ON THE EXISTENCE OF A PBD$(30, \{4, 5, 7, 8^*\})$

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ABSTRACT. In the present paper we will prove by an exhaustive search method that a pairwise balanced design on 30 points with blocks of size 4,5,7 and exactly one block of size 8 does not exist.

1. INTRODUCTION

Let K be a set of positive integers. Then a *pairwise balanced design* PBD(v, K) of order v with block sizes from K is a pair $(V, \mathcal{B})$, where V is a finite set (the *point set*) of cardinality v and $\mathcal{B}$ is a family of subsets (called *blocks*) of V which satisfy the following properties:

(i) every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$;
(ii) if $B \in \mathcal{B}$, then $|B| \in K$.

Let k be a positive integer. PBD$(v, K \cup k^*)$ denotes a PBD containing a block of size k. If $k \notin K$, this indicates that there is only one block of size k in the PBD. We refer the reader to [2] and [4] for undefined terms as well as a general overview of design theory.

The motivation for considering the existence question of a PBD$(30, \{4, 5, 7, 8^*\})$ comes from a problem on self orthogonal Latin Squares. A self orthogonal Latin Square $\mathfrak{L}$ of order n (short $SOLS[n]$) is a $n \times n$ square whose elements from $\mathbb{Z}_n$ fulfill the following conditions:

- $\mathfrak{L}(x_1, y) \neq \mathfrak{L}(x_2, y)$ and $\mathfrak{L}(x, y_1) \neq \mathfrak{L}(x, y_2)$ for all $x, x_1, x_2, y, y_1, y_2 \in \mathbb{Z}_n$;
- $(\mathfrak{L}(x_1, y_1), \mathfrak{L}(y_1, x_1)) \neq (\mathfrak{L}(x_2, y_2), \mathfrak{L}(y_2, x_2))$ for all $x_1, x_2, y_1, y_2 \in \mathbb{Z}_n$.

Already Brayton, Coppersmith and Hoffman [3] were able to prove that $SOLS[v]$ exist if and only if $v \neq 2, 3, 6$. Drake and Lenz [7] showed that the existence of a PBD(v, L) with block sizes k from $L := \mathbb{N} \setminus \{2, 3, 6\}$ implies the existence of an $SOLS[v]$ with sub-SOLS of order k. This is one reason why the research on PBD(v, L) with $L = \mathbb{N} \setminus \{2, 3, 6\}$ attracted the interest of design theorists.

Drake and Larson [5] proved the existence of PBD(v, L) for all v except $v = 30$. Settling the case $v = 30$ would complete the determination of the essential elements of the PBD-closed set L. In a research Drake and Larson [6] worked on conditions for the existence of PBD$(30, K), K \subseteq L$ and showed that K has to be a subset of $\{4, 5, 7, 8\}$. Furthermore, they proved that the block size 8 occurs either exactly once or not at all in a PBD$(30, \{4, 5, 7, 8\})$. A recent investigation was described by Berg [1].

This article is restricted to PBD$(30, \{4, 5, 7, 8^*\})$, the pairwise balanced design with 30 points and exactly one block of size 8. In Section 2 we will briefly present the most important results from [6] that lead to the prestructure of a PBD$(30, \{4, 5, 7, 8^*\})$, which

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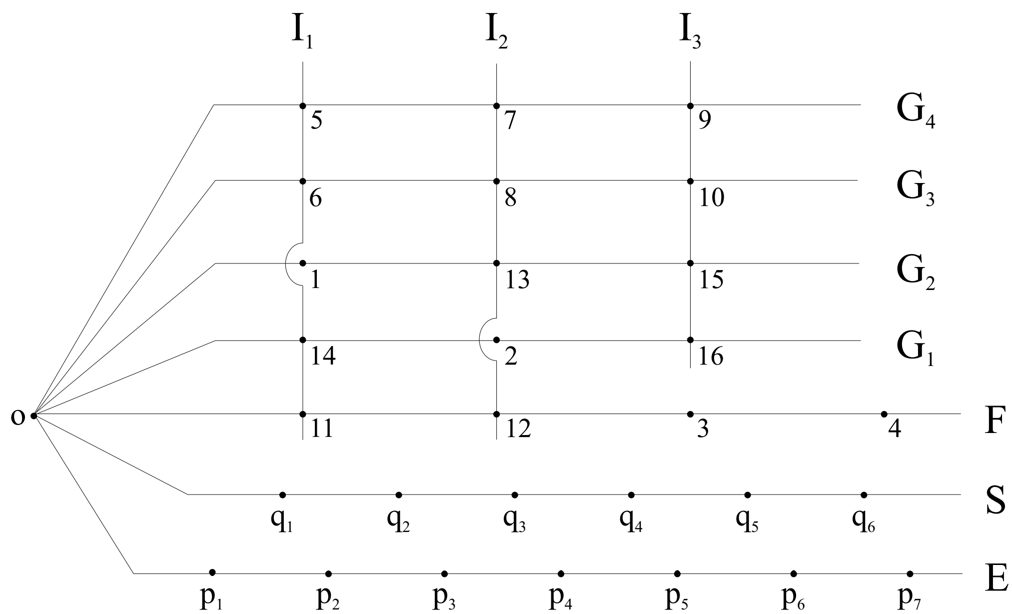

FIGURE 1. Prestructure of a PBD$(30, \{4, 5, 7, 8^*\})$ (from [6])

provides the foundation for the following work. In Section 3 it will be shown that the existence of a PBD$(30, \{4, 5, 7, 8^*\})$ is equivalent to the existence of a certain 16-tupel of permutations of the numbers $1, 2, \ldots, 7$. This relation is exploited for an exhaustive search technique in Section 4 that proves the nonexistence of a PBD$(30, \{4, 5, 7, 8^*\})$.

2. Preliminaries

In this section, we cite results from Drake and Larson [6] and define the terminology which will be used in the sequel. Throughout Section 2 we assume the existence of a PBD$(30, \{4, 5, 7, 8^*\})$, say Σ.

A block of size k is said to be a k-block. Let b_k denote the number of k-blocks in Σ. Then (b_8, b_7, b_5, b_4) is called the *block type* of Σ. A point x from Σ has *point type* $8^a 7^b 5^c 4^d$ if x is contained in exactly a 8-blocks, b 7-blocks, c 5-blocks and d 4-blocks.

Drake and Larson proved there is only one possible block type for Σ, namely $(1, 1, 14, 41)$, and that Σ contains one point o of type $8^1 7^1 5^1 4^4$, two points p_1, p_2 of type $8^1 7^0 5^4 4^2$, five points $p_3, p_4, \ldots, p_7$ of type $8^1 7^0 5^1 4^6$, six points $q_1, q_2, \ldots, q_6$ of type $8^0 7^1 5^2 4^5$, four points $1, 2, 3, 4$ of type $8^0 7^0 5^5 4^3$ and twelve points $5, 6, \ldots, 16$ of type $8^0 7^0 5^2 4^7$. The blocks that pass through the point o are denoted by E (8-block), S (7-block), F (5-block) and G_1, G_2, G_3, G_4 (4-blocks). In Σ there have to be exactly three 4-blocks, say I_1, I_2, I_3, which are disjoint from $E \cup S$ and partition the set $\{5, 6, \ldots, 16\}$. All these blocks, which we call the *prestructure* of Σ, and their incidences (determined by Drake and Larson) are exhibited in Figure 1. For further reference to this structure we define a matrix $A = (A_{ij})_{i=1,\ldots,16; j=1,\ldots,16}$ by

$$A_{ij} = \begin{cases} 1, & \text{if } \{i, j\} \text{ is a subset of one of the blocks } F, I_1, I_2, I_3, G_1, G_2, G_3, G_4; \\ 0, & \text{otherwise.} \end{cases}$$

For convenience we use the notation $P = \{p_1, p_2, \ldots, p_7\}$, $Q = \{q_1, q_2, \ldots, q_6\}$, $Z = \{1, 2, 3, 4\}$, $Y = \{5, 6, \ldots, 16\}$ and $V = \{o\} \cup P \cup Q \cup Z \cup Y$.

Drake and Larson also found restrictions on the remaining 5-blocks and to the points of type $8^0 7^0 5^4 4^3$. Each of the eight 5-blocks through p_1, p_2 intersects Z in a single point, while the five 5-blocks through $p_3, p_4, \ldots, p_7$ intersect Z in two points.

3. Necessary conditions for the existence of a PBD(30, {4, 5, 7, 8*})

In this section, we investigate the equivalence of a PBD$(30, \{4, 5, 7, 8^*\})$ and a tupel of permutations. Let S_7 be the set of all permutations of the elements $1, 2, \ldots, 7$. The basic idea (see Streso [9]) is that every point from $Z \cup Y$ lies on exactly seven of the missing blocks, each block containing one of the seven elements from P and containing either one of the six elements from Q or missing the block S.

Theorem 3.1. *There exists a PBD$(30, \{4, 5, 7, 8^*\})$ if and only if there exists a tupel $\pi = (\pi_1, \pi_2, \ldots, \pi_{16})$ ($\pi_i \in S_7$) such that the following conditions are all satisfied*

(i) $|\{k : \pi_i(k) = \pi_j(k), k \in \{1, 2, \ldots, 7\}\}| = 1 - A_{ij}$ *for all* $i, j \in \{1, 2, \ldots, 16\}, i \neq j$;

(ii) *the* 7×7 *matrix* M *given by*

$$M_{ij} = |\{k : \pi_k(i) = j, k \in \{1, 2, \ldots, 16\}\}|$$

is one of the following 3 matrices:

$$M_1 = \begin{pmatrix} 3 & 3 & 3 & 3 & 2 & 2 & 0 \\ 3 & 3 & 3 & 3 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 & 3 & 2 & 3 \\ 2 & 2 & 2 & 2 & 3 & 2 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 4 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 3 & 3 & 3 & 3 & 2 & 2 & 0 \\ 3 & 3 & 3 & 2 & 3 & 2 & 0 \\ 2 & 2 & 2 & 3 & 2 & 2 & 3 \\ 2 & 2 & 2 & 2 & 3 & 2 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 4 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 3 & 3 & 3 & 3 & 2 & 2 & 0 \\ 3 & 3 & 2 & 2 & 3 & 3 & 0 \\ 2 & 2 & 3 & 2 & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 & 2 & 2 & 3 \\ 2 & 2 & 2 & 2 & 3 & 2 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 4 \end{pmatrix}.$$

Proof. Assume that there exists a tupel π that satisfies conditions (i) and (ii). Define blocks as follows: $B_{ij} = \{p_i, q_j\} \cup \{k : \pi_k(i) = j\}$ for $i = 1, 2, \ldots, 7; j = 1, 2, \ldots, 6$ and $B_l = \{p_l\} \cup \{k : \pi_k(l) = 7\}$ for $l = 3, 4, \ldots, 7$. We claim that these 47 blocks together with the blocks from the prestructure form the family $\mathcal{B}$ of a PBD. Indeed, condition (i) ensures that every pair of points from $Z \cup Y$ occurs either in one of the blocks B_{ij}, B_l or in a block from the prestructure. Every pair of points (z, p) from $Z \times P$ or (z, q) from $Z \times Q$ occurs in B_{ij}, B_l since π_z is a permutation from S_7. Furthermore, since M is one of M_1, M_2, M_3 every pair of points from $P \times Q$ occurs in some B_{ij} and the size of a block B_{ij} or B_l is either 4 or 5. Thus, $(V, \mathcal{B})$ is the desired PBD$(30, \{4, 5, 7, 8^*\})$.

Now assume that $\Sigma = (V, \mathcal{B})$ is a PBD$(30, \{4, 5, 7, 8^*\})$. Let $B^{ik} \in \mathcal{B}$ be the unique block containing both p_i and k and define for $k = 1, \ldots, 16$ a permutation π_k by

$$\pi_k(i) = \begin{cases} 7 & \text{if } B^{ik} \cap Q = \emptyset, \\ j & \text{if } B^{ik} \cap Q = \{q_j\}. \end{cases} \tag{1}$$

It is easy to see that the tupel $\pi_\Sigma = (\pi_1, \ldots, \pi_{16})$ satisfies condition (i) of the theorem, since Σ fulfills condition (i) of the PBD definition. Unfortunately, π_Σ will generally not provide one of the desired matrices M_1, M_2 or M_3. But, we will show that there exists a PBD Σ' isomorphic to Σ that yields one of M_1, M_2 or M_3.

Let the points in Σ be labeled with $o, p_1, \ldots, p_7, q_1, \ldots, q_6, 1, \ldots, 16$ such that the point types and prestructure incidences are as in Section 2 and in Figure 1. Moreover, let p_7 be the unique point of type $8^1 7^0 5^1 4^6$ whose 5-block does not intersect the 7-block S, and let $Q_i = \{q_k : \exists B \in \mathcal{B} \text{ with } \{p_i, q_k\} \subset B \text{ and } |B| = 5\}$ for $i = 1, \ldots, 7$ and $P_j = \{p_k : \exists B \in \mathcal{B}$ with $\{p_k, q_j\} \subset B$ and $|B| = 5\}$ for $j = 1, \ldots, 6$. We define in each of the three possible cases (a) $|Q_1 \cap Q_2| = 4$, (b) $|Q_1 \cap Q_2| = 3$, (c) $|Q_1 \cap Q_2| = 2$ a bijective function $\varphi : V \to V$ that will yield the isomorphism.

Let φ be the identity on the subset $\{o, p_1, p_2, p_7, 1, \ldots, 16\}$ and, with the notation $\varphi(X) = \{\varphi(x) : x \in X\}$, let in case (a) $\varphi(Q_1) = \{q_1, \ldots, q_4\}$, $\varphi(q_x) = q_5$ and $\varphi(q_y) = q_6$ where $Q \setminus Q_1 = \{q_x, q_y\}$, $\varphi(P_x) = \{p_3, p_4\}$, $\varphi(P_y) = \{p_5, p_6\}$; let in case (b) $\varphi(Q_1 \cap Q_2) = \{q_1, q_2, q_3\}$, $\varphi(Q_1 \setminus Q_2 = \{q_x\}) = \{q_4\}$, $\varphi(Q_2 \setminus Q_1 = \{q_y\}) = \{q_5\}$, $\varphi(Q \setminus (Q_1 \cup Q_2) = \{q_z\}) = \{q_6\}$, $\varphi(P_x \setminus \{p_1\}) = \{p_3\}$, $\varphi(P_y \setminus \{p_2\}) = \{p_4\}$, $\varphi(P_z) = \{p_5, p_6\}$; while in case (c) let the bijection φ be defined such that $\varphi(Q_1 \cap Q_2) = \{q_1, q_2\}$, $\varphi(q_w) = q_3$, $\varphi(q_x) = q_4$, $\varphi(q_y) = q_5$, $\varphi(q_z) = q_6$ where $Q_1 \setminus Q_2 = \{q_w, q_x\}$ and $Q_2 \setminus Q_1 = \{q_y, q_z\}$, $\varphi(P_w \setminus \{p_1\}) = \{p_3\}$, $\varphi(P_x \setminus \{p_1\}) = \{p_4\}$, $\varphi(P_y \setminus \{p_2\}) = \{p_5\}$, $\varphi(P_z \setminus \{p_2\}) = \{p_6\}$.

Now $\Sigma' = (V, \mathcal{B}')$ with $\mathcal{B}' = \{\varphi(B) : B \in \mathcal{B}\}$ is also a PBD$(30, \{4, 5, 7, 8^*\})$. If we define for Σ' a tupel of permutations $\pi_{\Sigma'}$ as in (1), it is easy to check that $\pi_{\Sigma'}$ satisfies condition (i) and (ii) of the theorem, since Σ' fulfills the conditions of the PBD definition. This completes the proof. □

4. Exhaustive search method

In this section, we describe the way in which an exhaustive search technique (backtracking) was applied to search for a PBD$(30, \{4, 5, 7, 8^*\})$. We do this by systematically building up feasible partial tupel. For more information on search techniques used in design theory see for example [8].

We call a tupel $(\pi_1, \pi_2, \ldots, \pi_m)$ of length m a *feasible tupel* with respect to M_x ($x \in \{1, 2, 3\}$) if

(i) $|\{k : \pi_i(k) = \pi_j(k), k \in \{1, 2, \ldots, 7\}\}| = 1 - A_{ij}$ for all $i, j \in \{1, 2, \ldots, m\}, i \neq j$;

(ii) for the 7×7 matrix M given by

$$M_{ij} = |\{k : \pi_k(i) = j, k \in \{1, 2, \ldots, m\}\}|$$

$M \leq M_x$ holds, i.e. $M_{ij} \leq (M_x)_{ij}$ for $i, j \in \{1, \ldots, 7\}$.

The backtracking algorithm is now given below.

Backtracking algorithm to find a feasible tupel of length 16

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procedure Search(m,$(\pi_1, \pi_2, \ldots, \pi_m)$,$M_x$)
begin
  if $m = 16$
  then print$(\pi_1, \pi_2, \ldots, \pi_{16})$; STOP
  else
    for each $\pi_{m+1} \in S_7$ **do**
      if $(\pi_1, \pi_2, \ldots, \pi_{m+1})$ is a feasible tupel with respect to M_x

8. **then** Search($m+1$,$(\pi_1, \pi_2, \ldots, \pi_{m+1})$,$M_x$)
9. **end**

We made improvements in execution time by rejecting feasible tupel which are isomorphic to feasible tupel that have been generated earlier in the search. We remark that the results from Section 2 concerning 5-blocks and points from Z are not used so far and that in the proof of Theorem 3.1 there is some degree of freedom left to choose the bijection φ. This allows us to state the following lemma:

Lemma 4.1. *There exists a PBD*$(30, \{4, 5, 7, 8^*\})$ *if and only if there exists a tupel* $\pi = (\pi_1, \pi_2, \ldots, \pi_{16})$ $(\pi_i \in S_7)$ *such that the conditions (i) and (ii) from Theorem 3.1 and the following conditions are all satisfied*

(iii) $\{\pi_1(k), \pi_2(k), \pi_3(k), \pi_4(k)\} = \{j \in \{1, \ldots, 6\} : M_{kj} = 3\}$ *for* $k = 1, 2$, $|\{\pi_1(k), \pi_2(k), \pi_3(k), \pi_4(k)\} \setminus \{j \in \{1, \ldots, 6\} : M_{kj} = 3\}| = 2$ *for* $k = 3, 4, 5, 6$ *and* $|\{\pi_1(7), \pi_2(7), \pi_3(7), \pi_4(7)\} \setminus \{7\}| = 2$*;*

(iv) (a) *if* $M = M_1$*, then* $\pi_1(3) < \pi_1(4)$ *and* $\pi_1(5) < \pi_1(6)$*,* (b) *if* $M = M_2$*, then* $\pi_1(5) < \pi_1(6)$*;*

(v) *let* $\pi_1(i) = 1, \pi_1(j) = 2, \pi_1(k) = 3$ *and* $\pi_1(\ell) = 4$*,* (a) *if* $M = M_1$*, then* $i < j < k < \ell$*,* (b) *if* $M = M_2$*, then* $i < j < k$*,* (c) *if* $M = M_3$*, then* $i < j$*;*

(vi) $\pi_1 \leq_{lex} \pi_2$*,* $\pi_3 \leq_{lex} \pi_4$*,* $\pi_5 \leq_{lex} \pi_6$*, where* $\leq_{lex}$ *denotes the lexicographic ordering.*

Proof. We only need to show that the existence of a PBD$(30, \{4, 5, 7, 8^*\})$ Σ implies that there is a tupel π that satisfies all conditions. With regard of the proof of Theorem 3.1 we may assume that π_Σ satisfies (i) and (ii).

Let $k = 1$ or 2. Every 5-block containing p_k contains exactly one point from S and exactly one point from Z. For each such block $\{p_k, q_j, z, \ldots\}$ we have $M_{kj} = 3$ (since it is a 5-block) and $\pi_z(k) = j$ with $z \in Z = \{1, 2, 3, 4\}$. Therefore, $\{\pi_1(k), \pi_2(k), \pi_3(k), \pi_4(k)\} = \{j \in \{1, \ldots, 6\} : M_{kj} = 3\}$. Now, let $k = 3, 4, 5$ or 6. The 5-block containing p_k contains exactly one point q_j from S and exactly two points z_1, z_2 from Z. Thus, $M_{kj} = 3$ and $\pi_{z_1}(k) = \pi_{z_2}(k) = j$. All pairs of points from Z are covered by 5-blocks and, therefore, $|\{\pi_1(k), \pi_2(k), \pi_3(k), \pi_4(k)\} \setminus \{j\}| = 2$. Now, by the choice of p_7 in the proof of Theorem 3.1 the 5-block containing p_7 misses S and contains exactly two points z_1, z_2 from Z. Thus, $\pi_{z_1}(k) = \pi_{z_2}(k) = 7$ and $\{\pi_1(7), \pi_2(7), \pi_3(7), \pi_4(7)\} \setminus \{7\}$ contains exactly two elements proving claim (iii).

For the proof of (iv) and (v) we consider only case (a) $M = M_1$. The arguments in cases (b) and (c) are similar. In the proof of Theorem 3.1 we defined φ to be a bijection which maps $P_x \to \{p_3, p_4\}$ and $P_y \to \{p_5, p_6\}$. Obviously, we can redefine φ on the sets P_x, P_y such that $\pi_1(3) < \pi_1(4)$ and $\pi_1(5) < \pi_1(6)$. Also, φ bijectively mapping $Q_1 \to \{q_1, \ldots, q_4\}$ can be defined such that $i < j < k < \ell$ if $\pi_1(i) = 1, \pi_1(j) = 2, \pi_1(k) = 3, \pi_1(\ell) = 4$. This redefinition will not affect conditions (i),(ii) and (iii).

Finally, for the proof of (vi) we redefine φ on the set $Z \cup Y$. If $\pi_1 \geq_{lex} \pi_2$ define $\varphi_1(1) = 2, \varphi_1(2) = 1, \varphi_1(5) = 7, \varphi_1(6) = 8, \varphi_1(7) = 5, \varphi_1(8) = 6, \varphi_1(11) = 12, \varphi_1(12) = 11, \varphi_1(13) = 14, \varphi_1(14) = 13, \varphi_1(15) = 16, \varphi_1(16) = 15$ and $\varphi_1(x) = x$ for $x = 3, 4, 9, 10$, otherwise define φ_1 to be the identity on $Z \cup Y$. If $\pi_3 \geq_{lex} \pi_4$ define $\varphi_2(3) = 4, \varphi_2(4) = 3$ and $\varphi_2(x) = x$ for $x \in (Z \cup Y) \setminus \{3, 4\}$, otherwise define φ_2 to be the identity on $Z \cup Y$. If $\pi_5 \geq_{lex} \pi_6$ define $\varphi_3(5) = 7, \varphi_3(6) = 8, \varphi_3(7) = 5, \varphi_3(8) = 6, \varphi_3(9) = 10, \varphi_3(10) = 9$ and

$\varphi_3(x) = x$ for $x \in (Z \cup Y) \setminus \{5,6,7,8,9,10\}$, otherwise define φ_3 to be the identity on $Z \cup Y$. Now, define φ on $Z \cup Y$ by $\varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3$. Again, it is simple to check that this redefinition proves claim (vi) and will not affect conditions (i),. . .,(v). □

We call a feasible tupel $(\pi_1, \pi_2, \ldots, \pi_m)$ with respect to M_x a *proper feasible tupel* if

(iii) $\bigcup_{i=1}^{n}\{\pi_i(k)\} \subseteq \{j \in \{1,\ldots,6\} : M_{kj} = 3\}$ for $n = \min\{4, m\}$ and $k = 1, 2$, $|\bigcup_{i=1}^{n}\{\pi_i(k)\} \setminus \{j \in \{1,\ldots,6\} : M_{kj} = 3\}| \leq 2$ for $k = 3, 4, 5, 6$ and $|\bigcup_{i=1}^{n}\{\pi_i(7)\} \setminus \{7\}| \leq 2$, with equality in each case if $m \geq 4$;
(iv) (a) $x = 1$ implies $\pi_1(3) < \pi_1(4)$ and $\pi_1(5) < \pi_1(6)$, (b) $x = 2$ implies $\pi_1(5) < \pi_1(6)$;
(v) (a) $x = 1$ implies $i < j < k < \ell$, (b) $x = 2$ implies $i < j < k$, (c) $x = 3$ implies $i < j$ where $\pi_1(i) = 1, \pi_1(j) = 2, \pi_1(k) = 3$ and $\pi_1(\ell) = 4$;
(vi) $m \geq 2$ implies $\pi_1 \leq_{lex} \pi_2$, $m \geq 4$ implies $\pi_3 \leq_{lex} \pi_4$ and $m \geq 6$ implies $\pi_5 \leq_{lex} \pi_6$.

Now replace line 7. of the algorithm above by:

7'. **if** $(\pi_1, \pi_2, \ldots, \pi_{m+1})$ is a proper feasible tupel with respect to M_x

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Running the improved algorithm with Search(0,(),M_1), Search(0,(),M_2) and Search(0,(),M_3) in two independent implementations did not reveal a proper feasible tupel of length 16 and thus we conclude:

Theorem 4.2. *A PBD$(30, \{4, 5, 7, 8^*\})$ does not exist.*

We remark that the search without applying Lemma 4.1 would have taken years. With the improved algorithm we found 340 proper feasible tupel of length four in Case (a) which are checked to be non completable in 1 hour on a 650 MHz PC. In Cases (b) and (c) we considered 765 and 3822 proper feasible tupel of length four in 2 and 11 hours, respectively.

5. Conclusions

We can conclude that a PBD$(30, \{4, 5, 7, 8^*\})$ does not exist, but we cannot say anything about the existence of PBD$(30, \{4, 5, 7, 8\})$ in general. This means it is necessary to take a closer look at PBD$(30, \{4, 5, 7\})$. This has been done, in part, by the second author and shall be presented briefly.

Drake and Larson [6] showed that the block type (b_7, b_5, b_4) of a PBD$(30, \{4, 5, 7\})$ is one of the following: (3, 24, 22), (3, 15, 37), (1, 27, 24), (1, 24, 29) or (1, 15, 44). Then a PBD$(30, \{4, 5, 7\})$ with three 7-blocks can have two different basic structures: the 7-blocks either meet all in one point or they intersect in pairs in three different points. The latter case has been studied closer so that the prestructure looks as shown in Figure 2.

The points $a, b, s_6 = t_6$ have point type $7^2 5^2 4^3$ and $1, 2, \ldots, 12$ have type $7^0 5^5 4^3$ or $7^0 5^2 4^7$. The points p_i, s_i and t_i $(i = 1, \ldots, 5)$ have either type $7^1 5^2 4^5$ or $7^1 5^5 4^1$. One can also show that each point on a 7-block can be connected by a single 4- or 5-block to none, one or two points on the other 7-blocks. In order to fill in the blocks missing in Figure 2 one might try to complete Table 1 with each entry being a pair (i, j) where $i, j \in \{0, 1, \ldots, 6\}$. So, an entry (i, j) in row r_k and column ℓ for example corresponds to a block containing s_i, t_j, r_k, ℓ (if $i, j \neq 0$). The last two rows are necessary for those blocks which do not intersect R. If the filled table satisfies some obvious conditions it is equivalent to a PBD$(30, \{4, 5, 7\})$. For a more detailed description see [9]. Unfortunately

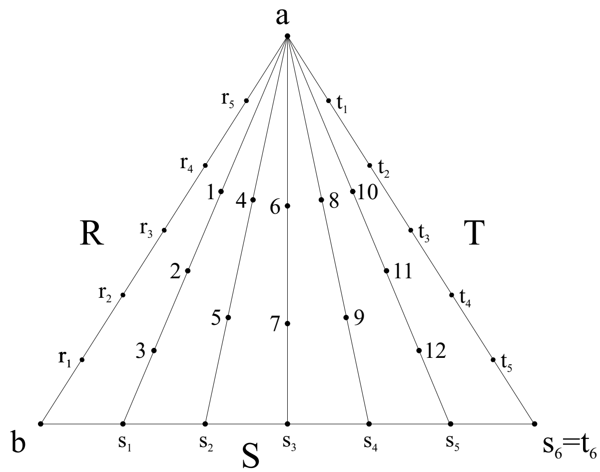

FIGURE 2. Prestructure of a PBD$(30, \{4, 5, 7\})$ with three 7-blocks intersecting in pairs in three points (from [9])

	1	2	3	4	5	6	7	8	9	10	11	12
a	(1,0)	(1,0)	(1,0)	(2,0)	(2,0)	(3,0)	(3,0)	(4,0)	(4,0)	(5,0)	(5,0)	(5,0)
b												
r_1												
r_2												
r_3												
r_4												
r_5												
7												
8												

TABLE 1. 4- and 5-blocks in a $PBD(30, \{4, 5, 7\})$ with three 7-blocks intersecting in pairs in three points (from [9])

an exhaustive search with this approach did not lead to a result because of the time consuming complexity of this structure. However, the idea might be useful with additional conditions or as a model for different structures of a PBD$(30, \{4, 5, 7\})$.

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