Super-simple $(\mathbf{v}, \mathbf{5}, \mathbf{2})$–designs

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Abstract

In this paper we study the spectrum of super–simple $(v, 5, 2)$–designs. We show that a super–simple $(v, 5, 2)$–design exists if and only if $v \equiv 1$ or $5 \pmod{10}$, $v \neq 5, 15$, except possibly when $v \in \{75, 95, 115, 135, 195, 215, 231, 285, 365, 385, 515\}$.

1 Introduction and Motivation

A (v, k, λ)–design is a pair $(V, \mathcal{B})$, where V is a v–element set of *points* and $\mathcal{B}$ is a collection of k–element subsets of V called *blocks* such that every pair of points is in exactly λ blocks. A (v, k, λ)–design $(V, \mathcal{B})$ is *super–simple* if $|B_1 \cap B_2| \leq 2$ for all blocks $B_1, B_2 \in \mathcal{B}$, $B_1 \neq B_2$. In this article we are interested in super–simple $(v, 5, 2)$–designs.

The concept of super-simple designs was introduced by Gronau and Mullin in [7]. In the papers by Adams, Bryant and Khodkar [3], Khodkar [13] and Chen [5] the spectrum of super-simple $(v, 4, \lambda)$–designs was determined for $2 \leq \lambda \leq 4$. Hartmann [9], [12] and [11] proved that the usual necessary conditions are asymptotically sufficient for arbitrary k and λ.

It seems natural to ask for the existence of (v, k, λ)–designs in which the blocks are not only different (such designs are called simple), but are also "as far apart from each other as possible", i.e. any 2 blocks share at most two points. Furthermore super-simple designs appear as suborthogonal double covers of special graphs, see Gronau, et al. [8].

We will frequently use Wilson's fundamental construction in which a primary ingredient is the group divisible design. A *group divisible design* (GDD) is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a set of points, $\mathcal{G} = [G_1, G_2, ..., G_N]$ is a partition of V into subsets which we call the *groups* and $\mathcal{B}$ is a collection of subsets of V such that every pair of points is either in a group or in exactly λ blocks. The type of a GDD is the multiset $\{|G_1|, |G_2|, \ldots, |G_N|\}$ of group sizes. It is our custom to write the type of a GDD as $g_1^{n_1} g_2^{n_2} \cdots g_m^{n_m}$ if there are n_i groups of size g_i, $i = 1, 2, \ldots, m$.

If $\mathcal{K}$ is the set of block sizes, (i.e. $|B| \in \mathcal{K}$, for all $B \in \mathcal{B}$) then we say the design is a $(\mathcal{K}, \lambda)$–GDD If $\mathcal{K} = \{k\}$, we write (k, λ)–GDD instead of $(\{k\}, \lambda)$–GDD. When $\lambda = 1$ it is omitted. Thus a k–GDD is a $(k, 1)$–GDD. A special type of GDD is the transversal design. A *transversal design* $\mathrm{TD}(k, g)$ is a k–GDD of type g^k.

A $(v, 5, 2)$–design or $(k, 2)$–GDD $(X, \mathcal{B})$ is super–simple if $|B_1 \cap B_2| \leq 2$ for all blocks $B_1, B_2 \in \mathcal{B}$, $B_1 \neq B_2$.

The necessary conditions for the existence of a $(v, 5, 2)$–design is that $v \equiv 1$ or $5 \bmod 10$. It is well-known that there is no $(15, 5, 2)$–design. For nontrivial $(v > 1)$ super-simple (v, k, λ)–designs we have $v \geq \lambda(k-2) + 2$, see [7]. Hence, the necessary condition for the existence of a super-simple $(v, 5, 2)$–design is that $v \equiv 1$ or $5 \bmod 10$ and $v \neq 5, 15$. In this article we show that this necessary condition is sufficient, except possibly when $v \in \{75, 95, 115, 135, 195, 215, 231, 285, 365, 385, 515\}$.

2 Ingredients

In this section, we give some direct constructions of super–simple designs that will be used in the recursive constructions given in section 3.

Lemma 2.1 *There exists a super–simple* $(5, 2)$*–GDD of type* v^5 *when* $v \not\equiv 2 \pmod 4$*.*

Proof: Let $v = p_1^{r_1} p_2^{r_2} \ldots p_n^{r_n}$ be the usual factorization of v into primes. Since $v \not\equiv 2 \pmod 4$ we have $p_i^{r_i} \geq 3$ for any prime p_i. From the existence of a generalized Hadamard matrix of order $p_i^{r_i}$ and the product construction of TDs, see Hartmann [10], there exists a super–simple $(5, 2)$–GDD of type v^5 when $v \not\equiv 2 \pmod 4$. $\square$

Given a subgroup $G \leq \mathsf{Sym}(X)$, the set–system generated by the base blocks $\mathcal{D} \subseteq \mathcal{P}(X)$ is the collection of blocks given by

$$\mathcal{D}^G = \bigcup_{B \in \mathcal{D}} B^G.$$

An orbit B^G is said to be a *short orbit* of the group G, if $|B^G| < |G|$.

Theorem 2.2 *Let G be an Abelian group, and $\mathcal{D}$ be a set of base blocks generating a $(k,1)$–GDD of type g^u with no short orbits. Then, $\mathcal{D} \cup -\mathcal{D}$ generates a super–simple $(k,2)$–GDD of type g^u.*

Proof: Obviously $\mathcal{D} \cup -\mathcal{D}$ generates a $(k,2)$–GDD of type g^u, we need only show that it is super–simple. Any two blocks from $\mathcal{D}^G$ intersect in at most one point, since $\mathcal{D}$ generates a $(k,1)$–GDD of type g^u. Also, any two blocks from $(-\mathcal{D})^G$ intersect in at most one point. Thus, if two blocks A and B intersect in 3 or more points, then we may assume that $A \in \mathcal{D}$, $B \in (-\mathcal{D})^G$ and $-B \in (\mathcal{D})^G$. Then $-B + g = D$ for some $g \in G$ and $D \in \mathcal{D}$. Suppose $x, y, z \in A \cap B$ and $x \neq y \neq z \neq x$. Then $-x+g, -y+g, -z+g \in D$ and

$$\{x,y\} = \{(-x+g)+x+y-g, (-y+g)+x+y-g\} \subseteq D + (x+y-g).$$

Hence $A = D+(x+y-g) = -B+(x+y)$, because distinct blocks in $\mathcal{D}^G$ meet in at most one point. Similarly, $A = -B+(x+z)$ and $A = -B+(y+z)$. But, then $B+(y-z) = B$ contrary to the assumption that any base block generates $|G|$ blocks. $\square$

We can obtain many super–simple $(v,5,2)$–designs in this manner, e.g. the following

Lemma 2.3 *There exists a super–simple $(v,5,2)$–design, if $v = 21, 41, 61$.*

Proof: There are base blocks that generate a cyclic $(v,5,1)$–design without using any short orbits for $v = 21, 41, 61$, see Abel [1]. $\square$

Lemma 2.4 *There exists a super–simple (5,2)-GDD of type 4^6.*

Proof: A super–simple (5,2)-GDD of type 4^6 on $V = \mathbb{Z}_{24}$ is obtained by developing the two base blocks

$$\{0,1,2,4,17\}, \text{ and } \{0,3,8,13,17\}$$

modulo 24. The groups are: $\{0,6,12,18\} + x : x = 0,1,2,\ldots,5$. $\square$

Lemma 2.5 *There exists a super–simple* $(5,2)$*-GDD of type* 10^n *when* $n = 9, 13, 17$.

Proof: There exists a cyclic 5-GDD of type 10^n, with no short orbits, for each $n = 9, 13, 17$, see Yin et al. [16]. Apply Theorem 2.2. □

Lemma 2.6 *There exists a super–simple (5,2)-GDD of type* $4^5 2^1$.

Proof: Let $V = \mathbb{Z}_{20} \cup \{\infty_1, \infty_2\}$. The groups are $\{0,1,2,3\}$, $\{4,5,6,7\}$, $\{8,9,10,11\}$, $\{12,13,14,15\}$, $\{16,17,18,19\}$, $\{\infty_1, \infty_2\}$. Develop the 8 base blocks:

$$\begin{array}{cccc}\{0,4,9,14,17\}, & \{0,5,9,12,19\}, & \{0,6,11,15,18\}, & \{1,7,10,14,19\}, \\ \{\infty_1,0,6,10,12\}, & \{\infty_1,1,7,11,13\}, & \{\infty_2,2,5,9,14\}, & \{\infty_2,3,4,8,15\}\end{array}$$

with the automorphism $X \mapsto X+4$ to obtain a super–simple (5,2)–GDD of type $4^5 2^1$. □

Lemma 2.7 *There exists a super–simple (5,2)-GDD of type* 2^{11}.

Proof: Let $V = \mathbb{Z}_{22}$ and the groups are

$$\{i, 11+i\} : i = 0, 1, 2, ..., 10$$

Develop the 2 base blocks:

$$\{0,1,2,5,10\}, \text{ and } \{0,2,6,9,16\}$$

modulo 22 to obtain a super–simple (5,2)–GDD of type 2^{11}. □

Lemma 2.8 *There exists a super–simple* $(v,5,2)$*–design when* $v = 25, 45, 65$.

Proof:

If $v = 25$, then let $V = \mathbb{Z}_{25}$. The 3 base blocks are

$$\{0,2,8,10,16\}, \{0,1,5,19,23\} \text{ and } \{0,1,2,3,4\}$$

with the group of order 50 generated by the 2 permutations

(0,1,2,3,4)(5,6,7,8,9)(10,11,12,13,14)(15,16,17,18,19)(20,21,22,23,24)
(1,5)(2,10)(3,15)(4,20)(7,11)(8,16)(9,21)(13,17)(14,22)(19,23).

The first two base blocks have an orbit of size 25, each block is generated twice, whereas the last base block have an orbit of size 10, each block is generated five times. We take exactly one copy of any block and get the 60 blocks of the design.

If $v = 45$, then let $V = \{0,1\} \times \mathbb{Z}_{22} \cup \{\infty\}$. The base blocks are

$$\begin{array}{ll}
\{(0,0),(0,2),(0,3),(0,10),(1,3)\}, & \{(0,0),(1,4),(1,5),(1,18),(1,20)\}, \\
\{(0,0),(0,1),(1,7),(1,10),(1,18)\}, & \{(0,0),(0,11),(0,16),(1,5),(1,8)\}, \\
\{(0,0),(0,9),(0,13),(0,16),(1,11)\}, & \{(0,0),(1,8),(1,4),(1,9),(1,13)\}, \\
\{(0,0),(0,20),(1,10),(1,12),(1,0)\}, & \{(0,0),(0,18),(0,8),(1,15),(1,21)\}, \\
\{(0,0),(0,17),(1,1),(1,16),(\infty)\}. &
\end{array}$$

If $v = 65$, then let $V = \{0,1\} \times \mathbb{Z}_{32} \cup \{\infty\}$. The base blocks are

$$\begin{array}{ll}
\{(0,0),(0,2),(0,22),(0,13),(1,8)\}, & \{(0,0),(0,19),(0,26),(0,31),(1,19)\}, \\
\{(0,0),(0,15),(0,26),(0,22),(1,9)\}, & \{(0,0),(0,16),(0,24),(1,13),(1,14)\}, \\
\{(0,0),(0,30),(0,1),(1,2),(1,8)\}, & \{(0,0),(0,18),(0,15),(1,11),(1,31)\}, \\
\{(0,0),(0,9),(1,31),(1,7),(1,23)\}, & \{(0,0),(0,14),(1,17),(1,3),(1,24)\}, \\
\{(0,0),(0,28),(1,1),(1,24),(1,5)\}, & \{(0,0),(1,4),(1,11),(1,26),(1,6)\}, \\
\{(0,0),(1,15),(1,2),(1,12),(1,16)\}, & \{(0,0),(1,18),(1,20),(1,23),(1,29)\}, \\
\{(0,0),(0,5),(1,0),(1,17),(\infty)\}. &
\end{array}$$

□

A *parallel class* in a design is a collection of blocks that partition the points of the design. If all the blocks can be partitioned into parallel classes we say that the design is *resolvable*. An *idempotent transversal design* is a transversal design with a parallel classes. An idempotent $\mathrm{TD}(k,n)$ exists whenever a $\mathrm{TD}(k+1,n)$ exists. Simply delete one of the groups, the blocks that contained a fixed point in the deleted group are now a parallel class in the new $\mathrm{TD}(k,n)$. A *2-parallel class* in a design is a collection of blocks that contain each point exactly twice. If all the blocks of the design can be partitioned into 2-parallel classes we say that the design is *2-resolvable.*

Lemma 2.9 *There exists a super–simple* $(5,2)$*–GDD of type* $1^{16}5^1$.

Proof: The 16 groups of size 1 are the 16 points of the super–simple $(16,4,2)$–design exhibited in Figure 1. The 40 blocks of this design can be partitioned into 5 2-parallel classes C_1, C_2, C_3, C_4, C_5 consisting of 8 blocks each. The remaining group consists of 5 new points x_1, x_2, x_3, x_4, x_5 and we

$\{0,2,9,11\}$,	$\{4,6,13,15\}$,	$\{5,7,12,14\}$,	$\{1,3,8,10\}$,
$\{0,1,2,3\}$,	$\{4,9,13,14\}$,	$\{5,8,11,12\}$,	$\{6,7,10,15\}$,
$\{0,3,5,15\}$,	$\{4,8,9,12\}$,	$\{6,10,11,14\}$,	$\{1,2,7,13\}$,
$\{0,4,5,6\}$,	$\{1,8,13,15\}$,	$\{2,7,11,14\}$,	$\{3,9,10,12\}$,
$\{0,4,7,10\}$,	$\{2,5,6,8\}$,	$\{3,11,12,13\}$,	$\{1,9,14,15\}$,
$\{0,7,8,9\}$,	$\{1,5,10,14\}$,	$\{2,6,12,13\}$,	$\{3,4,11,15\}$,
$\{0,8,13,14\}$,	$\{2,10,12,15\}$,	$\{3,6,7,9\}$,	$\{1,4,5,11\}$,
$\{0,10,11,13\}$,	$\{1,4,7,12\}$,	$\{2,5,9,15\}$,	$\{3,6,8,14\}$,
$\{0,1,6,12\}$,	$\{2,3,4,14\}$,	$\{5,9,10,13\}$,	$\{7,8,11,15\}$,
$\{0,12,14,15\}$,	$\{1,6,9,11\}$,	$\{2,4,8,10\}$,	$\{3,5,7,13\}$

Figure 1: A special $(16,4,2)$–design

take as the blocks the sets $A \cup \{x_i\}$, where $A \in C_i$ and $i = 1,2,3,4,5$. It is easily checked that the result is a $(5,2)$–GDD of type $1^{16}5^1$. □

Lemma 2.10 *There exists a super–simple* $(v,5,2)$*–design when* $v = 85$*.*

Proof: Take a super–simple $(5,2)$–GDD of type 16^5. Add 5 new points and construct on each 4 of the 5 groups a super–simple $(5,2)$–GDD of type $1^{16}5^1$. This GDD exists by Lemma 2.9. On the last group and 5 new points construct a super–simple $(21,5,2)$–design, which exists by Lemma 2.3. □

Lemma 2.11 *There exists a super–simple* $(5,2)$*-GDD of type* 6^5*.*

Proof: A super–simple $(5,2)$-GDD of type 6^5 on $V = \{0,1,2,3,4\} \times \mathbb{Z}_6$ is obtained by developing the second coordinate of following 12 base blocks modulo 6:

$$\begin{array}{ll}
\{(0,0),(1,0),(2,0),(3,0),(4,0)\} & \{(0,1),(1,3),(2,2),(3,4),(4,0)\} \\
\{(0,2),(1,0),(2,1),(3,5),(4,2)\} & \{(0,3),(1,1),(2,5),(3,4),(4,2)\} \\
\{(0,4),(1,3),(2,5),(3,2),(4,1)\} & \{(0,5),(1,5),(2,3),(3,1),(4,1)\} \\
\{(0,0),(1,2),(2,3),(3,2),(4,3)\} & \{(0,1),(1,2),(2,4),(3,0),(4,5)\} \\
\{(0,2),(1,5),(2,2),(3,3),(4,4)\} & \{(0,3),(1,4),(2,1),(3,1),(4,4)\} \\
\{(0,4),(1,1),(2,0),(3,3),(4,5)\} & \{(0,5),(1,4),(2,4),(3,5),(4,3)\}
\end{array}$$

The groups are $\{(i,0),(i,1),(i,2),(i,3),(i,4),(i,5)\}$, $i = 0,1,2,3,4$. □

Lemma 2.12 *There exists a super–simple* $(5,2)$*-GDD of type* 6^6*.*

Proof: This GDD can be generated cyclicly on the point set $V = \mathbb{Z}_{36}$ by developing the following 3 base blocks:

$$\{0,1,2,5,15\}, \{0,2,9,17,25\}, \text{ and } \{0,3,10,14,19\}.$$

modulo 24. The groups are $\{0,6,12,18,24,30\} + x : x = 0,1,2,\dots,6$. $\square$

Lemma 2.13 *There exists a super–simple* $(5,2)$*-GDD of type* $6^5 8^1$*.*

Proof: First we generate a super–simple $(4,2)$-GDD of type 6^5 that has a partition into 2-parallel classes, i.e. is 2-resolvable. Let

$$g = (0,1,2,\dots,23)(24,25,26)(27,28,29),$$

$H = id, g^8, g^{16}$ and let C be the set of 15 blocks generated by the base blocks

$$\{5,6,23,28\}, \{1,3,4,28\}, \{1,10,20,23\}, \{0,10,19,24\} \text{ and } \{0,6,13,25\}$$

under the subgroup H. Then

$$\bigcup_{i=0}^{7} C^{g^i}$$

is a super–simple $(4,2)$-GDD of type 5^6 that can be resolved into the 2–parallel classes:

$$C, C^g, C^{g^2}, C^{g^3}, \dots, C^{g^7}.$$

The groups of this GDD are

$$\{0,4,8,12,16,20\}, \{1,5,9,13,17,21\}, \{2,6,10,14,18,22\},$$
$$\{3,7,11,15,19,23\}, \{24,25,26,27,28,29\}$$

These are also the groups of size 6 in the desired $(5,2)$-GDD of type $6^5 8^1$. The final group of the desired GDD consists of eight new points x_0,x_1,x_2,$\dots$, x_7 and the blocks of the desired GDD are the 120 subsets of the form $A \cup \{x_i\}$ where $A \in C^{g^i}$ and $i = 0,1,2,\dots,7$. $\square$

Lemma 2.14 *(Abel and Greig) [2] There exists a RBIBD*$(v,5,1)$ *whenever* $v \equiv 5 \pmod{20}$ *except possibly when* $v = 45, 225, 345, 465, 645$*.*

3 Recursive Construction

In this section we complete our proof of the sufficiency of the necessary conditions. Our principal tool is to apply Wilson's Fundamental Construction. For example we have the following lemmas.

Lemma 3.1 *If there is a* $(\{5,6\},1)$*–GDD of type* $g_1 g_2 \cdots g_N$ *and there are super–simple* $(4g_i+1,5,2)$*–designs for each* i*,* $i=1,2,\ldots,N$*, then there exists a super–simple* $(4\sum_{i=1}^{N} g_i+1,5,2)$*–design.*

Proof: Let $(V,\mathcal{G},\mathcal{B})$ be a $(\{5,6\},1)$–GDD of type $g_1 g_2 \cdots g_N$. Give weight 4 to all of the points. That is replace each point $x \in V$ with four new points x_1,x_2,x_3,x_4. Now replace each block $B \in \mathcal{B}$ with a super–simple $(5,2)$–GDD of type $4^{|B|}$ with groups $[\{x_1,x_2,x_3,x_4\} : x \in B]$. These GDDs exist by Lemmas 2.1 and 2.4. Finally add a new point ∞ to the already chosen $4\sum_{i=1}^{N} g_i$ points and replace each group $G \in \mathcal{G}$, by the blocks of a super-simple $(4|G|+1,5,2)$–design on the point set $G \cup \{\infty\}$. It is elementary to check that the result is a super–simple $(4\sum_{i=1}^{N} g_i+1,5,2)$–design. $\square$

Lemma 3.2 *If there is a resolvable* $(5n,5,1)$*–design, a super–simple* $(4n+1,5,2)$*–design and a super–simple* $(4x+1,5,2)$*–design,* $x<\frac{5n-1}{4}$*, then there exists a super–simple* $(4(5n+x)+1,5,2)$*–design.*

Proof: A resolvable $(5n,5,1)$–design, $(V,\mathcal{B})$, has $\frac{5n-1}{4}$ parallel classes. Let $P_0,P_1,P_2,...,P_x$ be $x+1$ of them. Let $X=\{\infty_1,\infty_2,\ldots,\infty_x\}$ be a set of x new points. We construct a $(\{5,6\},1)$–GDD on $V \cup X$ of type $5^n x^1$ by taking the groups to be the blocks in P_0 and the set X and the set of blocks to be

$$\{B\cup\{\infty_i\} : B \in P_i, i=1,2,\ldots x\} \cup \{B \in \mathcal{B} : B \notin P_i, i=0,1,2,\ldots x\}.$$

Now applying Lemma 3.1 and using the super-simple $(21,5,2)$–design of Lemma 2.3 we obtain a super–simple $(4(5n+x)+1,5,2)$–design. $\square$

We divide the problem into 4 cases according to the congruence class of v modulo 20.

3.1 $v \equiv 1 \pmod{20}$

In this section we settle sufficiency when $v \equiv 1 \pmod{20}$.

Lemma 3.3 *If $v \equiv 1 \pmod{20}$, then there exists a super–simple $(v, 5, 2)$.*

Proof: Let $v = 20t + 1$. If $1 \leq t \leq 3$, see Lemma 2.3. If $t = 4$, there exists a $(81, 5, 1)$–design over $EA(81)$ with no short orbits, see Beth et al. [4], so, applying Theorem 2.2 we obtain a super–simple $(81, 5, 2)$–design.

Note that the TD exists for the orders we are interested in. If $5 \leq t \leq 24$ and $t \notin \{11, 16, 17, 18, 19\}$, then there exists a super–simple $(v, 5, 2)$–design because we may apply Theorem 2.2 to the cyclic $(v, 5, 1)$-BIBD with no short orbits given in Abel [1].

For the other values of t take a $\mathrm{TD}(6, n)$ and remove $n - x$ points from the last group to obtain a $(\{5, 6\}, 1)$–GDD of type $n^5 x^1$, $1 \leq x < n$. Apply Lemma 3.1.

If $t = 11$, there exists a $(\{5, 6\}, 1)$–GDD of type 5^{11}, see Greig [6], and if $t = 19$, we remove a point from a $(96, 6, 1)$–design to obtain a $(6, 1)$–GDD of type 5^{19}. We again apply Lemma 3.1 to obtain super–simple $(20t + 1, 5, 2)$–designs, for $t = 11$ and $t = 19$.

If $16 \leq t \leq 18$, then choose $n = 15$ and $x = 5, 10, 15$ and apply Lemma 3.1

If $t \geq 25$, then choose $n = 25 + 5y$ for $y \geq 0$ with $x = 0, 5, 10, 15, 20$ and apply Lemma 3.1. □

3.2 $v \equiv 5 \pmod{20}$

We next deal with the case when $v \equiv 5 \pmod{20}$.

Lemma 3.4 *If $v \equiv 5 \pmod{100}$, then there exists a super–simple $(v, 5, 2)$–design.*

Proof: If $v \equiv 5 \pmod{100}$, then $\frac{v}{5} \equiv 1 \pmod{20}$. Thus by Lemma 3.3 there exists a super–simple $(\frac{v}{5}, 5, 2)$–design, and by Lemma 2.1 there exists a super–simple $(5, 2)$–GDD of type $(\frac{v}{5})^5$. So we can construct an $(\frac{v}{5}, 5, 2)$–design on each of the groups of the GDD to obtain a super–simple $(v, 5, 2)$–design. □

Lemma 3.5 *If $v \equiv 5 \pmod{20}$ and $v \neq 5, 285, 365$ and 385, then there exists a super–simple $(v, 5, 2)$–design.*

Proof: If $v = 25, 45, 65, 85$, then there exists a super–simple $(v, 5, 2)$–design by Lemma 2.8 and Lemma 2.10. If $v = 105$, the there exists a super–simple $(105, 5, 2)$–design by Lemma 3.4. Take a $\mathrm{TD}(6, n)$ and remove points in one group to obtain a $(\{5, 6\}, 1)$–GDD of type n^5x^1. If $n \geq 30$ and $n \equiv 0 \pmod 5$, we take $k = 6, 11, 16, 21, 26$ and apply Lemma 3.1 to prove that whenever $v \geq 625$ and $v \equiv 5 \pmod{20}$, then there exists a super–simple $(v, 5, 2)$–design. Also, using Lemma 3.1 but taking $k = 6, 11, 16, 21$ and $n = 25$ constructs a super–simple $(v, 5, 2)$–design for $v = 525, 545, 565, 585$. If $v = 205, 305, 405, 505$ and 605, Lemma 3.4 shows that a super–simple $(v, 5, 2)$–design exists. If $v = 125, 225, 325, 425$, a super–simple $(v, 5, 2)$–design can be shown to exist by a method similar to Lemma 3.4. The required ingredients include the super–simple $(v, 5, 2)$–designs for $v = 25, 45, 65$, and 85 constructed in Lemma 2.8 and Lemma 2.10.

A $(\{5, 6\}, 1)$–GDD of type 6^6, 11^6, 16^6 and 21^6 can be obtained by either taking a $\mathrm{TD}(6, n)$ or removing a block in a $\mathrm{TD}(6, n+1)$. Apply Lemma 3.1 to obtain super–simple $(v, 5, 2)$–designs when $v = 145, 245, 345, 445$.

If $v = 165$, take a $(5, 1)$–GDD of type 4^{10}. Give weight 4 to each point and add five new points. On each of 9 of the 10 groups union the 5 new points Construct a super–simple $(5, 2)$–GDD of type $1^{16}5^1$. Construct a super–simple $(21, 5, 2)$–design on the last group and the 5 new points.

If $v = 185$, there exists a $(\{5, 6\}, 1)$–GDD of type 5^86^1, see Ling [14]. Apply Lemma 3.1.

If $v = 265$, consider an idempotent $\mathrm{TD}(6, 11)$. By adding the groups as blocks we get a pairwise balanced design with blocks of sizes 6 and 11, having a parallel class of blocks of size 6. Thus if we interpret the blocks of a parallel class as groups it is also a $\{6, 11\}$–GDD of type 6^{11}. Apply Lemma 3.1.

There exist resolvable $(5n, 5, 1)$–designs for $n = 21$ by Lemma 2.14. Thus we may apply Lemma 3.2 with $x = 11$ and 16 and obtain super-simple $(v, 5, 2)$–designs, when $v = 465$ and 485.

□

3.3 $v \equiv 11 \pmod{20}$

In this section we settle the case when $v \equiv 11 \pmod{20}$.

Theorem 3.6 *Let $q \equiv 11 \pmod{20}$ be a prime power and let $\zeta \in \mathbb{F}_q$ be a 5-th root of unity. Then the orbit of*

$$\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$$

under the special affine group

$$\mathsf{SAF}(q) = \{X \mapsto \alpha^2 X + \beta : \alpha, \beta \in \mathbb{F}_q, \alpha \neq 0\}$$

is a super–simple $(p, 5, 2)$*–design.*

Proof: Write $q = 20k + 11$ and let $\mathcal{B}$ be the orbit of

$$B = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$$

under $G = \mathsf{SAF}(q)$, then the 2–homogeninity of $\mathsf{SAF}(q)$ guarantees that $\mathcal{B}$ is a $(q, 5, \lambda)$–design for some λ. We know that

$$|\mathcal{B}| = \frac{q(q-1)}{5 \cdot 4}\lambda$$

and on the other-hand

$$|\mathcal{B}| = \frac{|G|}{|G_B|}$$

where G_B is the stabilizer of B in G. Hence noting that $|G| = q(q-1)/2$ we see that

$$\lambda = \frac{10}{|G_B|}.$$

If ρ is a primitive root modulo q, then we may assume $\zeta = \rho^{(q-1)/5} = \rho^{2+4k}$. Thus ζ is a square modulo q and consequently the map $X \mapsto \zeta X \in \mathsf{SAF}(q)$. This map has order 5 and fixes B, so 5 divides $|G_B|$. The order of G is odd, so we have $|G_B| = 5$ and thus $\lambda = 2$.

Let $B_1, B_2 \in \mathcal{B}$ be the two blocks containing $\{0, 1\}$. If $x, y \in \mathbb{F}_q$ are any two other points, there is a unique $f \in \mathsf{SAF}(q)$ such that $f(\{0, 1\}) = \{x, y\}$.

Thus the two blocks that contain $\{x, y\}$ are $f(B_1)$ and $f(B_2)$. Hence, if two blocks intersect in $i \geq 2$ points then every pair of blocks intersect in 0, 1 or i points. Suppose $i > 2$ and let $B_1 = \{x_1, x_2, x_3, x_4, x_5\}$ be any block. Then there is a unique other block B_2 that contains $\{x_1, x_2\}$. It must intersect B_1 in $i > 2$ points, so B_2 contains an additional point x_3 of B_1. Similarly there is a unique other block B_3 that contains $\{x_4, x_5\}$ and it must contain a third point say x_3 of B_1. Since $B_1 \neq B_2$ we may assume $x_5 \notin B_2$. Similarly we may assume $x_1 \notin B_3$. Let $B_4 \neq B_1$ be the unique other block containing $\{x_1, x_5\}$. But $|B_1 \cap B_4| = i > 2$, so B_4 contains another point of B_1, but there is no point that can be chosen that avoids covering a pair more then twice. Thus $i = 2$, and the design is super–simple. □

Lemma 3.7 *If* $v \equiv 11 \pmod{20}$, $v \neq 231$, *then there exists a super-simple* $(v, 5, 2)$*–design.*

Proof: If $v = 11, 31, 71, 131, 151, 191, 211, 251, 271, 311, 331, 431, 491, 571$, a super–simple $(v, 5, 2)$–design exists by Theorem 3.6. If $v = 51$ we construct a super–simple (51,5,2) by developing the 5 base blocks

$$\{0, 1, 14, 31, 35\}, \{0, 1, 9, 23, 33\}, \{0, 11, 16, 18, 42\},$$
$$\{0, 7, 13, 36, 39\} \text{ and } \{0, 4, 10, 12, 15\}$$

under the the cyclic group $\mathbb{Z}_{51}$. If $v = 91$ or 171, then the designs exist by Lemma 2.5 for $n = 9$ or 17. Take a TD$(6, 5n)$ for $n = 1$ and $n \geq 3$. Give weight 4 to all points in 5 groups. Let $0 \leq k \leq \frac{5}{3}n$ and give weight 4 to $2k$ points, weight 2 to k points and weight 0 to the last $5n - 3k$ points of the last group. Applying Lemmas 2.1, 2.6 and 2.4 we get a super–simple $(5, 2)$-GDD of type $(20n)^5(10k)^1$.

Applying Lemma 3.1 when $k = 1, 3, 5, 7, 9$, $n \geq 1$ and $n \neq 6$ yield super-simple $(v, 5, 2)$–design for all $v \equiv 11 \pmod{20}$ and $v \geq 600$. For the pairs $(n, k) = (1, 1), (3, 5), (4, 1), (4, 5), (5, 1), (5, 3), (5, 5)$ we get the orders $v = 111, 351, 411, 451, 511, 531, 551$. Now we modify the last construction by giving k points of the last group the weight 4 and l the weight 2, where $0 \leq k + l \leq 5n$. Then we get super–simple $(5, 2)$-GDD of type $(20n)^5(4k+2l)^1$. For the pairs $(n, k, l) = (4, 17, 1), (5, 22, 1)$ we get the orders $v = 471, 591$. Thus, we have all orders except when $v = 231, 291, 371, 391$. Take a RBIBD$(20n+5, 5, 1)$ (see Lemma 2.14) and extend the parallel classes to obtain a 6-GDD of type $5^{4n+1}(5n)^1$. Give weight 4 to the points in all groups of size 5, and weight 0, 2 or 4 to the point in the last group, as above. This leads to a super–simple $(5, 2)$-GDD of type $(20)^{4n+1}(10k)^1$ when $10k \leq 5$. Add a new point. Construct on the groups of size 20 and the new point a super-simple $(21, 5, 2)$–design and on construct on the groups of size 10k a super-simple $(10k + 1, 5, 2)$–design. This yields a $(80n + 20 + 10k + 1, 5, 2)$–design. Apply this with $(n, k) = (3, 3), (4, 3), (4, 5)$ to obtain super-simple $(v, 5, 2)$–designs with $v = 291, 371, 391$. □

3.4 $v \equiv 15 \pmod{20}$

Here we settle the case when $v \equiv 15 \pmod{20}$.

Lemma 3.8 *There exists a super–simple* $(v, 5, 2)$*–design whenever* $v \equiv 15 \pmod{20}$ *and* $v \geq 535$ *and* $415 \leq v \leq 495$.

Proof: Take a RBIBD$(20n+5,5,1)$, which exists by Lemma 2.14 for all $n \neq 2, 11, 17, 23, 32$. This design has $5n+1$ parallel classes. For all but one of the parallel classes add a new point to each of the blocks in that parallel class. Remove the $4n+1$ blocks of the final parallel class. This yields a 6-GDD of type $5^{4n+1}(5n)^1$. The groups of size 5 are the $4n+1$ blocks of the final parallel class and the $5n$ new points form the last group. Give weight 6 to the points in the first $4n+1$ groups of size 5 and weight 0, 6 or 8 to the points in the last group to form a super-simple $(5,2)$-GDD of type $(30)^{4n+1}(x)^1$ where $24 \leq x \leq 40n$. Since there exists a super–simple $(x+1,5,2)$ when $x+1 \equiv 5 \pmod{20}$ and $x \neq 5, 285, 265, 285$ by Lemma 3.5 and a super–simple $(31,5,2)$–design by Lemma 3.7, we obtain a super–simple $(120n+30+x+1,5,2)$–designs by adding a new point and constructing super–simple $(31,5,2)$ or $(x+1,5,2)$–designs on the groups and the new point. If $n \geq 4$ and $x+1 = 25, 45, 65, 85, 105, 125$, we get all orders with $v \geq 535$. If $n = 11, 17, 23, 32$ apply the same method, but reduce n by one and increase x by 120. If $n = 3$, we use $x+1 = 25, 45, 65, 85, 105$ and obtain a super–simple $(v,5,2)$–design whenever $v \equiv 15 \pmod{20}$ and $415 \leq v \leq 495$. □

Lemma 3.9 *There exists a super–simple* $(v,5,2)$*–design whenever* $v \equiv 15 \pmod{20}$ *and* $v \neq 15, 75, 95, 115, 135, 195, 215, 515$.

Proof: The previous lemma shows that we need only consider $v \leq 395$. First we construct a super–simple (35,5,2)–design by developing the 7 base blocks

$$\{0,1,2,18,20\}, \{1,2,4,6,9\}, \{1,5,18,25,31\}, \{1,7,21,28,30\},$$
$$\{1,7,26,29,34\}, \{1,8,29,30,33\} \text{ and } \{1,9,27,32,33\}$$

under the group generated by $(0)(1,2,...,17)(18,19,...,34)$.

If $v = 55, 155, 175, 255, 275, 355$, then we use a super–simple $(5,2)-GDD$ of type $(\frac{v}{5})^5$, which exists by Lemma 2.1, with $\frac{v}{5} = 11, 31, 35, 51, 55, 71$. Construct on the each of groups a super–simple $(\frac{v}{5},5,2)$–designs.

Now we apply the construction used in the previous Lemma, i.e. we take a RBIBD$(20n+5,5,1)$, add new points points to all but one of $5n+1$ parallel classes and remove the blocks of the final parallel class; this yields a 6-GDD of type $5^{4n+1}(5n)^1$. Give weight 4 to the points in the first $4n+1$ groups of size 5 and weight 0, 2 or 4 to the points in the last group to form a super-simple $(5,2)$-GDD of type $(20)^{4n+1}(x)^1$ where $x \leq 20n$. There exists a $(21,5,2)$–designs by Lemma 3.3. Thus if there is a super–simple $(x+1,5,2)$ we can add a new point to this $(5,2)$-GDD and construct super–simple $(u,5,2)$–designs on the groups and the new point, to obtain a super–simple $(80n+20+x+1,5,2)$–design. Applying this with $(n,x)=(3,34),(3,54),(4,34),(4,54)$ to obtain super–simple $(v,5,2)$–designs with $v=295,315,375,395$.

Next take a $\mathrm{TD}(6,15)$ and give weight 4 to points in 5 groups and weight 4 to 8 points, weight 2 to 1 point and weight 0 to the remaining 6 points in the last group. This constructs a super–simple $(5,2)$-GDD of type $(60)^5x^1$. Since there is a super–simple $(35,5,2)$ and a super–simple $(61,5,2)$–design, then there exists a super–simple $(335,5,2)$–design.

A PBD on 47 points with block sizes 5,7 and 9 can be constructed from the projective plane of order 8 as follows: take a projective plane of order 8 and remove the points in a hyperoval as well as the points in 2 exterior lines to the hyperoval except for the point of the intersection. Every other line intersects these 26 points either 0,2 or 4 times. Removing these points yields a PBD on 47 points, furthermore, we can construct a GDD with block sizes 5,7,9 and group sizes 5,7,9 by looking at the lines through a point in the hyperoval. This gives a $\{5,7,9\}$-GDD of type $5^x7^y9^z$ for appropriate x,y,z. Give weight 5 to each point to obtain a super–simple $(5,2)$-GDD on 235 points with group sizes 25,35 and 45. There exists a super–simple $(5,2)$-GDD of type 5^q for $q=5,7,9$ by Lemma 2.1. Fill in the groups yields a super–simple $(235,5,2)$–design. $\square$

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