A short note on hit–and–miss hyperspaces

Dedicated to Professor Som Naimpally
on the occasion of his
70th birthday

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Abstract. Based on some set-theoretical observations, compactness results are given for general hit-and-miss hyperspaces. Compactness here is sometimes viewed splitting into "κ-Lindelöfness" and "κ-compactness" for cardinals κ. To focus only hit-and-miss structures, could look quite old-fashioned, but some importance, at least for the techniques, is given by a recent result, [8], of Som Naimpally, to who this article is hearty dedicated.

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1. Introduction

Let (X, τ) be a topological space. By $\mathfrak{P}(X)$, $\mathfrak{P}_0(X)$, $Cl(X)$ and $K(X)$ respectively we denote the power set, the power set without the empty set $\varnothing$, the family of all closed subsets and the set of all compact subsets of X. For $B \in \mathfrak{P}(X)$ and $\mathfrak{A} \subseteq \mathfrak{P}(X)$ we define $B^{-\mathfrak{A}} := \{A \in \mathfrak{A} | A \cap B \neq \varnothing\}$ (hit–set) and $B^{+\mathfrak{A}} := \{A \in \mathfrak{A} | A \cap B = \varnothing\}$ (miss–set). Specializing $\mathfrak{A} := Cl(X)$, we get the usual symbols B^-, B^+. By $\tau_{l,\mathfrak{A}}$ we denote the topology for $\mathfrak{A}$, generated by the subbase of all $G^{-\mathfrak{A}}, G \in \tau$. Now consider $\varnothing \neq \alpha \subseteq \mathfrak{P}(X)$; by $\tau_{\alpha,\mathfrak{A}}$ we denote the topology for $\mathfrak{A}$ which is generated from the subbase of all $B^{+\mathfrak{A}}, B \in \alpha$ and $G^{-\mathfrak{A}}, G \in \tau$. Of course, for every possible α we have $\tau_{l,\mathfrak{A}} \subseteq \tau_{\alpha,\mathfrak{A}}$; for $\alpha = Cl(X)$ we get the Vietoris topology and for $\alpha = K(X)$ we get the Fell topology for $\mathfrak{A}$. If $\alpha = \Delta \subseteq Cl(X)$, $\tau_{\alpha,\mathfrak{A}}$ is called Δ–topology by Beer and Tamaki [2], and was first introduced by Poppe [10].

By $\mathfrak{F}(X)$ and $\mathfrak{F}_0(X)$ we denote the set of all filters and ultrafilters, respectively, on a set X (a filter is not allowed to contain the empty set $\varnothing$); the symbol $\mathfrak{F}(\varphi)$ (resp. $\mathfrak{F}_0(\varphi)$) means the set of all filters (resp. ultrafilters) which contain a given filter φ; $\dot{x}$ is the filter generated by a singleton $\{x\}, x \in X$. The symbol q_τ denotes the convergence structure induced by a topology τ, i.e. $q_\tau := \{(\varphi, x) \in \mathfrak{F}(X) \times X | \varphi \supseteq \dot{x} \cap \tau\}$, so q_τ is a relation between filters and points of a set X.

If X is a set, $\tau, \mathfrak{A}$ are subsets of $\mathfrak{P}(X)$, then we call $\mathfrak{A}$ *weakly complementary w.r.t.* τ, iff for every subset $\sigma \subseteq \tau$ there exist a subset $\mathfrak{B} \subseteq \mathfrak{A}$, s.t. $\bigcup_{B \in \mathfrak{B}} B = X \setminus \bigcup_{S \in \sigma} S$.

Lemma 1.1. *Let X be a set, $\tau, \mathfrak{A} \subseteq \mathfrak{P}(X)$ and $K \subseteq X$. Then holds*

$$\bigcup_{i \in I} G_i \supseteq K \implies \bigcup_{i \in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$$

for every collection $G_i, i \in I, G_i \in \tau$.
If $\mathfrak{A}$ is weakly complementary w.r.t. τ, then for every collection $G_i, i \in I, G_i \in \tau$ the implication

$$\bigcup_{i \in I} G_i \supseteq K \impliedby \bigcup_{i \in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$$

holds, too.

Proof. Let $\bigcup_{i\in I} G_i \supseteq K$. $A \in K^{-\mathfrak{A}} \Rightarrow A \cap K \neq \varnothing \Rightarrow \varnothing \neq A \cap \bigcup_{i\in I} G_i \Rightarrow \exists i_0 \in I$: $A \cap G_{i_0} \neq \varnothing \Rightarrow A \in G_{i_0}^{-\mathfrak{A}} \Rightarrow A \in \bigcup_{i\in I} G_i^{-\mathfrak{A}}$.
Conversely, let $\mathfrak{A}$ be weakly complementary w.r.t. τ and $\bigcup_{i\in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$. Assume $\bigcup_{i\in I} G_i \not\supseteq K$. Then $X\setminus\bigcup_{i\in I} G_i \supseteq K\setminus\bigcup_{i\in I} G_i \neq \varnothing$ holds, so there is an $A \in \mathfrak{A}, A \subseteq X \setminus \bigcup_{i\in I} G_i$ with $A \cap K \setminus \bigcup_{i\in I} G_i \neq \varnothing$. Thus $A \in K^{-\mathfrak{A}}$, implying $A \in \bigcup_{i\in I} G_i^{-\mathfrak{A}}$. This yields $\exists i_0 \in I : A \cap G_{i_0} \neq \varnothing$ in contradiction to the construction of A. □

Corollary 1.2. *Let X be a set, $\tau, \mathfrak{A} \subseteq \mathfrak{P}(X)$ and $K \subseteq X$. Then holds*

$$\bigcup_{i\in I} G_i \supseteq K \iff \bigcup_{i\in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}} \tag{1.1}$$

for every collection $G_i, i \in I, G_i \in \tau$ if and only if $\mathfrak{A}$ is weakly complementary w.r.t. τ.

Proof. We only have to show, that $\mathfrak{A}$ is weakly complementary w.r.t. τ, if (1.1) holds. Assume, $\mathfrak{A}$ is not weakly complementary w.r.t. τ. Then there must be a collection $\{G_i | i \in I\} \subseteq \tau$, such that $\bigcup\{A | A \in \mathfrak{P}(X \setminus \bigcup_{i\in I} G_i) \cap \mathfrak{A}\} \not\supseteq X \setminus \bigcup_{i\in I} G_i$. Now, we chose $K := \left(X \setminus \bigcup_{i\in I} G_i\right) \setminus \bigcup\{A | A \in \mathfrak{P}(X \setminus \bigcup_{i\in I} G_i) \cap \mathfrak{A}\} \neq \varnothing$. Then no element of $\mathfrak{A}$, which meets K, can be contained in $X \setminus \bigcup_{i\in I} G_i$, i.e. every element of $K^{-\mathfrak{A}}$ meets $\bigcup_{i\in I} G_i$, too. So, it must meet a $G_{i_0}, i_0 \in I$ and consequently it is contained in $\bigcup_{i\in I} G_i^{-\mathfrak{A}}$. But, by construction, the collection $\{G_i | i \in I\}$ doesn't cover K, so (1.1) would fail. □

Obviously, if for every collection $\{G_i | i \in I\} \subseteq \tau$ the complement $X \setminus \bigcup_{i\in I} G_i$ itself belongs to $\mathfrak{A}$, or if all singletons $\{x\}, x \in X$ are elements of $\mathfrak{A}$, then $\mathfrak{A}$ is weakly complementary w.r.t. τ. So, if τ is a topology on X, $Cl(X)$ and $K(X)$ are weakly complementary w.r.t. τ.

Corollary 1.3. *Let (X, τ) be a topological space, $K \subseteq X$ and $\forall i \in I : G_i \in \tau$. Then holds*

$$\bigcup_{i\in I} G_i \supseteq K \iff \bigcup_{i\in I} G_i^{-} \supseteq K^{-}$$

We have yet another easy, but useful set-theoretical lemma:

Lemma 1.4. *Let X be a set, $\mathfrak{A} \subseteq \mathfrak{P}(X)$ and $\varphi \in \mathfrak{F}(X)$. Assume, $\mathfrak{A}$ is closed under finite unions of its elements. Then holds*

$$\varphi \cap \mathfrak{A} \neq \varnothing \iff \forall \psi \in \mathfrak{F}_0(\varphi) : \psi \cap \mathfrak{A} \neq \varnothing \ ,$$

i.e. a filter contains an $\mathfrak{A}$–set, iff each refining ultrafilter contains an $\mathfrak{A}$–set.

Proof. Suppose $\forall \psi \in \mathfrak{F}_0(\varphi) : \exists A_\psi \in \mathfrak{A} : A_\psi \in \psi$. Now, assume $\varphi \cap \mathfrak{A} = \varnothing$. From this automatically follows $X \notin \mathfrak{A}$.
Consider $\mathfrak{B} := \{X \setminus A | \ A \in \mathfrak{A}\}$. Because of the closedness of $\mathfrak{A}$ under finite unions, $\mathfrak{B}$ is closed under finite intersection of its elements, and $\varnothing \notin \mathfrak{B}$, because $X \notin \mathfrak{A}$. For any $F \in \varphi, B \in \mathfrak{B}$ we have $F \cap B \neq \varnothing$, because $F \cap B = \varnothing$ would imply $F \subseteq X \setminus B \in \mathfrak{A}$ and therefore $\varphi \cap \mathfrak{A} \neq \varnothing$. So, $\varphi \cup \mathfrak{B}$ is a subbase of a filter and consequently, there exists an ultrafilter ψ, containing $\varphi \cup \mathfrak{B}$, therefore containing φ and the complement of every $\mathfrak{A}$–set - in contradiction to $\forall \psi \in \mathfrak{F}_0(\varphi) : \psi \cap \mathfrak{A} \neq \varnothing$.
The other direction of the statement of the lemma is obvious. □

Definition 1.5. *Let κ be a cardinal. Then a topological space (X, τ) is called κ-compact, iff every open cover of X with cardinality at most κ admits a finite subcover.*
(X, τ) is called κ-Lindelöf, iff every open cover of X admits a subcover of cardinality at most κ.

A filter is called κ-generated, iff it has a base of cardinality at most κ. A filter φ is called *κ-completable*, iff every subset $\mathfrak{B} \subseteq \varphi$ with $card(\mathfrak{B})$ at most κ fulfills $\bigcap_{B\in\mathfrak{B}} B \neq \varnothing$. It is called *$\kappa$-complete*, iff $\bigcap_{B\in\mathfrak{B}} B \in \varphi$ holds under this condition.

Proposition 1.6. *A topological space (X, τ) is κ-compact, if and only if every κ-generated filter on X has a convergent refining ultrafilter.*

Proof. Let (X, τ) be κ-compact and φ a filter on X with a base $\mathfrak{B}$ of cardinality at most κ. Assume, all refining ultrafilters of φ would fail to converge in X. Then for each element $x \in X$, all refining ultrafilters of φ contain the complement of an open neighbourhood of x. But the set of complements of open neighbourhoods of a point x is closed w.r.t. finite unions, thus by Lemma 1.4, φ contains the complement of an open neighbourhood of x. So, for each $x \in X$ there must exist $O_x \in \tau \cap \dot{x}$ and $B_x \in \mathfrak{B}$, s.t. $B_x \subseteq X \setminus O_x$, implying $\overline{B_x} \subseteq X \setminus O_x$ and thus $X \setminus \overline{B_x} \supseteq O_x$. Now, for each $B \in \mathfrak{B}$ we define $O_B := X \setminus \overline{B}$ and find, that $\{O_B |\ B \in \mathfrak{B}\}$ is an open cover of X, because of the preceeding facts. So, there must exist a finite subcover $O_{B_1} \cup \cdots \cup O_{B_n} = X$, implying $\bigcup_{i=1}^{n}(X \setminus \overline{B_i}) = X$, just meaning $\bigcap_{i=1}^{n} \overline{B_i} = \varnothing$, which is impossible, because all B_i belong to the filter φ. So, the assumption must be false; there must exist convergent refining ultrafilters of φ.
Otherwise, let all κ-generated filters on X have a convergent refining ultrafilter. Assume, there would exist an open cover $\mathfrak{C} := \{O_i \in \tau |\ i \in I\}, \bigcup_{i\in I} O_i = X, card(I) \leq \kappa$ such that all finite subcollections fail to cover X (implying κ to be infinite). But the set of all finite subcollections of the infinite collection $\mathfrak{C}$ of cardinality at most κ has cardinality at most κ, too. So, $\mathfrak{B} := \{X \setminus \bigcup_{k=1}^{n} O_{i_k} |\ n \in I\!N, i_k \in I\}$ is a filterbasis of cardinality at most κ, thus there must exist an ultrafilter $\varphi \supseteq \mathfrak{B}$, which converges in X - leading to the usual contradiction, because every $x \in X$ is contained in an open $O_x \in \mathfrak{C}$ and $X \setminus O_x$ belongs to $\mathfrak{B} \subseteq \varphi$. $\square$

Analogously we get a characterization of κ-Lindelöf-spaces.

Proposition 1.7. *If (X, τ) is κ-Lindelöf, then every κ-completable filter on X has a convergent refining ultrafilter.*
If κ is an infinite cardinal and every κ-complete filter on a topological space (X, τ) has a convergent refining ultrafilter, then (X, τ) is κ-Lindelöf.

Of course, every κ-complete filter is κ-completable, so we may say, that a topological space (X, τ) is κ-Lindelöf, if and only if each κ-complete filter on X has a convergent refinement.

2. Compactness Properties for Hyperspaces

Lemma 2.1. *Let κ be a cardinal, (X, τ) a topological space and let $\mathfrak{A} \subseteq \mathfrak{P}(X)$ be weakly complementary w.r.t. τ. If $\mathfrak{A}_0 := \mathfrak{A} \setminus \{\varnothing\}$ is κ-Lindelöf (resp. κ-compact) in $\tau_{l,\mathfrak{A}_0}$, then (X, τ) is κ-Lindelöf (resp. κ-compact).*

Proof. If $\mathfrak{A}$ is weakly complementary w.r.t. τ, then $\mathfrak{A}_0$ is, too. So, Corollary 1.2 is applicable. Let $\{G_i | i \in I\}$ be an open cover (resp. an open cover with cardinality at most κ) of X. By Corollary 1.2, then $\{G_i^{-\mathfrak{A}_0} | i \in I\}$ is an open cover of $X^{-\mathfrak{A}_0} = \mathfrak{A}_0$ (resp. of card. at most κ), so there exists a subset $J \subseteq I$ of cardinality at most κ (resp. a finite subset J), s.t. $\bigcup_{j\in J} G_j^{-\mathfrak{A}_0} \supseteq \mathfrak{A}_0 = X^{-\mathfrak{A}_0}$, implying $\bigcup_{j\in J} G_j \supseteq X$ by Corollary 1.2. $\square$

Of course, the assumed topology $\tau_{l,\mathfrak{A}_0}$ is not really hit-and-miss, because the miss-sets are missed. But every proper hit-and-miss topology would be stronger and therefore it would enforce the desired properties for (X, τ) as well.

Lemma 2.2. *Let (X, τ) be a κ-compact (resp. κ-Lindelöf) topological space and assume $Cl(X) \subseteq \mathfrak{A} \subseteq \mathfrak{P}(X)$. Then $\mathfrak{A}_0 := \mathfrak{A} \setminus \{\varnothing\}$ is κ-compact (resp. κ-Lindelöf) in $\tau_{l,\mathfrak{A}_0}$.*

Proof. Let $\hat{\varphi}$ be a κ-generated (resp. κ-complete) filter on $\mathfrak{A}_0$. Then, for an arbitrary $h \in \mathcal{A} := \{g \in X^{\mathfrak{P}_0(X)} |\ \forall M \in \mathfrak{P}_0(X) : g(M) \in M\}$ the image $h(\hat{\varphi})$ is a κ-generated (resp. κ-complete) filter on X and consequently it has a τ-convergent refining ultrafilter ψ_h. Furthermore, there must exist an ultrafilter $\hat{\psi} \supseteq \hat{\varphi}$, s.t. $h(\hat{\psi}) = \psi_h$. So, the set

$$A := \{a \in X |\ \exists f \in \mathcal{A} : (f(\hat{\psi}), a) \in q_\tau\}$$

is not empty and consequently the closure $\overline{A}$ belongs to $\mathfrak{A}_0$. Now, for any $O \in \tau$ with $\overline{A} \in O^{-\mathfrak{A}_0}$ ($\Leftrightarrow \overline{A} \cap O \neq \varnothing$) we get $A \cap O \neq \varnothing$ (because of the closure-properties). Now, the assumption $O^{-\mathfrak{A}_0} \notin \hat{\psi}$ would imply $O^{+\mathfrak{A}_0} \in \hat{\psi}$, yielding $\forall f \in \mathcal{A} : X \setminus O \in f(\hat{\psi})$, thus $\forall f \in \mathcal{A} : \forall b \in A \cap O : (f(\hat{\psi}), b) \notin q_\tau$ - in contradiction to the construction of A. Thus, $O \in \tau, \overline{A} \in O^{-\mathfrak{A}_0}$ always imply $O^{-\mathfrak{A}_0} \in \hat{\psi}$ and consequently $\hat{\psi}$ $\tau_{l,\mathfrak{A}_0}$-converges to $\overline{A}$. □

Definition 2.3. *Let (X, τ) be a topological space. A subset $A \subseteq X$ is called* weakly relatively complete *in X, iff*

$$\forall \varphi \in \mathfrak{F}(A) \cap q_\tau^{-1}(X) : \mathfrak{F}(\varphi) \cap q_\tau^{-1}(A) \neq \varnothing\ ,$$

i.e. every filter φ on A, which converges in X, has a refinement, converging in A.

Proposition 2.4. *Let (X, τ) be a topological space and $A \subseteq X$. Then holds:*

(a) *A is weakly relatively complete in X, iff $\mathfrak{F}_0(A) \cap q_\tau^{-1}(X) = \mathfrak{F}_0(A) \cap q_\tau^{-1}(A)$, i.e. every ultrafilter on A, which converges in X, converges in A.*
(b) *If A is closed in X, then A is weakly relatively complete in X.*
(c) *If A is compact, then A is weakly relatively complete in X.*
(d) *If (X, τ) is compact and A is weakly relatively complete in X, then A is compact.*
(e) *If (X, τ) is Hausdorff, then every weakly relatively complete subset $A \subseteq X$ is closed in (X, τ).*
(f) *A is compact iff A is weakly relatively complete and relatively compact.*
(g) *If (X, τ) is κ-compact and A is weakly relatively complete in (X, τ), then A is κ-compact.*
(h) *If (X, τ) is κ-Lindelöf and A is weakly relatively complete in (X, τ), then A is κ-Lindelöf.*
(i) *Weak relative completeness is transitive, i.e. for all $A \subseteq B \subseteq X$ with B weakly relatively complete in (X, τ) and A weakly relatively complete in $(B, \tau_{|B})$, the subset A is weakly relatively complete in (X, τ).*

There is also a useful description by coverings for weak relative completeness.

Lemma 2.5. *Let (X, τ) be a topological space and $A \subseteq X$. Then the following are equivalent:*

(1) *A is weakly relatively complete in X.*
(2) *For every open cover $\mathfrak{A}$ of A and every element x of X, there is an open neighbourhood $U_{x,\mathfrak{A}}$ of x, s.t. $U_{x,\mathfrak{A}} \cap A$ is covered by finitely many members of $\mathfrak{A}$.*
(3) *For every open cover $\mathfrak{A}$ of A there exists an open cover $\mathfrak{A}' \supseteq \mathfrak{A}$ of X, such that the intersection of every member of $\mathfrak{A}'$ with A can be covered by finitely many members of $\mathfrak{A}$, i.e. $\forall O \in \mathfrak{A}' : \exists n \in I\!N, P_1, ..., P_n \in \mathfrak{A} : \bigcup_{i=1}^n P_i \supseteq O \cap A$ holds.*

Proof. (1)⇒(2): Let $\mathfrak{A} \subseteq \tau$ with $\bigcup_{P \in \mathfrak{A}} P \supseteq A$ be given. For every $x \in A$ we can chose a single member of $\mathfrak{A}$ as open neighbourhood, whose intersection with A is covered by itself. So, assume

$$\exists x \in X \setminus A : \forall U_x \in \underline{U}(x) \cap \tau : \forall n \in I\!N, P_1, ..., P_n \in \mathfrak{A} : U_x \cap A \not\subseteq \bigcup_{i=1}^n P_i \tag{2.2}$$

Then $\mathfrak{B} := \{(U \cap A) \setminus \bigcup_{i=1}^{n} P_i |\ U \in \underline{U}(x) \cap \tau, n \in I\!N, P_i \in \mathfrak{A}\}$ would be closed under finite intersections and thus there would exist an ultrafilter φ on A with $\varphi \supseteq \mathfrak{B}$. By construction $\varphi \to x$ must hold for this ultrafilter, and now by the weak relative completeness of A it follows $\exists a \in A : \underline{U}(a) \subseteq \varphi$. But $\mathfrak{A}$ is an open cover of A, so there is an open set $P \in \mathfrak{A}$ with $a \in P$, implying $P \in \varphi$ – in contradiction to the construction of φ. Thus (2.2) is false and we have

$$\forall x \in X \setminus A : \exists U_x \in \underline{U}(x) \cap \tau : \exists n \in I\!N, P_1, ..., P_n \in \mathfrak{A} : U_x \cap A \subseteq \bigcup_{i=1}^{n} P_i$$

(2)⇒(3): Note, that (3) is fulfilled with $\mathfrak{A}' := \{U_x |\ x \in X \setminus A\} \cup \mathfrak{A}$.
(3)⇒(1): For a given ultrafilter φ on A with $\varphi \to x \in X$ assume $\varphi \notin q_\tau^{-1}(A)$. Then $\forall a \in A : \exists U_a \in \underline{U}(a) \cap \tau : U_a^c = X \setminus U_a \in \varphi$. With these neighbourhoods define $\mathfrak{A} := \{U_a |\ a \in A\}$, which is an open cover of A. By (2) there is an open cover $\mathfrak{A}' \supseteq \mathfrak{A}$ of X such that $\forall O \in \mathfrak{A}' : \exists n \in I\!N, P_1, ..., P_n \in \mathfrak{A} : \bigcup_{i=1}^{n} P_i \supseteq O \cap A$ holds. Now, $\varphi \to x$ implies $\exists O \in \mathfrak{A}' : O \in \varphi$ (especially $A \cap O \neq \varnothing$ follows), and then we have $\exists n \in I\!N, P_1, ..., P_n \in \mathfrak{A} : O \cap A \subseteq \bigcup_{i=1}^{n} P_i$, implying $\exists j \in \{1, ..., n\} : P_j \in \varphi$ – in contradiction to the construction of $\mathfrak{A}$. So, the assumption $\varphi \notin q_\tau^{-1}(A)$ must be false, showing, that every ultrafilter on A, which converges in X, converges in A. □

Theorem 2.6. *Let (X, τ) be a topological space, and let $\alpha \subseteq \mathfrak{P}(X)$ consist of weakly relatively complete subsets of X. Then holds for any $\mathfrak{A}$ with $Cl(X) \subseteq \mathfrak{A} \subseteq \mathfrak{P}(X)$: $(\mathfrak{A}_0, \tau_\alpha)$ is compact $\iff$ (X, τ) is compact.*

Proof. According to Lemma 2.1 we need only to show that $(\mathfrak{A}_0, \tau_\alpha)$ is compact, if (X, τ) is compact. So, assuming (X, τ) to be compact, by Proposition 2.4 every weakly relatively complete subset of X is compact, and we have $\alpha \subseteq K(X)$. Now we will use Alexander's Lemma: let $\underline{U}$ be a cover of $\mathfrak{A}_0$, consisting of subbase elements $K_i^{+\mathfrak{A}_0}, G_j^{-\mathfrak{A}_0}$ with K_i compact and G_j open.
$A := X \setminus (\bigcup\{G | G^{-\mathfrak{A}_0} \in \underline{U}\})$ is closed.
By construction, $A \notin G^{-\mathfrak{A}_0}$ for any $G^{-\mathfrak{A}_0} \in \underline{U}$, so for $A \neq \varnothing$ there must exist some $K_0^{+\mathfrak{A}_0} \in \underline{U}$ with $A \in K_0^{+\mathfrak{A}_0}$, yielding that $K_0 \subseteq \bigcup\{G | G^{-\mathfrak{A}_0} \in \underline{U}\}$; K_0 compact $\Rightarrow$ $\exists G_1, ..., G_n \in \underline{U}$ with $K_0 \subseteq \bigcup_{k=1}^{n} G_k$, but then $\{K_0^{+\mathfrak{A}_0}\} \cup \{G_1^{-\mathfrak{A}_0}, ..., G_n^{-\mathfrak{A}_0}\}$ is a cover of $\mathfrak{A}_0$.
If $A = \varnothing$, then $\bigcup\{G_i | G_i^{-\mathfrak{A}_0} \in \underline{U}\} = X$, so from the compactness of X the existence of some $G_1^{-\mathfrak{A}_0}, ..., G_n^{-\mathfrak{A}_0} \in \underline{U}$ with $X = \bigcup_{k=1}^{n} G_k$ follows. By Lemma 1.1 then $\bigcup_{k=1}^{n} G_k^{-\mathfrak{A}_0} = \mathfrak{A}_0$ holds. □

Many known theorems of compactness w.r.t. the Fell– or the Vietoris–topology follow immediately from the above result.

Lemma 2.7. *Let (X, τ) be a topological space, $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ with $Cl(X) \subseteq \mathfrak{A}$ and $\alpha \subseteq Cl(X)$. If $R \subseteq X$ is relatively compact in X, then $\mathfrak{P}_0(R) \cap \mathfrak{A}$ is relatively compact in $(\mathfrak{A}, \tau_\alpha)$.*

Proof. Let $\mathfrak{B} := \{O_i^{-\mathfrak{A}} |\ i \in I, O_i \in \tau\} \cup \{C_j^{+\mathfrak{A}} |\ j \in J, C_j \in \alpha\}$ be an open cover of $\mathfrak{A}$ by subbase elements of τ_α. Let $O := \bigcup_{i \in I} O_i$.
If $O = X$, then there exists finitely many $i_1, ..., i_n \in I$ with $\bigcup_{k=1}^{n} O_{i_k} \supseteq R$, because R is relatively compact, and thus $\bigcup_{k=1}^{n} O_{i_k}^{-\mathfrak{A}} \supseteq R^{-\mathfrak{A}} \supseteq \mathfrak{P}_0(R) \cap \mathfrak{A}$, by Lemma 1.1.
If $O \neq X$, then $X \setminus O$ is nonempty and closed, but not covered by the $O_i^{-\mathfrak{A}}$ from $\mathfrak{B}$. Thus, there must exist a $j_0 \in J$ with $X \setminus O \in C_{j_0}^{+\mathfrak{A}}$, implying $C_{j_0} \subseteq O$. Now, we have $\mathfrak{P}_0(R) \cap \mathfrak{A} = (\mathfrak{P}_0(R) \cap C_{j_0}^{+\mathfrak{A}}) \cup (\mathfrak{P}_0(R) \cap C_{j_0}^{-\mathfrak{A}})$, and, of course, $\mathfrak{P}_0(R) \cap C_{j_0}^{+\mathfrak{A}}$ is covered just by $C_{j_0}^{+\mathfrak{A}} \in \mathfrak{B}$. So, we have to find a finite subcover for $(\mathfrak{P}_0(R) \cap C_{j_0}^{-\mathfrak{A}})$,

if this is not empty. Observe, that $R \cap C_{j_0}$ is relatively compact in X, because it is a subset of R. Furthermore, $\{O_i|\ i \in I\} \cup \{X \setminus C_{j_0}\}$ is an open cover of X. Thus we find again finitely many $i_1, .., i_n \in I$, s.t. $\bigcup_{k=1}^{n} O_{i_k} \supseteq R \cap C_{j_0}$ (because $X \setminus C_{j_0}$ can be removed from any cover of $R \cap C_{j_0}$ without loosing the covering property). Therefore $\bigcup_{k=1}^{n} O_{i_k}^{-\mathfrak{A}} \supseteq (R \cap C_{j_0})^{-\mathfrak{A}}$, by Lemma 1.1. But $\mathfrak{P}_0(R) \cap C_{j_0}^{-\mathfrak{A}} \subseteq (R \cap C_{j_0})^{-\mathfrak{A}}$ holds, because any subset of R, which hits C_{j_0}, automatically hits $R \cap C_{j_0}$. □

3. Compact Unions

As an interesting application of a simple set-theoretical property, concerning the $^+$-operator, we want to take a brief look at the naturally arising question, whether a union of compact sets itself is compact. Michael showed in [6] that a union of closed sets is compact, if the unifying family is compact w.r.t. the Vietoris-topology. Now, the Vietoris-topology is induced by the upper-Vietoris τ_V^+(miss sets: $A^{+\mathfrak{A}}$ with $A^c \in \tau$) and τ_l, but τ_l is not sufficient to enforce compactness of a union of compact sets, as the following example shows: Let $X := I\!R$, endowed with euclidian topology, $\mathfrak{M} := \{[-m, m]|\ m \in I\!N\}$. Then $\bigcup_{M \in \mathfrak{M}} M = I\!R$, is obviously not compact. But every cover of $\mathfrak{M}$ with elements of the defining subbase for τ_l must especially cover the element $\{0\} = [0, 0]$ of $\mathfrak{M}$, so it must contain a set O^- with $0 \in O$. Now, every element of $\mathfrak{M}$ contains the point 0, thus $\mathfrak{M} \subseteq O^-$ follows. So, $\mathfrak{M}$ is compact in τ_l by Alexander's subbase Lemma.
And unifying compact sets, τ_l is not necessary, too, as we will see.

Proposition 3.1. *Let X be a set, $\mathfrak{X} \subseteq \mathfrak{P}(X)$ and $\mathfrak{M} \subseteq \mathfrak{X}$. Then holds*

$$\bigcup_{i \in I} C_i^{+\mathfrak{X}} \supseteq \mathfrak{M} \Longrightarrow \bigcup_{i \in I} C_i^c \supseteq \bigcup_{M \in \mathfrak{M}} M$$

for every collection $C_i, i \in I$.

Proof. For every $M \in \mathfrak{M}$ there must exist an $i_M \in I$ with $M \in C_{i_M}^{+\mathfrak{X}}$, because of $\bigcup_{i \in I} C_i^{+\mathfrak{X}} \supseteq \mathfrak{M}$. Thus $M \subseteq C_{i_M}^c \subseteq \bigcup_{i \in I} C_i^c$. □

In [5] it was shown

Lemma 3.2. *Let (X, τ) be a topological space and $\mathfrak{M} \subseteq K(X)$ compact w.r.t. the upper–Vietoris topology. Then*

$$K := \bigcup_{M \in \mathfrak{M}} M$$

is compact w.r.t. τ.

Applying our simple set-theoretical statement, we get a similar result for unions of relatively compact subsets.

Lemma 3.3. *Let (X, τ) be a topological space, let $\mathfrak{X}$ be the family of all relatively compact subsets of X and let $\mathfrak{M} \subseteq \mathfrak{X}$ be relatively compact in $\mathfrak{X}$ w.r.t. the upper Vietoris topology. Then*

$$R := \bigcup_{M \in \mathfrak{M}} M$$

is relatively compact in (X, τ).

Proof. Let $\bigcup_{i \in I} O_i \supseteq X$ with $O_i \in \tau, i \in I$ an open cover of X. Because of the relative compactness of all $P \in \mathfrak{X}$, there is a finite subcover $O_{i_P^1}, ..., O_{i_P^{n_P}}$ for every $P \in \mathfrak{X}$, i.e. $O_P := \bigcup_{k=1}^{n_P} O_{i_P^k} \supseteq M$. Of course, $O_P \in \tau$ and so $(O_P)^c$ is closed w.r.t. τ. Furthermore, $P \cap O_P^c = \varnothing$, implying $P \in (O_P^c)^{+\mathfrak{X}}$. Thus we have $\mathfrak{X} \subseteq \bigcup_{P \in \mathfrak{X}} (O_P^c)^{+\mathfrak{X}}$, where the $(O_P^c)^{+\mathfrak{X}}$ are open w.r.t. the upper–Vietoris topology. Because of the relative compactness of $\mathfrak{X}$ w.r.t. the upper–Vietoris topology, there must exist finitely many $P_1, ..., P_n \in \mathfrak{X}$ with $\mathfrak{M} \subseteq \bigcup_{j=1}^{n} (O_{P_j}^c)^{+_\mathfrak{X}}$. Now, from Proposition 3.1 we get $R = \bigcup_{M \in \mathfrak{M}} M \subseteq \bigcup_{j=1}^{n} O_{P_j}$, where every O_{P_j} is a finite union of members of the original cover $\{O_i | i \in I\}$ by construction. □

Corollary 3.4. *Let (X, τ) be a topological space and let $\mathfrak{M} \subseteq \mathfrak{P}_0(X)$ consist of relatively compact subsets of X. If $\mathfrak{M}$ is compact w.r.t. the upper–Vietoris topology, then*

$$R := \bigcup_{M \in \mathfrak{M}} M$$

is relatively compact in (X, τ).

Proof. $\mathfrak{M}$ is compact and therefore relatively compact in every set, which contains $\mathfrak{M}$, especially in the family of all relatively compact subsets of X. So, Lemma 3.3 applies. □

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