1}, j_{v-1}+1), \ldots, (i_{v-1}, j'), (i_{v-1} \pm 1, j'), \ldots,$$
$$(i_v \pm 1, j'',), (i_v, j''), (i_v, j''+1), \ldots, (i_v, j_v).$$

This is illustrated in Figure 5. Let Q be the $((i_{v-1}, j'), (i_v, j''))$ subpath.

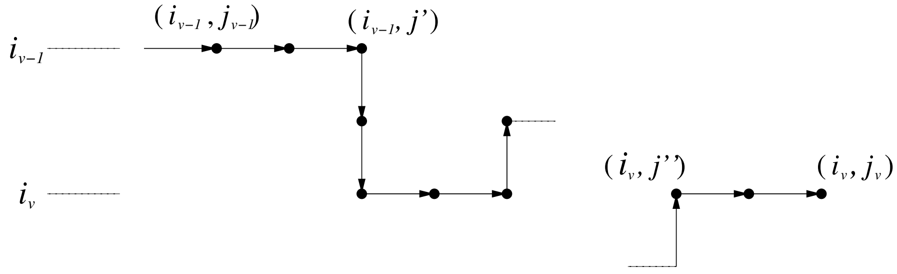

Figure 5: Illustration for the proof of Lemma 1.

1. Suppose that $g(i_v, j_v) = -1$ and $l_{i_v} > j_v$. We prove by induction going backwards along Q, that $l_i > j$ for every $(i, j) \in Q$, in particular $l_{i_{v-1}} > j' \geq j_{v-1}$. For $(i, j) = (i_v, j'')$,
$$l_i = l_{i_v} > j_v \geq j''.$$
For an arc $((i, j-1), (i, j))$, $l_i > j - 1$ follows from $l_i > j$. For an arc $((i-1, j), (i, j))$ of Q, by (10) we have $a_{i-1,j} \leq a_{i,j}$. The TGC implies $s_{i-1,j} = 0$, and together with the ICC and $l_i > j$ this implies $l_{i-1} > j$. The analogue argument works for arcs $((i+1, j), (i, j))$.

2. Suppose that $g(i_{v-1}, j_{v-1}) = -1$ and $r_{i_{v-1}} < j_{v-1}$. By induction along Q, we show that $r_i < j$ for every $(i, j) \in Q$, in particular $r_{i_v} < j'' \leq j_v$. For $(i, j) = (i_{v-1}, j')$,
$$r_i = r_{i_{v-1}} < j_{v-1} \leq j'.$$
For an arc $((i, j), (i, j+1))$ of Q, $r_i < j + 1$ follows from $r_i < j$. For an arc $((i-1, j), (i, j))$ of Q, by (9) we have $a_{i,j} < a_{i-1,j}$. The TGC implies $s_{i,j} = 0$, and together with the ICC and $r_{i-1} < j$ this implies $r_i < j$. The analogue argument works for arcs $((i, j), (i-1, j))$.

Now we are prepared to show by induction on v, that $r_{i_v} < j_v$ for $p < v \leq q$, in particular $r_{i_q} < j_q$, contradicting $s_{i_q,j_q} = 1$. Let $v = p+1$. From $s_{i_v,j_v} = 0$ it follows that either $l_{i_v} > j_v$ or $r_{i_v} < j_v$. But $g(i_v, j_v) = -1$, and by Claim 1, $l_{i_v} > j_v$ implies $l_{i_p} > j_p$, contradicting $s_{i_p,j_p} = 1$. Hence $r_{i_v} < j_v$. Let $p + 2 \leq v < q$. If $g(i_v, j_v) = 1$, by Claim 2, $r_{i_{v-1}} < j_{v-1}$ yields $r_{i_v} < j_v$. So suppose that $g(i_v, j_v) = -1$. From $s_{i_v,j_v} = 0$ we obtain $l_{i_v} > j_v$ or $r_{i_v} < j_v$. By Claim 1, $l_{i_v} > j_v$ implies $l_{i_{v-1}} > j_{v-1}$, contradicting $r_{i_{v-1}} < j_{v-1}$. Hence $r_{i_v} < j_v$. Finally, Claim 2 with $v = q$ yields $r_{i_q} < j_q$. ∎

Lemma 2. *Let P be a $(0,1)-$path in G and let g be defined according to (5). Then*
$$\sum_{(i,j) \in [m] \times [n]} g(i, j) a_{i,j} = w(P).$$

Proof. Again, we denote the vertices (i,j) with $g(i,j) \neq 0$ by $(i_1,j_1),\dots,(i_t,j_t)$ in the order in which they occur on P. Also, let P_k be the $((i_{k-1},j_{k-1}),(i_k,j_k))$–subpath of P for $k \in [2,t]$ and let P_1 be the $(0,(i_1,j_1))-$subpath of P. Now the lemma is an immediate consequence of the following claim which is easy to verify: For $k \in [t]$, we have (with $a_{i_0,j_0} = 0$),

$$g(i_k,j_k) = 1 \qquad \Rightarrow \qquad w(P_k) = a_{i_k,j_k} - a_{i_{k-1},j_{k-1}}, \tag{11}$$

$$g(i_k,j_k) = -1 \qquad \Rightarrow \qquad w(P_k) = 0. \tag{12}$$

Observe that w.l.o.g. we suppose that $g(i_t,j_t) = 1$. Assume that $g(i_t,j_t) = -1$. Then $a_{i,j} = 0$ for every (i,j) on the $((i_t,j_t),1)$–subpath of P, and we can put $g(i,n) = 1$ for the unique i with $(i,n) \in P$ preserving the feasibility of g and the objective value. ■

4 The algorithm

We construct a segmentation of A according to the following algorithm:

$A^{(0)} := A$, $k := 0$.
while $A^{(k)} \neq 0$ **do**
  Determine an $A^{(k)}-$segment $S^{(k+1)}$ such that
  $A^{(k+1)} := A^{(k)} - S^{(k+1)}$ is nonnegative and $c\left(A^{(k+1)}\right) = c\left(A^{(k)}\right) - 1$.
  $k := k+1$.

Suppose that we have a method to construct $S^{(k+1)}$ from $A^{(k)}$, such that the given conditions are fulfilled. The next lemma implies that the algorithm yields only A–segments.

Lemma 3. *Suppose that $A = \sum_{k=1}^{t} S^{(k)}$, and for every $k \in [t]$, $S^{(k)}$ is an $A^{(k-1)}$–segment, where $A^{(0)} = A$ and for $k \geq 1$, $A^{(k)} := A - \sum_{k'=1}^{k} S^{(k)}$. Then for $k \in [t]$, every $A^{(k-1)}$–segment (in particular $S^{(k)}$) is also an A–segment.*

Proof. We use induction on k. The case $k = 1$ is clear by hypothesis. So let $k > 1$. Using that $S^{(k-1)}$ is an $A^{(k-2)}-$segment, we obtain

$$a_{i,j}^{(k-2)} \leq a_{i+1,j}^{(k-2)} \;\Rightarrow\; a_{i,j}^{(k-1)} \leq a_{i+1,j}^{(k-1)} \quad (i \in [m-1],\; j \in [n]),$$

$$a_{i,j}^{(k-2)} \leq a_{i-1,j}^{(k-2)} \;\Rightarrow\; a_{i,j}^{(k-1)} \leq a_{i-1,j}^{(k-1)} \quad (i \in [2,m],\; j \in [n]),$$

and this implies that every $A^{(k-1)}-$segment is also an $A^{(k-2)}-$segment and by induction an $A-$segment. ■

Now we suppose that $A^{(k)}$ is given and we describe how the segment $S^{(k+1)}$ is determined. Let w_k denote the weight function on the arcs of G with respect to $A^{(k)}$ and put, for $(i,j) \in [m] \times [n]$,

$$\alpha_1^{(k)}(i,j) = \max\{w_k(P) \ : \ P \text{ is a } (0,(i,j)) - \text{path in } G\}, \tag{13}$$

$$\alpha_2^{(k)}(i,j) = \max\{w_k(P) \ : \ P \text{ is a } ((i,j),1) - \text{path in } G\}, \tag{14}$$

$$\alpha^{(k)}(i,j) = \alpha_1^{(k)}(i,j) + \alpha_2^{(k)}(i,j). \tag{15}$$

We define the matrix $S^{(k+1)}$ by

$$s_{i,j}^{(k+1)} = \begin{cases} 1 & \text{if } a_{i,j}^{(k)} > 0, \ \alpha^{(k)}(i,j) = c\left(A^{(k)}\right), \ \alpha_1^{(k)}(i,j) = a_{i,j}^{(k)}, \\ 0 & \text{otherwise.} \end{cases}$$

In order to show that this matrix has the required properties, it is clearly sufficient to consider the case $k = 0$. So let $w = w_0$ and $S = S^{(1)}$. The nonnegativity of $A - S$ follows directly from the definition of S.

Lemma 4. *S is an $A-$segment.*

Proof. In order to show that there are l_i, r_i $(i \in [m])$ satisfying (1) and (2), we just have to observe that there are no $i \in [m]$, $j, j' \in [n]$, with $j' > j+1$, such that $s_{i,j} = s_{i,j'} = 1$, $s_{i,j+1} = 0$. Assume the contrary. Then we have

- $a_{i,j+1} = 0$ (contradiction to $\alpha_1(i,j') = a_{i,j'}$) or
- $\alpha_1(i,j+1) > a_{i,j+1}$ (contradiction to $\alpha_1(i,j') = a_{i,j'}$) or
- $\alpha(i,j+1) < c(A)$ (contradiction to $\alpha_1(i,j') = a_{i,j'}$ and $\alpha(i,j) = c(A)$).

So we have to check that the ICC and the TGC are satisfied. Assume that the ICC (3) is violated. Then there are rows i, i', such that $l_i > r_{i'} + 1$ and $l_{i''} = r_{i''} + 1$ for all rows i'' between i and i'. For symmetry reasons we have to consider only the case $i < i'$, which is illustrated in Figure 6. By definition of S, $a_{i,j} = 0$ for $j < l_i$. We use induction on i'' to show that

$$a_{i'',j} = 0 \quad \text{for } i \leq i'' < i', \ 1 \leq j < l_i.$$

Suppose $i < i'' < i'$, $1 \leq j < l_i$ and $a_{i'',j} > 0$. Using the induction hypothesis, we obtain that any $(0,(i'',j))$–path of weight $\alpha_1(i'',j)$ concatenated with the path

$$(i'',j), (i''-1,j), \ldots, (i,j), (i,j+1), \ldots, (i,l_i)$$

yields a $(0,(i,l_i))$–path of weight $\alpha_1(i'',j) - a_{i'',j} + a_{i,l_i}$. This implies

$$\alpha_1(i'',j) = a_{i'',j} \quad \text{and} \quad \alpha(i'',j) = c(A),$$

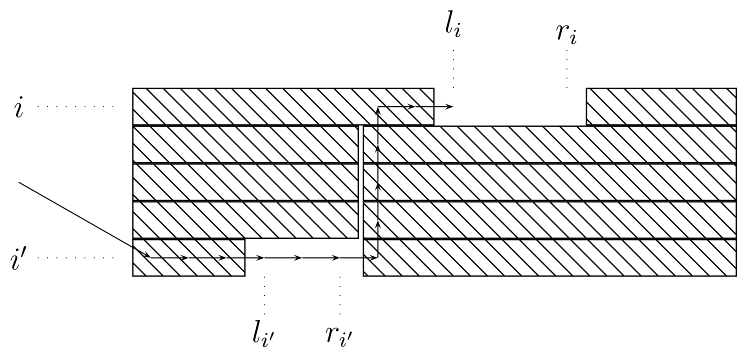

Figure 6: Illustration of an ICC–violation

hence $s_{i'',j} = 1$, a contradiction. Let P be a $(0, (i', r_{i'} + 1))$–path of weight $\alpha_1(i', r_{i'} + 1)$ concatenated with the path

$$(i', r_{i'} + 1), (i' - 1, r_{i'} + 1), \ldots, (i, r_{i'} + 1), (i, r_{i'} + 2), \ldots, (i, l_i).$$

Then $w(P) = \alpha_1(i', r_{i'} + 1) - a_{i',r_{i'}+1} + a_{i,l_i}$. This implies

$$\alpha_1(i', r_{i'} + 1) = a_{i',r_{i'}+1} \quad \text{and} \quad \alpha(i', r_{i'} + 1) = c(A),$$

and in particular $a_{i',r_{i'}+1} \geq a_{i',r_{i'}} > 0$. But then $s_{i',r_{i'}+1} = 1$, a contradiction. Now assume that the TGC (4) is violated. By symmetry it is sufficient to consider the case

$$a_{i,j} \leq a_{i+1,j}, \quad s_{i,j} = 1, \quad s_{i+1,j} = 0.$$

We have that

$$\alpha_1(i, j) \geq \alpha_1(i + 1, j) + w((i + 1, j), (i, j)) = \alpha_1(i + 1, j) + a_{i,j} - a_{i+1,j}.$$

This implies $\alpha_1(i + 1, j) = a_{i+1,j}$ and $\alpha(i + 1, j) = c(A)$. Since also $a_{i+1,j} \geq a_{i,j} > 0$, we obtain $s_{i+1,j} = 1$, a contradiction. ∎

Let $A' = A - S$, denote the weight function on G with respect to A' by w' and put

$$\alpha'_1(i, j) = \max\{w'(P) \ : \ P \text{ is a } (0, (i, j)) - \text{path in } G\}, \tag{16}$$
$$\alpha'_2(i, j) = \max\{w'(P) \ : \ P \text{ is a } ((i, j), 1) - \text{path in } G\}, \tag{17}$$
$$\alpha'(i, j) = \alpha'_1(i, j) + \alpha'_2(i, j). \tag{18}$$

It is clear that for an arc $e \in E$, we can have $w'(e) \neq w(e)$ only if $s_{i,j} = 1$ for exactly one vertex of e. In order to prove $c(A') = c(A) - 1$, we need the following lemma, whose proof is trivial.

Lemma 5. *Let $e \in E$, and put, for brevity of notation, $s_{i,0} = s_{i,n+1} = 0$ $(i \in [m])$.*

1. *If $e = ((i, j-1), (i, j))$, $s_{i,j-1} = 0$ and $s_{i,j} = 1$, then $w'(e) = w(e) - 1$.*
2. *If $e = ((i, j-1), (i, j))$, $s_{i,j-1} = 1$ and $s_{i,j} = 0$, then*

$$w'(e) = \begin{cases} w(e) = 0 & \text{if } a_{i,j-1} > a_{i,j}, \\ w(e) + 1 & \text{if } a_{i,j-1} \leq a_{i,j}. \end{cases}$$

3. *If $e = ((i \pm 1, j), (i, j))$, $s_{i\pm1,j} = 0$ and $s_{i,j} = 1$, then $w'(e) = w(e) = 0$.*
4. *If $e = ((i \pm 1, j), (i, j))$, $s_{i\pm1,j} = 1$ and $s_{i,j} = 0$, then $w'(e) = w(e) + 1$.*

For arcs on paths of weight $c(A)$ we can say even more.

Lemma 6. *Let P be a $(0, 1)$–path in G with $w(P) = c(A)$, and let $e \in E$ be an arc of P.*

1. *If $e = ((i, j-1), (i, j))$, $s_{i,j-1} = 1$ and $s_{i,j} = 0$, then $w'(e) = w(e) = 0$.*
2. *If $e = ((i \pm 1, j), (i, j))$, $s_{i\pm1,j} = 1$ and $s_{i,j} = 0$, then $a_{i,j} = \alpha_1(i, j) = 0$.*

Proof.

1. Assume the contrary and let P_1 be the $(0, (i, j))$–subpath of P. By Lemma 5, $a_{i,j} \geq a_{i,j-1} > 0$. Using $w(P) = c(A)$, we obtain

$$\alpha_1(i, j) = w(P_1) = a_{i,j} \quad \text{and} \quad \alpha(i, j) = c(A),$$

hence $s_{i,j} = 1$, contradicting the assumption.

2. Assume the contrary for $e = ((i-1, j), (i, j))$ and let P_1 be the $(0, (i, j))$–subpath of P. (The case $e = ((i+1, j), (i, j))$ is treated similarly.) Using $w(P) = c(A)$, we obtain

$$\alpha_1(i, j) = w(P_1) = a_{i,j} \quad \text{and} \quad \alpha(i, j) = c(A).$$

So by construction of S, $s_{i,j} = 0$ implies $a_{i,j} = 0$.

■

Lemma 7.

$$s_{i,j} = 1 \;\Rightarrow\; \alpha'_1(i, j) = \alpha_1(i, j) - 1 = a'_{i,j} \qquad (i \in [m],\; j \in [n]).$$

Proof. Clearly, $\alpha'_1(i, j) \geq a'_{i,j}$. Assume that $s_{i,j} = 1$ and $\alpha'_1(i, j) > a'_{i,j}$, and let P be a $(0, (i, j))-$path with $w'(P) = \alpha'_1(i, j)$. W.l.o.g. we suppose that (i, j) is the first vertex on P with $s_{i,j} = 1$ and $\alpha'_1(i, j) > a'_{i,j}$.

Case 1: $s_{p,q} = 0$ for every $(p,q) \in P \setminus \{(i,j)\}$.

If e is an arc of P, $w'(e) \neq w(e)$ is possible only if e is the last arc of P. If the last arc is $e = ((i,j-1),(i,j))$,

$$w'(P) = w(P) - 1 \leq \alpha_1(i,j) - 1 = a'_{i,j}.$$

So suppose that the last arc of P is $e = ((i-1,j),(i,j))$. (The case $e = ((i+1,j),(i,j))$ is treated similarly.) Then $w'(P) = w(P)$, and from $w'(P) > a_{i,j} - 1$, $w(P) \leq a_{i,j}$, it follows that

$$w(P) = w'(P) = a_{i,j}.$$

Now for the first $(p,q) \in P$ with $a_{p,q} > 0$ we have $\alpha(p,q) = c(A)$ and $\alpha_1(p,q) = a_{p,q}$, hence $s_{p,q} = 1$. Consequently, $a_{p,q} = 0$ for every $(p,q) \in P \setminus \{(i,j)\}$, thus $a_{i,j} = w(P) = 0$, a contradiction.

Case 2: There is some $(p,q) \in P \setminus \{(i,j)\}$ with $s_{p,q} = 1$.

Let (i_0, j_0) be the last vertex (p,q) on $P \setminus \{(i,j)\}$ with $s_{p,q} = 1$, and denote by P_1 and P_2 the $(0,(i_0,j_0))-$ and the $((i_0,j_0),(i,j))-$subpath of P, respectively. Because $w'(P_1) = a'_{i_0,j_0}$, w.l.o.g. we suppose that P_1 is the path $0,(i_0,0),\dots,(i_0,j_0)$. The only arcs e of P_2 for which $w'(e) \neq w(e)$ is possible are the first arc e_1 and the last arc e_2. By Lemma 5, we have $w'(e_2) \leq w(e_2)$ and $w'(e_1) \leq w(e_1) + 1$. So

$$w'(P) = w'(P_1) + w'(P_2) \leq a_{i_0,j_0} - 1 + w(P_2) + 1 = w(P) \leq a_{i,j},$$

and we conclude $w'(P) = w(P) = a_{i,j}$ and $w'(e_1) = w(e_1) + 1$. Since for the concatenation Q of P with an $((i,j),1)$–path of weight $\alpha_2(i,j)$ we have $w(Q) = c(A)$, we can apply Lemma 6 to e_1, and obtain $e_1 = ((i_0,j_0),(i_0 \pm 1, j_0))$. By symmetry, we only have to consider the case $e_1 = ((i_0,j_0),(i_0+1,j_0))$. By Lemma 6,

$$\alpha_1(i+1,j) = a_{i+1,j} = 0.$$

But now the path

$$0,(i_0+1,0),(i_0+1,1),\dots,(i_0+1,j_0)$$

concatenated with the $((i_0+1,j_0),(i,j))$–subpath of P yields a $(0,(i,j))$–path Q with $w'(Q) = w'(P)$ and $s_{p,q} = 0$ for every $(p,q) \in Q \setminus \{(i,j)\}$, and we are in Case 1.

■

Lemma 8. $c(A') = c(A) - 1$.

Proof. Suppose that $c(A') > c(A)-1$ and let P be a path with $w'(P) = c(A')$. It is easy to see that P contains a vertex (i,j) with $s_{i,j} = 1$: if Q is a path with $w(Q) = c(A)$ then for the first $(i,j) \in Q$ with $a_{i,j} > 0$ we have $s_{i,j} = 1$, so for every path Q with $s_{i,j} = 0$ for every $(i,j) \in Q$ we have $w'(Q) = w(Q) < c(A)$. Let (i,j) be the last vertex on P with $s_{i,j} = 1$ and denote by P_1 and P_2 the $(0,(i,j))-$ and the $((i,j),1)-$subpath of P, respectively. By Lemma 7, $w'(P_1) = a'_{i,j}$. From

$$c(A) \leq w'(P) = w'(P_1) + w'(P_2) = \alpha_1(i,j) - 1 + \alpha'_2(i,j)$$

it follows that $\alpha'_2(i,j) > \alpha_2(i,j)$. Since $w'(e) = w(e)$ for all arcs e of P_2 except possibly for the first one, we obtain

$$\alpha'_2(i,j) = w'(P_2) \leq w(P_2) + 1 \leq \alpha_2(i,j) + 1,$$

hence

$$w(P_2) = \alpha_2(i,j), \quad w(P) = c(A),$$

and by Lemma 6, for the first arc e of P_2, we have $e = ((i,j),(i \pm 1, j))$. If $e = ((i,j),(i+1,j))$, Lemma 6 also yields

$$\alpha_1(i,j) = a_{i+1,j} = 0,$$

and the path

$$0, (i+1,0), (i+1,1), \ldots, (i+1,j)$$

concatenated with the $((i+1,j),1)$–subpath of P yields a path Q of weight $c(A)$ with $s_{p,q} = 0$ for every $(p,q) \in Q$, and this contradiction proves the lemma. ■

Proof of Theorem 1. Another formulation of the statement of the theorem is that the minimal TNMU of a segmentation of A equals $c(A)$. That the TNMU is at least $c(A)$, follows from Lemmas 1 and 2 by duality. By Lemmas 3 and 4, our algorithm yields A–segments $S^{(1)}, \ldots, S^{(t)}$, such that

$$A = \sum_{k=1}^{t} S^{(k)} \tag{19}$$

and by repeated application of Lemma 8,

$$c(A^{(k)}) = c(A) - k \qquad (k \in [t]).$$

So $c(A^{(c(A))}) = 0$, hence $t = c(A)$, and (19) is a segmentation of A with $c(A)$ monitor units. ■

Kamath *et al.* [10] presented an algorithm generating unidirectional schedules, i.e. with the leaves moving only from left to right, and they show that it is TNMU–optimal among all unidirectional schedules. Here we show that the segments determined by our algorithm can be realized by a sequence of leaf positions in which the leaves move only from left to right. Observe that the leaf positions are not uniquely determined by S: in rows with at least one nonzero entry we have to put

$$l_i = \min\{j \ : \ s_{i,j} = 1\}, \quad r_i = \max\{j \ : \ s_{i,j} = 1\}.$$

while in rows that are completely zero we have $r_i = l_i - 1$ but there might be several possible values for l_i. We solve this ambiguity by taking the leftmost of the possible leaf positions. More precisely, for $i \in [m]$ with $s_{i,j} = 0$ for all $j \in [n]$, we put $l_i = \max\{l_{i'}, l_{i''}\}$ and $r_i = l_i - 1$, where

$$i' = \begin{cases} 0 & \text{if } \forall k < i, j \in [n] \ s_{k,j} = 0, \\ \max\{k < i \ : \ \exists j \in [n] \ s_{k,j} = 1\} & \text{otherwise,} \end{cases}$$
$$i'' = \begin{cases} m+1 & \text{if } \forall k > i, j \in [n] \ s_{k,j} = 0, \\ \min\{k > i \ : \ \exists j \in [n] \ s_{k,j} = 1\} & \text{otherwise,} \end{cases}$$

and $l_0 = l_{m+1} = 1$. This is illustrated in Figure 7. The next theorem states

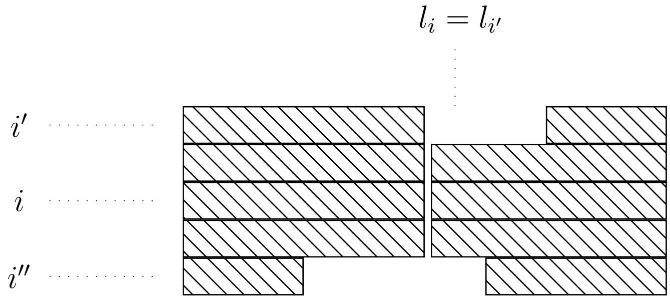

Figure 7: The choice of the leaf positions for zero rows.

that the schedule obtained in this way is unidirectional, so the TNMU must be same as for the schedule from the algorithm of Kamath *et al.*

Theorem 2. *Let $A = \sum_{k=1}^{t} S^{(k)}$ be a segmentation determined by our algorithm and let $l_i^{(k)}$, $r_i^{(k)}$ ($i \in [m]$, $k \in [t]$) be the parameters determined as above. Then*

$$l_i^{(k+1)} \geq l_i^{(k)}, \ r_i^{(k+1)} \geq r_i^{(k)} \quad (i \in [m], \ k \in [t-1]).$$

In order to prove this theorem we need the following lemma.

Lemma 9.

$$j \geq l_i \;\Rightarrow\; \alpha_2'(i,j) \geq \alpha_2(i,j) \quad (i \in [m],\ j \in [n]).$$

Proof. By Lemma 5, the only arcs e with $w'(e) = w(e) - 1$ are the arcs $((i,j),(i,j+1))$ with $j+1 = l_i \leq r_i$. Let (i_0, j_0) be a vertex with $j_0 > l_{i_0}$ and let P be any $((i_0,j_0),1)$–path. We denote the vertices (i,j) with $(i,j),(i,j+1) \in P$ and $l_i = j+1$ by $(i_1,j_1),\dots,(i_t,j_t)$. Then it is easy to check that on the $((i_{k-1},j_{k-1}),(i_k,j_k))$–subpath ($k \in [t]$) there must be an arc $e = ((i,j),(i\pm 1,j))$ with $s_{i,j} = 1$ and $s_{i\pm1,j} = 0$. For these arcs we have $w'(e) = w(e) + 1$, hence $w'(P) \geq w(P)$ and this concludes the proof. ∎

Proof of Theorem 2. We claim that $a_{i,j}^{(k-1)} = 0$ for $j < l_i^{(k)}$ ($i \in [m]$, $k \in [t]$). If $s_{i,j}^{(k)} = 1$ for some $j \in [n]$, obviously

$$l_i^{(k)} = \min\{j \;:\; a_{i,j}^{(k-1)} > 0\}.$$

So suppose that $s_{i,j}^{(k)} = 0$ for every $j \in [n]$. By symmetry we can assume that $l_i^{(k)} = l_{i'}^{(k)}$ where i' is the maximal row index $i' < i$ such that $s_{i',j}^{(k)} = 1$ for some $j \in [n]$. Now assume that $a_{i,j}^{(k-1)} > 0$ for some $j < l_i^{(k)}$. By induction on $i - i'$ we suppose that $a_{i_0,j}^{(k-1)} = 0$ for $i' < i_0 < i$. Now we obtain a $(0,(i',l_{i'}^{(k)}))$–path

$$P = (0,(i,0),(i,1),\dots,(i,j),(i-1,j),\dots,(i',j),(i',j+1),\dots,(i',l_{i'}))$$

with $w_{k-1}(P) = a_{i',l_{i'}}^{(k-1)}$, hence

$$\alpha_1^{(k-1)}(i,j) = a_{i,j}^{(k-1)} \quad \text{and} \quad \alpha^{(k-1)}(i,j) = c(A^{(k-1)}),$$

consequently $s_{i,j}^{(k)} = 1$, and this contradiction establishes the claim.

Case 1: $l_i^{(k+1)} \leq r_i^{(k+1)}$.

In this case $l_i^{(k+1)} < l_i^{(k)}$ leads to the contradiction ($j := l_i^{(k+1)}$)

$$a_{i,j}^{(k)} = a_{i,j}^{(k-1)} = 0, \qquad s_{i,j}^{(k+1)} = 1.$$

So $l_i^{(k+1)} \geq l_i^{(k)}$, and if $r_i^{(k)} = l_i^{(k)} - 1$ also $r_i^{(k+1)} \geq r_i^{(k)}$. Assume that

$$r_i^{(k)} \geq l_i^{(k)} \quad \text{and} \quad r_i^{(k+1)} < r_i^{(k)}.$$

Then for $j = r_i^{(k)}$ we have

$$\alpha_1^{(k)}(i,j) = a_{i,j}^{(k)} > 0 \quad \text{and} \quad \alpha^{(k)}(i,j) = c(A^{(k)}),$$

hence $s_{i,j}^{(k+1)} = 1$, a contradiction to $j > r_i^{(k+1)}$.

Case 2: $l_i^{(k+1)} = r_i^{(k+1)} + 1$.

Case 2.1: $l_i^{(k)} \le r_i^{(k)}$.

For $l_i^{(k)} \le j \le r_j^{(k)}$, we have

$$\alpha_1^{(k)}(i,j) = a_{i,j}^{(k)} \quad \text{and} \quad \alpha^{(k)}(i,j) = c(A^{(k)}).$$

Together with $s_{i,j}^{(k+1)} = 0$ this implies

$$a_{i,j}^{(k)} = 0 \quad \text{and} \quad \alpha_2^{(k)}(i,j) = c(A^{(k)}).$$

Now by Lemma 9 applied to $A^{(k)}$ and $A(k+1)$, and using $c(A^{(k+1)}) = c(A^{(k)}) - 1$, we obtain

$$l_i^{(k+1)} > r_i^{(k)} \ge l_i^{(k)}, \quad r_i^{(k+1)} \ge l_i^{(k+1)} - 1 \ge r_i^{(k)}.$$

Case 2.2: $l_i^{(k)} = r_i^{(k)} + 1$.

We suppose that $l_i^{(k)} = l_{i'}^{(k)}$ where i' is the maximal row index $i' < i$ with $l_{i'}^{(k)} \le r_{i'}^{(k)}$. Then $l_{i''}^{(k)} = l_{i'}^{(k)}$ for every $i'' \in [i', i]$. We show by induction on $i'' - i'$ that

$$l_{i''}^{(k+1)} \ge l_{i''}^{(k)} \quad \text{for } i'' \in [i', i].$$

For $i'' = i'$ this is clear by Case 2.1, so let $i'' > i'$. If $l_{i''}^{(k+1)} \le r_{i''}^{(k+1)}$ we are in Case 1, and if $l_{i''}^{(k+1)} = r_{i''}^{(k+1)} + 1$, we obtain by induction and ICC,

$$l_{i''}^{(k+1)} = r_{i''}^{(k+1)} + 1 \ge l_{i''-1}^{(k+1)} \ge l_{i''-1}^{(k)} = l_{i''}^{(k)}.$$

■

5 Minimizing the number of segments

The problem of minimizing the number of segments is NP–hard even for a single row intensity matrix [8]. So it is natural to look for a heuristic approach that yields segmentations with a small number of segments in a reasonable time even if optimality is not always reached. In [7] we used a greedy strategy in order to find a segmentation with minimal TNMU and a small NS for MLCs with ICC but neglecting the TGC. This method can be modified to respect the TGC. In order to characterize the maximal coefficient u for which there is an A–segment S, such that uS can be a term in a segmentation of A with minimal TNMU, we need a kind of converse to Lemma 3.

Lemma 10. *Let $A = \sum_{k=1}^{t} u_k S^{(k)}$ be a segmentation of A (i.e. the $S^{(k)}$ are A–segments), and put $A^{(0)} = A$ and $A^{(k)} = A - \sum_{k'=1}^{k} u_{k'} S^{(k')}$ for $k \in [t]$. Then, for every $k \in [t]$ we have*

- $s_{i,j}^{(k)} = 1$ *and* $s_{i+1,j}^{(k)} = 0 \;\Rightarrow\; a_{i,j}^{(k-1)} \geq a_{i+1,j}^{(k-1)} + u \quad (i \in [m-1],\ j \in [n])$,
- $s_{i,j}^{(k)} = 1$ *and* $s_{i-1,j}^{(k)} = 0 \;\Rightarrow\; a_{i,j}^{(k-1)} \geq a_{i-1,j}^{(k-1)} + u \quad (i \in [2,m],\ j \in [n])$.

Informally speaking, if we consider the sequence of matrices starting with A and subtracting one by one the $S^{(k)}$ taking $S^{(k)}$ exactly u_k times, the lemma claims that in each step we subtract an A'–segment, where A' is the resulting matrix after the previous step.

Proof. Assume the contrary and let k be the first index where one of the two claims fails to be true. By symmetry, we assume

$$s_{i,j}^{(k)} = 1, \quad s_{i+1,j}^{(k)} = 0, \quad a_{i,j}^{(k-1)} < a_{i+1,j}^{(k-1)} + u.$$

Since $S^{(k)}$ is an A–segment, the TGC implies $a_{i,j} > a_{i+1,j}$. From our assumption we obtain

$$a_{i,j}^{(k)} < a_{i+1,j}^{(k)},$$

hence

$$s_{i,j}^{(k')} = 0 \quad \text{and} \quad s_{i+1,j}^{(k')} = 1$$

for some $k' > k$, contradicting the assumption that $S^{(k')}$ is an A–segment. ■

We call a pair (u, S) of a positive integer u and an A–segment S an *admissible segmentation pair*, if

- $A - uS$ is nonnegative,
- $s_{i,j} = 1$ and $s_{i+1,j} = 0 \quad \Rightarrow \quad a_{i,j} \geq a_{i+1,j} + u \qquad (i \in [m-1],\ j \in [n])$,
- $s_{i,j} = 1$ and $s_{i-1,j} = 0 \quad \Rightarrow \quad a_{i,j} \geq a_{i-1,j} + u \qquad (i \in [2,m],\ j \in [n])$,
- $c(A - uS) = c(A) - u$.

Now we proceed exactly as in [7]: we find an admissible segmentation pair (u, S) with maximal u and continue with $A - uS$ until we reach the zero matrix. How the pair (u, S) is determined is described here only informally, see [7] for the details. First we compute numbers $u(i, l, r)$ for $i \in [m]$, $l \in [n+1]$, $r \in [0, n]$ which are upper bounds for u in an admissible segmentation pair (u, S) with $l_i = l$ and $r_i = r$. These upper bounds come from the following lemma (see [7] for the proof).

Lemma 11. *Let (u, S) be an admissible segmentation pair with $l_i = l$ and $r_i = r$. Then $u \leq v_2(i, l, r)$ where*

$$v_2(i, l, l-1) = c(A) - \alpha_1(i, l-1) - \max\{0, d_{i,l}\} - \alpha_2(i, l),$$

and if $r \geq l$ then $v_2(i, l, r) = \min\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where

$$\begin{aligned}
\gamma_1 &= c(A) - \alpha_1(i, l-1) - \alpha_2(i, l), \\
\gamma_2 &= c(A) - \alpha_1(i, l-1) - \sum_{j=l+1}^{r} \max\{0, d_{i,j}\} - \alpha_2(i, r+1), \\
\gamma_3 &= c(A) - \alpha_1(i, l-1) - d_{i,l} - \sum_{j=l+1}^{r} \max\{0, d_{i,j}\} - d_{i,r+1} - \alpha_2(i, r+1), \\
\gamma_4 &= \frac{1}{2}\left(c(A) - \alpha_1(i, l-1) - \sum_{j=l+1}^{r} \max\{0, d_{i,j}\} - d_{i,r+1} - \alpha_2(i, r+1)\right).
\end{aligned}$$

Now we start with some upper bound for u, search for an admissible S, and continue with $u - 1$ if our search is not successful. The search for S is done by a branch–and–bound method: we start in row 1, and assuming we have already constructed i rows of the segment we try to add row $i + 1$ by checking all possible pairs (l_{i+1}, r_{i+1}), i.e. all pairs with

$$\begin{aligned}
l_{i+1} \leq r_i + 1, &\quad r_{i+1} \geq l_i - 1, \\
u(i+1, l_{i+1}, r_{i+1}) &\geq u, \\
a_{i+1,j} \geq a_{i,j} + u &\qquad \text{if } l_{i+1} \leq j < l_i \text{ or } r_i < j \leq r_{i+1}, \text{ and} \\
a_i \geq a_{i+1} + u &\qquad \text{if } l_i \leq j < l_{i+1} \text{ or } r_{i+1} < j \leq r_i.
\end{aligned}$$

Example 2. For a benchmark matrix from [11] our algorithm yields the segmentation

$$\begin{pmatrix} 4&5&0&1&4&5 \\ 2&4&1&3&1&4 \\ 2&3&2&1&2&4 \\ 5&3&3&2&5&3 \end{pmatrix} = 2\begin{pmatrix} 0&0&0&0&1&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&1 \end{pmatrix} + 2\begin{pmatrix} 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 1&1&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0&1&0&0&0&0 \\ 0&1&1&1&0&0 \\ 0&1&1&0&0&0 \\ 1&1&1&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&1&1&1 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&0&1&1&1&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&1&1&1&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&1&1 \end{pmatrix}.$$

6 Test results

We implemented our algorithm in C++ on a 2 GHz workstation and computed segmentations for 15×15–matrices where the entries are chosen randomly from $\{0, 1, \dots, L\}$ (uniformly distributed) ($L = 3, 4, \dots, 16$). The results are shown in Table 1 where for each value of L the entries are averaged over 1000 matrices. For comparison, in Table 2 we include the results without TGC from [7]. Concerning the computation time, the algorithm seems to be practicable: the 1000 segmentations for $L = 16$ were computed in about 20 minutes, and the maximal time for one single matrix was 20 seconds.

L	TNMU	NS
3	16.6	15.5
4	21.2	18.0
5	25.8	20.5
6	30.3	22.6
7	34.9	24.3
8	39.2	25.7
9	43.6	27.0
10	48.2	28.3
11	52.9	29.5
12	57.2	30.5
13	61.7	31.4
14	66.0	32.2
15	70.6	33.1
16	74.8	33.9

Table 1: Test results with TGC for 15×15–matrices with random entries from $\{0, 1, \ldots, L\}$.

L	TNMU	NS
3	15.4	12.6
4	19.5	14.5
5	23.6	16.0
6	27.6	17.2
7	31.7	18.2
8	35.7	19.1
9	39.8	19.9
10	43.8	20.7
11	47.7	21.3
12	51.8	21.9
13	55.7	22.5
14	59.8	23.0
15	63.8	23.5
16	67.7	24.0

Table 2: Test results without TGC for 15×15–matrices with random entries from $\{0, 1, \ldots, L\}$.

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