# Linear combinations of shifted eigenfunctions of two-scale difference equations

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**Abstract.** In this paper, we continue our considerations in [3] on eigenfunctions of two-scale difference equations. Using the results in [3] concerning equivalent eigenfunctions and equivalent characteristic polynomials, we derive sum relations for shifts of an eigenfunction in the interval $(-1, 1)$. In particular, we deal with the case that the characteristic polynomial contains a cyclic factor. We give necessary and sufficient conditions for the linear independence of shifts of an eigenfunction, and we determine a basis for the coefficient vector in the case of linear dependence. Here the representation of the characteristic polynomial by means of the corresponding minimal polynomial is basically used. Our main emphasis is laid on linear combinations of such shifts yielding polynomials, where both the possible coefficients and the possible polynomials are characterized. The results are specialized to the constant polynomial equal to 1, i.e. to partitions of unity. But also linear combinations of shifts of an eigenfunction, yielding certain distributions, are investigated.

**Keywords:** *Two-scale difference equations, distributional solutions, eigenfunctions, Appell polynomials, sums of shifted eigenfunctions, cyclic zeros and cyclic polynomials, linear independence, partitions of unity.*

**AMS subject classification:** 39A13, 39B32, 46F10, 26C15, 33C65, 15A18

## 1. Preliminaries

Two-scale difference equations

$$\lambda\varphi\left(\frac{t}{2}\right) = \sum_{\nu=0}^{N} c_\nu \varphi(t-\nu) \qquad (t \in \mathbb{R}) \tag{1.1}$$

with complex coefficients and $\lambda c_0 c_N \neq 0$, $N \in \mathbb{N}_0$, appear in wavelet theory, multiresolution analysis and subdivision schemes, cf. [11], [8]. The solutions of (1.1) are sought in the class of generalized functions consisting of complex valued continuous functions and their derivatives of finite order in the distributional sense [1]. In [3] we have denoted non-zero solutions $\varphi$ of (1.1) as *eigenfunctions* if they satisfy the two "boundary" conditions:

**(i)** $\varphi(t) = 0$ for $t < 0$,

**(ii)** $\varphi(t)$ is equal to a polynomial for $t > N$.

These conditions are quite natural since we want to include compactly supported solutions of (1.1), if such exist, and we demand always that integrals of eigenfunctions with **(i)** shall again be eigenfunctions. Under the normalization

$$\sum_{\nu=0}^{N} c_\nu = 1 \tag{1.2}$$

the eigenvalues of (1.1) are exactly the numbers $\lambda_n = 2^n$ $(n \in \mathbb{Z})$ and they are all simple, i.e. the eigenfunctions are uniquely determined up to a constant factor, cf. [12], [13] and [6].

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An eigenfunction $\varphi$ belonging to $\lambda_n$ vanishes for $t > N$ when $n < 0$, and it is equal to a polynomial of degree $n$ when $n \geq 0$. The eigenfunction belonging to $\lambda_n$ $(n \in \mathbb{Z})$ and normalized by the condition

**(iii)** $\qquad \varphi^{(n)}(t) = 1$ for $t > N$

is denoted by $\varphi_n$. As usual, $\varphi^{(n)}$ denotes, for $n \geq 0$, the distributional derivative of order $n$ and, for $n < 0$, the $(-n)$-times iterated integral satisfying **(i)**. For all $n \in \mathbb{Z}$ it holds $\varphi_n' = \varphi_{n-1}$ and $\varphi_n^{(-1)} = \varphi_{n+1}$.

The main goal of this paper is to investigate linear combinations of shifts of an eigenfunction. For this reason we list such notations and results of [3], partly in an improved version, which we shall need afterwards. At the beginning we point out three

**Corrections 1.1**

**1.** In [ 3 : (3.1) ] replace $\tilde{\phi}_{-1}$ by $\tilde{\varphi}_{-1}$.

**2.** In [ 3 : Definition 6.7 ] there must be added a convention concerning multiple cycles, cf. the later Definition 1.7.

**3.** In [ 3 : Algorithm II.1, p.480 ] the words "all quadratic factors by symmetric zeros, i.e." must be cancelled.

We always use the notation

$$p_n(t) = \varphi_n(t) \qquad (t > N) \tag{1.3}$$

for the polynomials in (ii) and

$$P(w) = \sum_{\nu=0}^{N} c_\nu w^\nu \tag{1.4}$$

for the *characteristic polynomial* corresponding to (1.1), where $P(0) \neq 0$ according to $c_0 \neq 0$, and $P(1) = 1$ according to (1.2). Conversely, every polynomial (1.4), having the properties $P(0) \neq 0$ and $P(1) = 1$, can be termed as characteristic polynomial since it generates a corresponding two-scale difference equation (1.1).

The polynomials (1.3) are *Appell polynomials.* For $n < 0$ they are equal to the zero polynomial, and for $n \geq 0$ they are defined by the generating function

$$e^{tz}\phi(z) = \sum_{n=0}^{\infty} p_n(t) z^n \qquad (t \in \mathbb{R}), \tag{1.5}$$

cf. [22], where

$$\phi(z) = \mathcal{L}\{\varphi_{-1}\} = \prod_{j=1}^{\infty} P\left(e^{-z/2^j}\right) \tag{1.6}$$

is the Laplace transform of the eigenfunction $\varphi_{-1}$ with $\phi(0) = 1$, cf. [2]. Note that the Appell polynomials $p_n$ are uniquely determined by the characteristic polynomial $P$. For $n \geq 0$ the main term of $p_n$ is equal to $\frac{1}{n!}t^n$, so that in particular $p_0 = 1$.

A pendant to (1.6) is the infinite product

$$S(w) = \prod_{j=0}^{\infty} Q\left(w^{2^j}\right) \tag{1.7}$$

with $Q(0) = 1$. If $Q$ is a holomorphic function for small $|w|$ then also $S$ is such a function, and $S(0) = 1$.

**Proposition 1.2** [ 3 : Proposition 7.6 ] *For a polynomial $Q$ the product* (1.7) *is a rational function if and only if $Q$ is of the form*

$$Q(w) = (1+w)^\alpha \frac{p(w^2)}{p(w)} \tag{1.8}$$

*where $p$ is a characteristic polynomial, and $\alpha \in \mathbb{N}_0$.*

The main topic of [3] consists in a comparison of (1.1) with a second two-scale difference equation

$$\tilde{\lambda}\tilde{\varphi}\left(\frac{t}{2}\right) = \sum_{\nu=0}^{\tilde{N}} \tilde{c}_\nu \tilde{\varphi}(t-\nu) \tag{1.9}$$

for which analogous notations and assumptions are used as for (1.1), in particular $\tilde{\lambda}\tilde{c}_0\tilde{c}_{\tilde{N}} \neq 0$, $\tilde{P}(1) = 1$ and $\tilde{\varphi}_m$ ($m \in \mathbb{Z}$) for the normalized eigenfunction belonging to the eigenvalue $\tilde{\lambda} = 2^m$, as well as $\tilde{p}_m(t)$ for the Appell polynomials belonging to

$$\tilde{\phi}(z) = \mathcal{L}\{\tilde{\varphi}_{-1}\} = \prod_{j=1}^{\infty} \tilde{P}\left(e^{-z/2^j}\right). \tag{1.10}$$

**Definition 1.3** [3: Definitions 3.1 and 4.10] For fixed $n, m \in \mathbb{Z}$ the eigenfunctions $\varphi_n$ and $\tilde{\varphi}_m$ are called *equivalent* $\varphi_n \sim \tilde{\varphi}_m$ if they satisfy the equality

$$\varphi_n(t) = r_0\tilde{\varphi}_m(t) \qquad (t < 1) \tag{1.11}$$

with a certain constant $r_0$. The characteristic polynomials $P$, $\tilde{P}$ are called *equivalent* $P \sim \tilde{P}$ if there exists a constant $\alpha \in \mathbb{Z}$ such that $\varphi_n \sim \tilde{\varphi}_m$ for

$$m - n = \alpha. \tag{1.12}$$

**Proposition 1.4** [3: Lemmas 3.2 and 4.7, Corollary 4.9, Propositions 3.3 and 3.5] *For equivalent eigenfunctions $\varphi_n$, $\tilde{\varphi}_m$ it is $r_0 \neq 0$ in* (1.11)*. Moreover, there exist further constants $r_k$ such that* (1.11) *can be extended to*

$$\varphi_n(t) = \sum_{k=0}^{\infty} r_k\tilde{\varphi}_m(t-k) \qquad (t \in \mathbb{R}). \tag{1.13}$$

*The generating function*

$$R(w) = \sum_{k=0}^{\infty} r_k w^k \tag{1.14}$$

*of the coefficients in* (1.13) *is a rational function satisfying the limit relation*

$$\lim_{w\to 1} \frac{R(w)}{(1-w)^\alpha} = 1 \tag{1.15}$$

*with $\alpha$ from* (1.12)*, and for $w = e^{-z}$ it is representable in the form*

$$R\left(e^{-z}\right) = \frac{z^\alpha \phi(z)}{\tilde{\phi}(z)} \tag{1.16}$$

*with* (1.6), (1.10) *and* (1.12)*. The relation* (1.13) *can be inverted by*

$$\tilde{\varphi}_m(t) = \sum_{k=0}^{\infty} s_k\varphi_n(t-k), \tag{1.17}$$

*with the coefficients $s_k$ from*

$$\frac{1}{R(w)} = \sum_{k=0}^{\infty} s_k w^k. \tag{1.18}$$

The function $R$, defined by (1.16), satisfies the homogeneous equation

$$2^\alpha P(w)R(w) = \tilde{P}(w)R(w^2) \tag{1.19}$$

which is called the *basic functional equation*.

**Proposition 1.5** [3: Proposition 4.4] *Let* (1.19) *with* $\alpha \in \mathbb{Z}$ *have a solution* $R$ *with the representations*

$$R(w) = w^{\alpha_0} R_0(w), \qquad R(w) = (1-w)^{\alpha_1} R_1(w), \tag{1.20}$$

*where the functions* $R_j$ *are continuous at* $w = j$ *with* $R_j(j) \neq 0$ $(j \in \{0,1\})$, *then* $R$ *is a rational function which is uniquely determined up to a constant factor,* $\alpha_0 = 0$ *and* $\alpha_1 = \alpha$.

**Definition 1.6** [3: Definition 5.3] A rational solution $R$ of the basic functional equation (1.19) is called the *canonical solution*, if it is normalized by (1.15), i.e. if

$$R(w) = (1-w)^{\alpha} R_1(w) \tag{1.21}$$

with $R_1(1) = 1$.

The main result concerning equivalent eigenfunctions, i.e. equivalent characteristic polynomials, and the basic functional equation reads:

**Theorem 1.7** [3: Theorem 5.1, Corollary 5.4] *With the notation* (1.12) *the following assertions are equivalent:*

**(a)** $\varphi_n \sim \tilde{\varphi}_m$, *i.e.* $P \sim \tilde{P}$,

**(b)** *the basic functional equation* (1.19) *has a non-zero rational solution* $R$.

*If these assertions are satisfied, then in* (1.13) *the coefficients* $r_k$, *belonging to the equivalent eigenfunctions* $\varphi_n$ *and* $\tilde{\varphi}_m$, *are the coefficients in* (1.14) *of the canonical solution* $R$ *of the basic functional equation* (1.19).

**Definition 1.8** [3: Definition 6.7] The set of non-vanishing pairwise distinct numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$ $(k \in \mathbb{N})$ is called a *cycle* with the *length* $k$ under a mapping $f : \mathbb{C} \mapsto \mathbb{C}$ if $f(\zeta_j) = \zeta_{j+1}$ for $j = 1, \ldots, k-1$ and $f(\zeta_k) = \zeta_1$, cf. [23]. The numbers $\zeta_1, \ldots, \zeta_\ell$ are called *cyclic under* $f$ if each $\zeta_j$ belongs to a cycle of $f$ possibly with a certain multiplicity $m$, where the corresponding cycles are to interpret as $m$ distinct cycles. In the case $f : w \mapsto w^2$ we usually drop "*under* $f$".

For a cycle $\zeta_1, \ldots, \zeta_k$ with the length $k$ (under $w \mapsto w^2$) we have $\zeta^{2^k} = \zeta$ for each $\zeta = \zeta_j$ $(j = 1, \ldots, k)$, and consequently in view of $\zeta \neq 0$ that $\zeta^{2^k - 1} = 1$, i.e. all cyclic numbers are roots of unity with odd root exponents.

**Lemma 1.9** [3: Lemma 6.8] *If* $p$ *and* $q$ *are polynomials with* $q(0) \neq 0$ *and*

$$p(w) = \frac{q(w^2)}{q(w)}, \tag{1.22}$$

*and if* $p$ *does not have symmetric zeros, then the zeros of* $q$ *are cyclic under* $w \mapsto w^2$ *and the zeros of* $p$ *are cyclic under* $w \mapsto -w^2$.

**Lemma 1.10** [3: Algorithm II] *If* $P$ *is a characteristic polynomial with symmetric zeros, then there exist two characteristic polynomials* $R$ *and* $\tilde{P}$ *such that*

$$P(w) = \frac{R(w^2)}{R(w)} \tilde{P}(w),$$

*and* $\tilde{P}$ *does not have symmetric zeros.*

**Definition 1.11** [3: Definition 6.1] A characteristic polynomial $P_0$ of degree $N_0$ is called *minimal* if $P \sim P_0$ implies that $N \geq N_0$.

Minimal characteristic polynomials are uniquely determined in every class of equivalent polynomials. They are related with two-scale symbols of scaling functions having minimum support in [9: p. 119].

**Proposition 1.12** [3: Proposition 6.9] *The characteristic polynomial $P_0$ is minimal if and only if the following conditions are satisfied:*

**(a)** *$P_0$ does not have symmetric zeros,*

**(b)** *$P_0$ does not have cyclic zeros under $w \mapsto -w^2$.*

**Proposition 1.13** [3: Proposition 6.2] *Every characteristic polynomial $P$ has a unique representation of the form*

$$P(w) = \left(\frac{1+w}{2}\right)^{\beta} \frac{p(w^2)}{p(w)} P_0(w) \tag{1.23}$$

*where $p$ is a characteristic polynomial, $P_0$ a minimal characteristic polynomial, and $\beta \in \mathbb{N}_0$.*

For a given characteristic polynomial $P$ the representation (1.23) can be found by means of [3: Algorithm II]. Conversely, every polynomial $P$ with a representation (1.23) is a characteristic polynomial where $P$ is itself a minimal polynomial if and only if both $\beta = 0$ and $p = 1$ in (1.23). If $p$ in (1.23) has $-1$ as a zero with the multiplicity $k$, then it must be $0 \le k \le \beta$ and $-1$ is a zero of $P$ with the multiplicity $\beta - k$.

**Proposition 1.14** [3: Corollary 7.5, 14: Theorem 2] *The shifts $\varphi_{-1}(\cdot + \nu)$ $(\nu \in \mathbb{Z})$ of the eigenfunction $\varphi_{-1}$ are linearly independent if and only if in the representation* (1.23) *of the characteristic polynomial $P$ it holds $p = 1$.*

**Proposition 1.15** [3: Theorem 7.2] *The equality*

$$\sum_{\nu=0}^{N} \varphi_{-1}(t+\nu) = 1 \qquad (|t| < 1) \tag{1.24}$$

*is valid, if and only if in the representation* (1.23) *of the characteristic polynomial $P$ it holds $\beta \ge 1$.*

For Lebesgue-integrable $\varphi_{-1}$ equality (1.24) is well-known, cf. [19], [12] and [4]. In view of Proposition 1.15 we have finally

**Proposition 1.16** [3: Example 2.2] *If $\beta \ge 1$ in the representation* (1.23) *of the characteristic polynomial $P$ and $n \in \mathbb{N}_0$ then it holds the equality*

$$\sum_{\nu=0}^{N} \varphi_{n-1}(t+\nu) = \sum_{k=0}^{n} \frac{a_{n-k}}{k!} B_k(t+N+1) \qquad (|t| < 1), \tag{1.25}$$

*where $B_k(\cdot)$ are the Bernoulli polynomials and $a_n = p_n(0)$ with* (1.5) *and* (1.6).

This paper is motivated by the question concerning the existence of equations of the form

$$\sum_{\nu=0}^{N} \xi_\nu \varphi_n(t+\nu) = Q(t) \qquad (|t| < 1) \tag{1.26}$$

with fixed $n \in \mathbb{Z}$ and a polynomial $Q$. The clue to the success consists in the use of equivalent eigenfunctions being polynomials for great $t$ according to **(ii)**, and in the fundamental factorization (1.23). Nontrivial equations (1.26) exist if and only if the characteristic polynomial is not a minimal polynomial. In the just mentioned case the coefficients $\xi_\nu$ in (1.26) only depend on $\beta$ and $p$ in (1.23), they are independent on the minimal polynomial $P_0$.

We begin with the basic Theorem 2.1 concerning sums of shifted eigenfunctions and with some simple applications. Theorem 3.6 presents additional details in the case of cyclic polynomials. Proposition 1.14 concerning the linear independence of shifts of $\varphi_{-1}$ is transferred to arbitrary eigenfunctions $\varphi_n$ in Theorem 4.4 for $n \in \mathbb{N}_0$, and in Corollary 4.8 for $n \in \mathbb{Z}$, however, in the interval (-1,1), cf. Remark 1.17 and [20]. Theorem 4.7 clarifies the case in which the shifts are linearly dependent. Theorem 5.1 gives a complete survey on the possible coefficients $\xi_\nu$ in (1.26) and Proposition 5.6 on the possible degrees of the polynomials $Q$. In Corollary 5.2.3 it turns out that the coefficients must be certain exponential polynomials in $\nu$.

The special case $Q = 1$ of (1.26), i.e. the partition of unity, is studied in detail, the corresponding results are listed in Theorem 6.1. By means of Theorem 6.3 it can be checked whether given numbers $\xi_\nu$ are possible as coefficients. Proposition 6.4 shows that $\xi_\nu = c$ is only possible in the case that both $n = -1$ and $c = 1$, i.e. in the case (1.24), Example 6.5 that $\xi_\nu = c\,\zeta^\nu$, $\zeta \neq 1$, is only possible if $n = 0$, $p\,(\zeta) = 0$ and $c = \frac{\zeta-1}{\zeta^{N+1}}$, and Proposition 6.6 concerns the case $\xi_\nu = \frac{1}{2}(1 + (-1)^\nu)$.

Proposition 7.2 deals with linear combinations of shifts of an eigenfunction which yield special distributions. Finally, Section 8 deals with different supplements to the foregoing results. In particular, Proposition 8.1 gives necessary and sufficient conditions such that the infinite product (1.7) represents a rational function.

**Remark 1.17** If $\varphi_n$ in (1.26) is Lebesgue-integrable or even continuous then it suffices to consider this equation only in the interval $[0, 1]$ or in $(0, 1)$ (where the term on the left-hand side with $\nu = N$ is a polynomial). But for distributions there do not exist equations in closed intervals, and in $(0, 1)$ we do not have any information concerning the boundary points. In view of the shifts at hand we need an interval with length greater than 1, and the simplest interval of this kind is $(-1, 1)$. In particular, this interval is important in order to find the compatibility conditions in Remark 7.3.3. Note that the validity of equalities in a finite interval is necessary but not sufficient for the corresponding validity for all real numbers, whereas for linear independence it is the opposite.

# 2. Sums of shifted eigenfunctions

In this section we consider linear combinations of shifts of an eigenfunction which are fundamental for later investigations. As before, $P$ and $\tilde{P}$ are two characteristic polynomials of degree $N$ and $\tilde{N}$, respectively.

**Theorem 2.1** *Assume that $P \sim \tilde{P}$. Let $R$ be the canonical solution of* (1.19) *with* (1.18)*. Then for $\ell \geq \tilde{N} + 1$ the normalized eigenfunctions of the two-scale difference equation* (1.1) *satisfy the equalities*

$$\sum_{\nu=0}^{\ell} s_{\ell-\nu}\varphi_n(t+\nu) = \tilde{p}_m(t+\ell) \qquad (|t| < 1) \tag{2.1}$$

*where $m = n + \alpha$, and where $\tilde{p}_m$ are the Appell polynomials belonging to the generating function*

$$\frac{e^{tz} z^\alpha \phi(z)}{R(e^{-z})} = \sum_{m=0}^{\infty} \tilde{p}_m(t) z^m \tag{2.2}$$

*with the function $\phi$ from* (1.6).

**Proof.** According to Theorem 1.7 we have (1.13) with $m = n + \alpha$ for all $n \in \mathbb{Z}$ and the coefficients from (1.14). By Proposition 1.4 this equation can be written as (1.17) with the coefficients from (1.18). Replacing $t$ by $t + \ell$ with $|t| < 1$ and considering (i), we get

$$\tilde{\varphi}_m(t+\ell) = \sum_{k=0}^{\ell} s_k \varphi_n(t+\ell-k) = \sum_{\nu=0}^{\ell} s_{\ell-\nu}\varphi_n(t+\nu).$$

Now, in view of $\ell \geq \tilde{N} + 1$ we have $\tilde{\varphi}_m(t+\ell) = \tilde{p}_m(t+\ell)$ for $t > -1$ with the Appell polynomials $\tilde{p}_m$ having the generating function

$$e^{tz}\tilde{\phi}(z) = \sum_{m=0}^{\infty} \tilde{p}_m(t) z^m. \tag{2.3}$$

According to (1.16) this expansion immediately implies (2.2) ■

As applications we consider at first the both special cases

$$P(w) = \left(\frac{1+w}{2}\right)^k \tilde{P}(w) \tag{2.4}$$

and

$$P(w) = \frac{1+w^k}{2}\tilde{P}(w) \tag{2.5}$$

where $\tilde{P}$ is an arbitrary characteristic polynomial and $k \in \mathbb{N}$, so that in both cases also $N = k + \tilde{N} \in \mathbb{N}$. In the case $P = \tilde{P}$ the assertions of Theorem 2.1 are trivial.

**Proposition 2.2** *If $-1$ is a zero of the characteristic polynomial $P$ with the multiplicity at least $k \geq 1$, then for $\ell \geq N + 1 - k$ the normalized eigenfunctions of the two-scale difference equation* (1.1) *satisfy the equalities*

$$\sum_{\nu=0}^{\ell} \binom{\ell+k-1-\nu}{k-1} \varphi_{m-k}(t+\nu) = \tilde{p}_{km}(t+\ell) \qquad (|t|<1) \tag{2.6}$$

*where $\tilde{p}_{km}$ are the Appell polynomials belonging to the generating function*

$$\frac{e^{tz}z^k\phi(z)}{(1-e^{-z})^k} = \sum_{m=0}^{\infty} \tilde{p}_{km}(t)z^m \tag{2.7}$$

*with the function $\phi$ from* (1.6).

**Proof.** If $-1$ is a zero of $P$ and its multiplicity at least $k$, then $P$ can be written in the form (2.4), and the corresponding basic functional equation (1.19) with $\alpha = k$ has the canonical solution $R(w) = (1-w)^k$, cf. Definition 1.6. Theorem 1.7 implies $P \sim \tilde{P}$, and in view of

$$\frac{1}{R(w)} = \frac{1}{(1-w)^k} = \sum_{\nu=0}^{\infty} \binom{k-1+\nu}{k-1} w^\nu$$

and $\tilde{N} = N - k$, Theorem 2.1 yields the assertions ■

**Remark 2.3** According to

$$\nu^\kappa = \sum_{j=0}^{\kappa} d_{j\kappa} \binom{\nu-N-1}{j} = \sum_{j=0}^{\kappa} d_{j\kappa}(-1)^j \binom{N+j-\nu}{j},$$

where $d_{j\kappa} = \Delta^j (N+1)^\kappa$ are the forward differences with $\Delta p(N) = p(N+1) - p(N)$, we obtain from (2.6) with $\ell = N$, $k = j+1$ and $m = j$ that

$$\sum_{\nu=0}^{N} \nu^\kappa \varphi_{-1}(t+\nu) = \sum_{j=0}^{\kappa} d_{j\kappa}(-1)^j \tilde{p}_{j+1,j}(t+N) \qquad (|t|<1) \tag{2.8}$$

for $0 \leq \kappa \leq k-1$. For $\kappa = 1$ the right-hand side of (2.8) is equal to $P'(1) - t$.

The next result is also a special case of Theorem 2.1, but we shall prove it in a direct way.

**Proposition 2.4** *If the characteristic polynomial $P$ has the form* (2.5)*, then the normalized eigenfunctions of the two-scale difference equation* (1.1) *satisfy the equalities*

$$\sum_{\mu=0}^{\ell} \varphi_{m-1}(t+\mu k) = \frac{1}{k}\tilde{p}_m(t+k\ell) \qquad (-1<t<k), \tag{2.9}$$

*where $\ell \geq \frac{N+1}{k} - 1$ and where $\tilde{p}_m$ are the Appell polynomials belonging to the generating function*

$$\frac{ke^{tz}z\phi(z)}{1-e^{-kz}} = \sum_{m=0}^{\infty} \tilde{p}_m(t)z^m \tag{2.10}$$

*with the function $\phi$ from* (1.6).

**Proof.** In the case (2.5) with $k \in \mathbb{N}$ equation (1.19) with $\alpha = 1$ has the canonical solution $R(w) = \frac{1}{k}(1 - w^k)$, and Theorem 1.7 implies

$$\varphi_{m-1}(t) = \frac{1}{k}\left(\tilde{\varphi}_m(t) - \tilde{\varphi}_m(t-k)\right). \tag{2.11}$$

By summation we obtain

$$\sum_{\mu=0}^{\ell} \varphi_{m-1}(t + \mu k) = \frac{1}{k}\left(\tilde{\varphi}_m(t + \ell k) - \tilde{\varphi}_m(t-k)\right),$$

and owing to (ii) and (1.3) applied to $\tilde{\varphi}_m$ this equation turns over into (2.9) for $N - k(\ell+1) \leq -1 < t < k$. Finally, (2.10) is valid analogously to (2.2) ■

For $k = 1$ the results of Proposition 2.2 and Proposition 2.4 coincide. In particular for $\ell = N$ and $m = 0$ the equalities (2.6) and (2.9) reduce to (1.24) so that $P(-1) = 0$ is a sufficient condition for the validity of (1.24), cf. Proposition 1.15.

In the case (2.5) with $\tilde{N} < k$ equality (2.11) with $m = 0$ implies the relation

$$k\varphi_{-1}\left(t + \frac{N}{2}\right) = 1 \qquad \left(|t| < \frac{k - \tilde{N}}{2}\right),$$

which is a certain pendant to (1.24).

Equality (2.9) is equivalent to the system of equalities

$$\sum_{\mu=0}^{\ell} \varphi_{m-1}(t + \mu k + j) = \frac{1}{k}\tilde{p}_m(t + \ell k + j) \qquad (|t| < 1)$$

for $\ell \geq \frac{N+1}{k} - 1$ and $j = 0, \dots, k-1$, from which for $m = 0$ it follows

$$\sum_{\mu=0}^{[\frac{N-j}{k}]} \varphi_{-1}(t + \mu k + j) = \frac{1}{k} \qquad (|t| < 1) \tag{2.12}$$

where we have used that $\varphi_{-1}(t) = 0$ for $t > N$ and $\tilde{p}_0(t) = 1$.

# 3. Cyclic polynomials

In order to give a further application of Theorem 2.1 we first recall some facts from [23] and [5]. Let $\mathcal{M}$ be a finite set of non-vanishing complex numbers, which is closed under the mapping $w \mapsto w^2$. Then $\mathcal{M}$ contains cyclic numbers, i.e. at least one cycle of pairwise distinct numbers $\zeta_1, \dots, \zeta_k$ in the sense of Definition 1.8. All $\zeta \in \mathcal{M}$ are roots of unity. We consider the square roots $\sqrt{\zeta}$, $-\sqrt{\zeta}$ of elements $\zeta \in \mathcal{M}$ and term the square roots not belonging to $\mathcal{M}$ *absent roots* of $\mathcal{M}$. The set $\Omega$ of the absent roots of $\mathcal{M}$ consists only of pre-periods, it has the same cardinality as $\mathcal{M}$, and the set $\mathcal{M} \cup \otimes$ is also closed under $w \mapsto w^2$. The following lemma is a supplement to [17: Lemma 2.3].

**Lemma 3.1** *Two polynomials $p$ and $q$ with $p(0) \neq 0$ are related by*

$$q(w) = \frac{p(w^2)}{p(w)}, \tag{3.1}$$

*if and only if the zero set of $p$ is closed under $w \mapsto w^2$, and the zeros of $q$ are the corresponding absent roots.*

**Proof.** Let $\mathcal{M} = \{\zeta_1, \dots, \zeta_\ell\}$ and $\Omega = \{\omega_1, \dots, \omega_\ell\}$ be given with the foregoing properties. We define

$$q(w) = \prod_{j=1}^{\ell}(w - \omega_j), \qquad p(w) = a\prod_{j=1}^{\ell}(w - \zeta_j) \tag{3.2}$$

with $a \neq 0$ so that

$$p(w^2) = a \prod_{j=1}^{\ell} \left(w - \sqrt{\zeta_j}\right) \prod_{j=1}^{\ell} \left(w + \sqrt{\zeta_j}\right). \tag{3.3}$$

Since the set of all zeros of (3.3) is equal to $\mathcal{M} \cup \otimes$, and since $q$ has the main coefficient 1, it follows that $p(w^2) = p(w)q(w)$.

Conversely, if (3.1) is given, then the zero set $\mathcal{M}$ of $p(w)$ must be a subset of the zero set of $p(w^2)$, i.e. $\mathcal{M}$ must be closed under $w \mapsto w^2$, and the zero set $\Omega$ of $q(w)$ must be the corresponding set of the absent roots ■

**Remark 3.2**

**1.** If the zero set $\mathcal{M}$ of $p$ contains not only cyclic numbers, but also pre-periods, then $q$ contains symmetric zeros and we can factorize $p = p_1 p_2$, where $p_1$ has the numbers of the pre-periods as zeros, and $p_2$ the cyclic numbers of $\mathcal{M}$. The corresponding factorization of (3.1)

$$q(w) = \frac{p_1(w^2)}{p_1(w)} \frac{p_2(w^2)}{p_2(w)}$$

has the following properties: The first factor of the right-hand side is a rational function containing the symmetric zeros of $q$, and the second factor is a polynomial, cf. Lemma 1.9.

**2.** If the zero set $\mathcal{M}$ of $p$ contains no pre-periods, then $\mathcal{M}$ is cyclic under $w \mapsto w^2$, and $\omega_j = -\zeta_j$ $(j = 1, \ldots, \ell)$ so that

$$q(w) = \prod_{j=1}^{\ell} (w + \zeta_j) = \frac{(-1)^\ell}{a} q(-w).$$

In this case (3.1) turns over into

$$p(w)p(-w) = Cp\left(w^2\right) \tag{3.4}$$

with $C = (-1)^\ell a$. But in (3.4) it is also $C = p(0)$, and for $p(1) \neq 0$ moreover $C = p(-1)$, i.e.

$$p(-1) = p(0). \tag{3.5}$$

Since a cycle cannot contain symmetric numbers, $q(w)$ cannot have symmetric zeros so that (3.4) sharpens (1.22) in Lemma 1.9.

In the sequel we write $R$ instead of $p$ since this polynomial shall be interpreted as canonical solution of the basic functional equation.

**Definition 3.3** A polynomial $R$ is called a *cyclic polynomial* if the zeros of $R$ are cyclic under $w \mapsto w^2$.

In particular, non-vanishing constant polynomials can be considered as trivial cyclic polynomials. For a cyclic polynomial

$$R(w) = \sum_{\nu=0}^{K} r_\nu w^\nu \tag{3.6}$$

we have $R(0) = r_0 \neq 0$. In the sequel we use the notation $\overline{R}(w) = \overline{R(\overline{w})}$.

**Proposition 3.4**

**1.** *A non-zero polynomial $R$ satisfies the equality*

$$R(w)R(-w) = R(0)R\left(w^2\right) \tag{3.7}$$

*if and only if $R$ is a cyclic polynomial.*

**2.** *If $R$ is a cyclic polynomial of degree $K$, then with* (3.6) *we have*

$$r_0 = (-1)^K r_K. \tag{3.8}$$

*Moreover, $\overline{R}$ is a cyclic polynomial, too, and it holds*

$$\overline{R}(w) = Cw^K R\left(\frac{1}{w}\right) \tag{3.9}$$

*where $C$ is a constant with $|C| = 1$. In particular, if $R$ has the form* (1.21) *with real $R_1(1) \neq 0$, then $C = (-1)^\alpha$.*

**3.** *A cyclic polynomial $R$ has the property*

$$R(-1) = 2^\alpha R(0) \tag{3.10}$$

*where $\alpha$ is defined by* (1.21) *with $R_1(1) \neq 0$.*

**Proof. 1.** The if-part of the first assertion follows from Remark 3.2.2, the inversion from Lemma 1.9.

**2.** Substituting (3.6) into (3.7) and comparing the main terms yields (3.8). In view of $\overline{\zeta} = \zeta^{-1}$ for cyclic zeros the polynomial $\overline{R}$ is also cyclic and both sides of (3.9) have the same zeros. This implies (3.9) with a certain constant $C$. Writing $R$ in the form (1.21) with $R_1(1) \neq 0$, then substituting into (3.9) yields

$$C = \lim_{w \to 1} \frac{\overline{R}(w)}{R(w)} = (-1)^\alpha \frac{\overline{R}_1(1)}{R_1(1)}$$

so that $|C| = 1$, and $C = (-1)^\alpha$ for real $R_1(1)$.

**3.** Using once more (1.21) with $R_1(1) \neq 0$ we see that

$$\frac{R\left(w^2\right)}{R(w)} = (1+w)^\alpha \frac{R_1\left(w^2\right)}{R_1(w)} \to 2^\alpha$$

as $w \to 1$, and (3.7) yields (3.10) ■

**Remark 3.5**

**1.** For an arbitrary cyclic polynomial $R$ it is $R(-1) \neq 0$ according to (3.10) and $R(0) \neq 0$. Hence (3.7) implies that $P(w) = \frac{R(-w)}{R(-1)}$ is a characteristic polynomial equivalent to 1, cf. Theorem 1.7. The constant $C$ in (3.9) can also be expressed by $C = (-1)^K \frac{\overline{r}_0}{r_0}$, cf. (3.8).

**2.** For real $R(1) \neq 0$ we have $C = 1$ in (3.9). This equality yields $\overline{r}_\nu = r_{K-\nu}$ for $\nu = 0, \ldots, K$, and the last one

$$\sum_{\nu=0}^{K} \operatorname{Im} r_\nu = 0,$$

and according to (3.8) moreover $\overline{r}_0 = (-1)^K r_0$ and $\overline{r}_K = (-1)^K r_K$. Hence, $r_0$ and $r_K$ are purely imaginary for odd $K$, and real for even $K$. For even $K = 2k$ we also have $\operatorname{Im} r_k = 0$.

**3.** In view of $\overline{\zeta} = \zeta^{-1}$ for arbitrary roots of unity the relation (3.9) is already valid if the zeros of $R$ are only closed under $w \mapsto w^2$ where $|C| = 1$, too. For real $R(1) \neq 0$ we have again $C = 1$, and it follows $R'(1) + \overline{R}'(1) = K$ so that $\operatorname{Re} R'(1) = \operatorname{Re} \overline{R}'(1) = \frac{K}{2}$. Moreover, for real $R(-1) \neq 0$ the degree $K$ must be even.

**4.** The polynomial $\frac{1}{C}\overline{R}$ in (3.9) is the reversed polynomial of $R$, cf. [3], [7].

In the following we consider one single cycle $\zeta, \zeta^2, \ldots, \zeta^{2^{K-1}}$, where $\zeta^{2^K} = \zeta$ with minimal $K$, and the corresponding cyclic polynomial

$$R(w) = r_K \prod_{j=1}^{K} \left(w - \zeta^{2^{j-1}}\right). \tag{3.11}$$

The coefficients $r_\nu$ of the corresponding expansion (3.6) can be calculated by means of the algebraic field theory, cf. [15]. In particular, for even $K = 2k$ all coefficients $r_\nu$ of $R$ are real if $\zeta^{2^k+1} = 1$, because then the zeros $\zeta_j = \zeta^{2^{j-1}}$ satisfy $\overline{\zeta}_\nu = \zeta_{k+\nu}$ for $\nu = 1, \ldots, k$. We restrict ourselves to the case $\zeta \neq 1$ so that $R(1) \neq 0$ and $K \geq 2$ as well as $R(-1) = R(0)$, cf. (3.10) with $\alpha = 0$. Hence, we can consider characteristic polynomials $P$, $\tilde{P}$, which are connected by

$$P(w) = \frac{R(-w)}{R(0)} \tilde{P}(w) \tag{3.12}$$

and by means of (3.7) this equality can be written as

$$P(w)R(w) = \tilde{P}(w) R\left(w^2\right), \tag{3.13}$$

i.e. as basic functional equation (1.19) with $\alpha = 0$. We normalize $R$ by $R(1) = 1$ so that $R$ is the canonical solution of (3.13). Now, Theorem 1.7 yields $P \sim \tilde{P}$, and Theorem 2.1 yields (2.1) with $m = n$ and $\ell \geq \tilde{N} + 1 = N - K + 1$ as well as (2.2) with $\alpha = 0$. The coefficients $s_\nu$ in (2.1) can be characterized in the following way, where we remark that according to $\zeta^{2^K-1} = 1$ and $K \geq 2$ there exists always a smallest odd integer $L \geq 3$ with $\zeta^L = 1$ where $L|(2^K - 1)$.

**Theorem 3.6** *Let $P$ be a characteristic polynomial of the form* (3.12)*, and $R$ the cyclic polynomial* (3.11) *with $R(1) = 1$ and $\zeta^L = 1$ $(L \geq 3)$. Then in* (1.18) *the coefficients $s_\nu$ are L-periodic and they satisfy the relations*

$$s_0 + s_1 + \cdots + s_{L-K} = 0, \tag{3.14}$$

$$s_2 + s_4 + \cdots + s_{2M} = 0 \tag{3.15}$$

*where* $M = [\frac{L-K}{2}]$,

$$s_\nu = -\overline{s}_{L-K-\nu} \qquad \textit{for} \quad \nu = 0, 1, \ldots, L - K \tag{3.16}$$

*and*

$$s_\nu = 0 \qquad \textit{for} \quad \nu = L - K + 1, \ldots, L - 1. \tag{3.17}$$

**Proof.** We define a polynomial $Q$ of degree $L - K$ by means of the factorization $1 - w^L = R(w)Q(w)$, so that

$$\frac{1}{R(w)} = \frac{Q(w)}{1 - w^L}. \tag{3.18}$$

This relation immediately proves

$$Q(w) = \sum_{\nu=0}^{L-K} s_\nu w^\nu, \tag{3.19}$$

(3.14), (3.17) and the periodicity $s_\nu = s_{L+\nu}$. The polynomial $Q$ in (3.18) is a cyclic polynomial with $Q(1) = 0$ and $Q'(1) = -L$. Hence, according to Proposition 3.4.2 with $\alpha = 1$, $Q$ has the property

$$\overline{Q}(w) = -w^{L-K} Q\left(\frac{1}{w}\right),$$

and this implies (3.16). Moreover, Proposition 3.4.3 yields $Q(-1) = 2Q(0)$, i.e.

$$s_0 - s_1 + \ldots + (-1)^{L-K} s_{L-K} = 2s_0.$$

Together with (3.14) this implies (3.15) ■

**Example 3.7** (Case $K = 2$, $L = 3$) The third roots of unity $\zeta = -\frac{1}{2} + \frac{i}{2}\sqrt{3}$ and $\zeta^2 = -\frac{1}{2} - \frac{i}{2}\sqrt{3}$ form a cycle of the length 2. Here

$$R(w) = \frac{1}{3}\left(1 + w + w^2\right)$$

has real coefficients and

$$Q(w) = 3(1 - w)$$

so that $s_0 = 3$, $s_1 = -3$, $s_2 = 0$ and $s_{\nu+3} = s_\nu$ for $\nu \in \mathbb{N}_0$.

**Example 3.8** (Case $K = 3$, $L = 7$) For $\zeta = e^{\frac{2\pi i}{7}}$ the numbers $\zeta$, $\zeta^2$ and $\zeta^4$ form a cycle of the length 3. Here we have

$$R(w) = \frac{i}{\sqrt{7}}\left(-1 - aw + \overline{a}w^2 + w^3\right)$$

with $a = \frac{1}{2}\left(1 + i\sqrt{7}\right)$ and

$$Q(w) = 7(1 - w)\overline{R}(w)$$

so that $s_0 = i\sqrt{7}$, $s_1 = \frac{1}{2}(7 - i\sqrt{7})$, $s_2 = -i\sqrt{7}$, $s_3 = \frac{-1}{2}(7 + i\sqrt{7})$, $s_4 = i\sqrt{7}$, $s_5 = s_6 = 0$, and $s_{\nu+7} = s_\nu$ for $\nu \in \mathbb{N}_0$. The conjugate polynomial $\overline{R}$ is the polynomial (3.11) with $\zeta^3$ instead of $\zeta$, i.e. with respect to the cycle $\zeta^3$, $\zeta^6$ and $\zeta^5$.

**Example 3.9** (Case $K = 4$, $L = 15$) For $\zeta = e^{\frac{2\pi i}{15}}$ the numbers $\zeta$, $\zeta^2$, $\zeta^4$ and $\zeta^8$ form a cycle of the length 4. Here

$$R(w) = -1 + bw + 2w^2 + \overline{b}w^3 - w^4$$

with $b = \frac{1}{2}\left(1 + i\sqrt{15}\right)$ and

$$Q(w) = \left(1 - w^5\right)\left(1 + w + w^2\right)\overline{R}(w)$$

so that $s_0 = -1$, $s_1 = \frac{-1}{2}(1+i\sqrt{15})$, $s_2 = \frac{1}{2}(3-i\sqrt{15})$, $s_3 = 3$, $s_4 = \frac{1}{2}(3+i\sqrt{15})$, $s_5 = \frac{1}{2}(1+i\sqrt{15})$, $s_{11-\nu} = -\overline{s}_\nu$ for $\nu = 0, \ldots, 5$, $s_{12} = s_{13} = s_{14} = 0$, and $s_{\nu+15} = s_\nu$ for $\nu \in \mathbb{N}_0$. The conjugate polynomial $\overline{R}$ is the polynomial (3.11) with $\zeta^7$ instead of $\zeta$, i.e. with respect to the cycle $\zeta^7$, $\zeta^{14}$, $\zeta^{13}$ and $\zeta^{11}$.

The last two examples were elaborated by means of the DERIVE system.

# 4. Linear independence

For stability questions of wavelet decompositions and subdivision schemes it is important that the shifts $\varphi_{-1}(\cdot + \nu)$ $(\nu \in \mathbb{Z})$ of the eigenfunction $\varphi_{-1}$ are *linearly independent*, cf. [10], [14], [20], [21]. In [10] there was given a necessary and sufficient condition for the linear independence of the shifts of a continuous eigenfunction $\varphi_{-1}$, even in the more dimensional case, and in [21] for the shifts of a distribution $\varphi_{-1}$. In [ 14 : Theorem 2 ] this condition was formulated by means of the corresponding symbol (which is proportional to our characteristic polynomial), and in [3] it was simplified by the foregoing Proposition 1.14. In [20] the local linear independence of shifts over any nonempty open subset of $\mathbb{R}$ and its connection to the global linear independence was discussed.

In this section we consider the linear independence over the interval $(-1, 1)$, cf. Remark 1.17. The shifts of an eigenfunction $\varphi_n$ with fixed $n \in \mathbb{Z}$ are called *linearly independent* if

$$\sum_{\nu=0}^{N} \xi_\nu \varphi_n(t + \nu) = 0 \qquad (|t| < 1) \tag{4.1}$$

implies $\xi_\nu = 0$ for $\nu = 0, 1, \ldots, N$, otherwise they are called *linearly dependent*. We recall that $N = \deg P$, cf. (1.4). For our intention we need the following two Lemmas.

**Lemma 4.1** *Let $y$ be a polynomial of degree $n$. Then $k+1$ shifts $y(t + t_\mu)$ are linearly independent in any interval if and only if both $k \le n$ and $t_\mu \ne t_\nu$ for $\mu \ne \nu$.*

**Proof.** Let $a_n$ be the main coefficient of $y$, and let $y_\mu(t) = y(t + t_\mu)$ $(\mu = 0, 1, \ldots, k)$. In the case $k = n$ the corresponding Wronskian has the value

$$W(y_0, \ldots, y_n) = a_n^{n+1} \prod_{j=1}^{n} j^j \prod_{\mu<\nu} (t_\mu - t_\nu).$$

which was calculated by means of the Vandermonde determinant. Hence, $t_\mu \ne t_\nu$ implies the linear independence of $y_0, \ldots, y_n$, and the more of $y_0, \ldots, y_k$ in the case $k < n$.

Conversely, if one of the conditions $k \le n$ and $t_\mu \ne t_\nu$ is violated, then $y_0, \ldots, y_k$ are linearly dependent ■

**Lemma 4.2** *The shifts $\varphi_{-1}(\cdot + \nu)$ $(\nu \in \{0, 1, \ldots, N\})$ of the eigenfunction $\varphi_{-1}$ are linearly independent if and only if in the representation* (1.23) *of the characteristic polynomial $P$ it holds $p = 1$.*

The validity of this lemma follows from Proposition 1.14 and the equivalence between the global to the local linear independence on the unit interval $(0, 1)$ and therefore also on $(-1, 1)$, cf. [18] and [20], p.2.

In the sequel assumptions and assertions concerning eigenfunctions are usually addressed to the representation (1.23) of the corresponding characteristic polynomial $P$. Once for all we introduce the notations $K = \deg p$ and

$$p(w) = \sum_{k=0}^{K} \varrho_k w^k \tag{4.2}$$

so that the degrees in (1.23) are connected by

$$N - N_0 = K + \beta, \tag{4.3}$$

and we introduce the

**Convention 4.3** *If in connection with an eigenfunction $\varphi_n$ of* (1.1) *we use notations as* p *and $P_0$, as well as $N$, $\beta$, $K$ and $N_0$, then we always mean these data from the corresponding characteristic polynomial $P$ in the representation* (1.23) *with* (4.2) *and* (4.3).

The representation (1.23) is equivalent to the basic functional equation (1.19) with $\alpha = \beta$, $\tilde{P} = P_0$ and the canonical solution

$$R(w) = (1-w)^{\beta} p(w) \tag{4.4}$$

of degree $K + \beta$, but there are further possibilities to connect (1.23) with (1.19) as we shall see at once.

**Theorem 4.4** *The shifts of the eigenfunction $\varphi_n$ $(n \in \mathbb{N}_0)$ are linearly independent if and only if $n \geq K - 1$ with $K$ from Convention* 4.3.

**Proof.** The representation (1.23) is equivalent to the basic functional equation (1.19) with $\alpha = 0$, the characteristic polynomial

$$\tilde{P}(w) = \left(\frac{1+w}{2}\right)^{\beta} P_0(w)$$

of degree $\tilde{N} = \beta + N_0$, and the canonical solution $R = p$ with (4.2). Owing to Theorem 1.7 the corresponding eigenfunctions $\varphi_n$ and $\tilde{\varphi}_n$ are related by

$$\varphi_n(t) = \sum_{k=0}^{K} \varrho_k \tilde{\varphi}_n(t-k) \qquad (t \in \mathbb{R}).$$

Therefore the left-hand side of (4.1) can be written as

$$\sum_{\nu=0}^{N} \xi_\nu \varphi_n(t+\nu) = \sum_{\mu=0}^{N} \eta_\mu \tilde{\varphi}_n(t+\mu) \qquad (|t|<1) \tag{4.5}$$

where

$$\eta_\mu = \sum_{\nu=\mu}^{\min(N, K+\mu)} \xi_\nu \varrho_{\nu-\mu} \tag{4.6}$$

for $\mu = 0, 1, \ldots, N$. Given $\varrho_k$ with $\varrho_0 \neq 0$, (4.6) is a linear system with a nonsingular upper triangular matrix. This means that (4.6) is a bijection between $\xi_\nu$ and $\eta_\mu$, i.e. the shifts of $\varphi_n$ are linearly independent if and only if the shifts of $\tilde{\varphi}_n$ in (4.5) are linearly independent.

In order to prove the theorem we have to assume that (4.1) is satisfied. Hence, by $n+1$ differentiations we obtain from (4.5)

$$\sum_{\mu=0}^{\tilde{N}} \eta_\mu \tilde{\varphi}_{-1}(t+\mu) = 0 \qquad (|t|<1)$$

since $\tilde{\varphi}_n(t)$ is a polynomial of degree $n$ for $t > \tilde{N}$. Now, Lemma 4.2 implies $\eta_0 = \ldots = \eta_{\tilde{N}} = 0$ so that (4.1) and (4.5) yield the linear combination

$$\sum_{\mu=\tilde{N}+1}^{N} \eta_\mu \tilde{\varphi}_n(t+\mu) = 0 \qquad (|t| < 1), \tag{4.7}$$

which remains to investigate. Since $\tilde{\varphi}_n(t)$ is a polynomial of degree $n$ for $t > \tilde{N}$, Lemma 4.1 yields that the $K = N - \tilde{N}$ shifts in (4.7) are linearly independent if and only if $K - 1 \leq n$. This proves the theorem ■

In the following we come back to the general case $n \in \mathbb{Z}$, and we say that (4.1) has the *solution* $\underline{\xi} = (\xi_0, \ldots, \xi_N)$ if the coordinates of the vector $\underline{\xi}$ satisfy (4.1). Integrating (4.1) several times, Theorem 4.4 implies:

**Corollary 4.5** *If the equation* (4.1) *has a non-zero solution* $\underline{\xi} = (\xi_0, \ldots, \xi_N)$*, then there exists an integer* $k$ *with* $n < k \leq K - 1$ *such that*

$$\sum_{\nu=0}^{N} \xi_\nu \varphi_k(t+\nu) = C \qquad (|t| < 1) \tag{4.8}$$

*with a constant* $C \neq 0$.

After normalizing (4.8) by $C = 1$ we shall come back to this equation in Section 6.

**Lemma 4.6** *If the equation* (4.1) *has* $k \geq 2$ *linearly independent solutions* $\underline{\xi} = (\xi_0, \ldots, \xi_N)$*, then there are at least* $k - 1$ *linearly independent vectors* $\underline{\xi}$ *satisfying*

$$\sum_{\nu=0}^{N} \xi_\nu \varphi_{n+1}(t+\nu) = 0 \qquad (|t| < 1).$$

**Proof.** By integration of (4.1) with $\underline{\xi} = \underline{\xi}_\mu = (\xi_{\mu 0} \ldots, \xi_{\mu N})$ $(\mu = 1, \ldots, k)$ we get

$$\sum_{\nu=0}^{N} \xi_{\mu\nu} \varphi_{n+1}(t+\nu) = d_\mu \qquad (|t| < 1) \tag{4.9}$$

with certain constants $d_\mu$. The assertion of the lemma is valid if at least $k - 1$ constants $d_\mu$ are vanishing. If this is not the case we can assume that $d_\mu \neq 0$ for $\mu = \ell, \ldots, k$ with $1 \leq \ell \leq k - 1$ and $d_\mu = 0$ otherwise. In this case the validity of the lemma can be seen if we replace (4.9) for $\mu \geq \ell$ by

$$\sum_{\nu=0}^{N} \left( \frac{\xi_{\mu\nu}}{d_\mu} - \frac{\xi_{k\nu}}{d_k} \right) \varphi_{n+1}(t+\nu) = 0 \qquad (|t| < 1) \qquad (\mu = \ell, \ldots, k-1) \,\blacksquare$$

For the next theorem we recall Convention 4.3.

**Theorem 4.7** *For a given eigenfunction* $\varphi_n$ *there exist exactly* $K - n - 1$ *linearly independent vectors* $\underline{\xi}$ *satisfying* (4.1) *when* $-\beta \leq n \leq K - 2$*, and exactly* $K + \beta$ *linearly independent vectors when* $n < -\beta$.

**Proof.** According to the equivalence between (1.23) and (1.19) with $\alpha = \beta$, $\tilde{P} = P_0$ and the canonical solution (4.4) Theorem 1.7 implies

$$\varphi_n(t) = \sum_{k=0}^{K+\beta} r_k \tilde{\varphi}_m(t-k) \qquad (t \in \mathbb{R}). \tag{4.10}$$

In the case $n < -\beta$, i.e. $m < 0$, it holds $\tilde{\varphi}_m(t) = 0$ both for $t < 0$ and $t > \tilde{N} = N - K - \beta$, and (4.10) with $-1 < t < N + 1$ can be written in matrix form

$$\psi(t) = M\tilde{\psi}(t) \qquad (|t| < 1) \tag{4.11}$$

where

$$\psi(t) = (\varphi_n(t), \varphi_n(t+1), \ldots, \varphi_n(t+N))^{\mathrm{T}},$$

$$\tilde{\psi}(t) = (\tilde{\varphi}_m(t), \tilde{\varphi}_m(t+1), \ldots, \tilde{\varphi}_m(t+\tilde{N}))^{\mathrm{T}}$$

and where $M$ is the $(N+1) \times (\tilde{N}+1)$-matrix

$$M = \begin{pmatrix} r_0 & & 0 \\ \vdots & \ddots & \\ & & r_0 \\ r_{K+\beta} & & \\ & \ddots & \vdots \\ 0 & & r_{K+\beta} \end{pmatrix}$$

In view of $r_0 \neq 0$ there exist exactly $K+\beta$ linearly independent vectors $\underline{\xi} = (\xi_0, \ldots, \xi_N)$ satisfying $\underline{\xi}M = \underline{0}$, since the last $K+\beta$ coordinates $\xi_{\tilde{N}+1}, \ldots, \xi_N$ can be chosen arbitrarily, and then the $\tilde{N}+1$ first coordinates $\xi_0, \ldots, \xi_{\tilde{N}}$ are uniquely determined by the recursive system $\underline{\xi}M = \underline{0}$. For each such vector $\underline{\xi}$ equality (4.11) implies (4.1). Conversely, $\underline{\xi}\psi = \underline{0}$ implies $\underline{\xi}M\tilde{\psi} = \underline{0}$. Since $\tilde{P}$ is a minimal polynomial, Theorem 4.4 implies $\underline{\xi}M = \underline{0}$ so that there are exactly $K+\beta$ linearly independent vectors $\underline{\xi}$ satisfying (4.1). In particular, this is valid in the case $n = -\beta - 1$.

As we just have shown, there exist exactly $K+\beta$ linearly independent solutions when $n = -\beta-1$. Hence it follows by Lemma 4.6 and Theorem 4.4 that there exist exactly $K-n-1$ linearly independent solutions when $-\beta \leq n \leq K-2$ ■

A basis for the linearly independent vectors $\underline{\xi}$ of Theorem 4.7 shall be given in Remark 5.4.1. In view of $K-n-1 \leq K+\beta-1$ for $n \geq -\beta$ we have the

**Corollary 4.8** *For an eigenfunction $\varphi_n$ there exist at most $K+\beta$ linearly independent vectors $\underline{\xi}$ satisfying* (4.1)*, so that the shifts of $\varphi_n$ are linearly independent when $K = \beta = 0$.*

# 5. Linear combinations yielding polynomials

The main results of this paper concern the equation (1.26), i.e. the equation

$$\sum_{\nu=0}^{N} \xi_\nu \varphi_n(t+\nu) = Q(t) \qquad (|t| < 1) \tag{5.1}$$

with fixed $n \in \mathbb{Z}$ and a polynomial $Q$. Our aim is to determine explicitly a complete system of linearly independent vectors $\underline{\xi} = (\xi_0, \ldots, \xi_N)$ satisfying (5.1) with any polynomial $Q$, and to determine the possible degrees of $Q$. For a given eigenfunction $\varphi_n$ we recall Convention 4.3, and in particular the relation (4.3) which implies $N \geq N_0$. For the polynomial (4.4) we use the expansion (1.18), i.e.

$$\frac{1}{(1-w)^\beta p(w)} = \sum_{j=0}^{\infty} s_j w^j, \tag{5.2}$$

and by means of the coefficients in (5.2) we build up the $(N+1)$-dimensional vectors

$$\left.\begin{array}{lcl} \underline{s}_1 & = & (\; s_{N_0+1} \quad s_{N_0} \quad \cdots \quad s_0 \quad 0 \quad \cdots \quad 0 \;) \\ \underline{s}_2 & = & (\; s_{N_0+2} \quad s_{N_0+1} \quad \cdots \quad s_0 \quad \cdots \quad 0 \;) \\ \vdots & & \qquad\qquad\qquad \ddots \qquad\qquad \ddots \\ \underline{s}_{N-N_0} & = & (\; s_N \quad s_{N-1} \quad \cdots \quad s_0 \;) \end{array}\right\}. \tag{5.3}$$

These vectors exist for $N > N_0$, i.e. for $P \neq P_0$, and they are linearly independent since $s_0 \neq 0$. If we extend $s_j$ for $j < 0$ by $s_j = 0$ then the vector $\underline{s}_\mu$ $(\mu = 1, \ldots, N-N_0)$ can be written as $\underline{s}_\mu = (s_{N_0+\mu}, \ldots, s_{N_0-N+\mu})$ or shortly $\underline{s}_\mu = (s_{N_0+\mu-\nu})_{\nu=0}^{N}$.

**Theorem 5.1** *For a given eigenfunction $\varphi_n$ ($n \in \mathbb{Z}$) there exist exactly $N - N_0$ linearly independent vectors $\underline{\xi} = (\xi_\nu)_{\nu=0}^N$ satisfying* (5.1) *with certain polynomials $Q$. In the case $N > N_0$ the $N - N_0$ vectors $\underline{s}_\mu$ ($\mu = 1, \ldots, N - N_0$) from* (5.3) *form a basis for the set of these vectors $\underline{\xi}$. Each possible polynomial $Q$ in* (5.1) *is a linear combination of the $N - N_0$ polynomials $\tilde{p}_{n+\beta}(t + N_0 + \mu)$ where $\tilde{p}_m$ are the Appell polynomials from* (2.3) *with* (1.10) *and $\tilde{P} = P_0$.*

**Proof.** First we show that there exist at most $N - N_0$ linearly independent vectors $\underline{\xi}$ satisfying (5.1) with a certain polynomial $Q$. Supposing there would exist $N - N_0 + 1$ such vectors where the degrees of the corresponding polynomials $Q$ are less or equal to $L$. Then by $k = L+1$ differentiations we would obtain $N - N_0 + 1$ linearly independent solutions of the equations

$$\sum_{\nu=0}^{N} \xi_\nu \varphi_{n-k}(t+\nu) = 0 \qquad (|t| < 1),$$

i.e. in view of (4.3), at least $K + \beta + 1$ linearly independent solutions of (4.1) with $n - k$ instead of $n$. But this is impossible by Corollary 4.8.

Hence, for $N = N_0$ there exist no non-zero vectors $\underline{\xi}$ satisfying (5.1) with a polynomial $Q$. In the following let be $N - N_0 \geq 1$. Using the equivalence between (1.23) and (1.19) with $\alpha = \beta$, $\tilde{P} = P_0$ and (4.4), we apply Theorem 2.1 with $m = n + \beta$ and $\ell = N_0 + \mu$ for $\mu \in \{1, \ldots, N - N_0\}$. Hence, (2.1) yields the $N - N_0$ equations

$$\sum_{\nu=0}^{N} s_{N_0+\mu-\nu} \varphi_n(t+\nu) = \tilde{p}_{n+\beta}(t + N_0 + \mu) \qquad (|t| < 1) \tag{5.4}$$

with the extended coefficients $s_j$ from (5.2), i.e. the $N - N_0$ linearly independent vectors $\underline{\xi} = \underline{s}_\mu$ from (5.3) satisfy an equation of the form (5.1). Consequently, these $N - N_0$ vectors form a basis for the set of all vectors satisfying (5.1), and $Q$ is a linear combination of the polynomials $\tilde{p}_{n+\beta}$ from (5.4) ∎

**Corollary 5.2**

**1.** *For a given eigenfunction $\varphi_n$ there exists an equation of the form* (5.1) *with a non-zero vector $\underline{\xi}$ if and only if the corresponding characteristic polynomial $P$ is not minimal.*

**2.** *The maximal degree of a polynomial $Q$ in* (5.1) *is equal to $n + \beta$ when $n + \beta \geq 0$, whereas for $n + \beta < 0$ the polynomial $Q$ in* (5.1) *must necessarily be the zero polynomial.*

**3.** *The coefficients $s_j$ are exponential polynomials in $j$, i.e. they are exponential functions with polynomial coefficients.*

Corollary 5.2.3. follows from a decomposition of the left-hand side of (5.2) in partial fractions which also explains the bases of the exponential functions and the degrees of the polynomial coefficients, cf. [10: Theorem 4.1].

Note that differentiations or integrations of (5.1) with respect to $t$ can only change $n$ and $Q$, but not the coefficients $\xi_\nu$ which remain invariant. In what follows we assume that $P$ is not a minimal polynomial, i.e. $N - N_0 = K + \beta \geq 1$, so that the vectors (5.3) exist. Moreover, we use the forward differences $\Delta \underline{s}_\mu = \underline{s}_{\mu+1} - \underline{s}_\mu$ and the notation $Q = Q_L$ which shall mean that $Q$ is a polynomial of degree $L$ when $L \geq 0$ and the zero polynomial when $L < 0$.

**Proposition 5.3** *For a given eigenfunction $\varphi_n$ with $n \geq -\beta - 1$ and an integer $k$ with both $0 \leq k \leq K + \beta - 1$ and $k \leq n + \beta + 1$ there exist exactly $K + \beta - k$ linearly independent vectors $\underline{\xi} = (\xi_\nu)_{\nu=0}^N$ satisfying* (5.1) *with a certain polynomial $Q = Q_L$ where $L \leq n + \beta - k$. The $K + \beta - k$ vectors $\Delta^k \underline{s}_\mu$, $\mu \in \{1, \ldots, K + \beta - k\}$, with $\underline{s}_\mu$ from* (5.3) *form a basis for the set of these vectors.*

**Proof.** From the equations (5.4) we take the $k$-fold differences with a fixed $k \in \{0, 1, \ldots, K + \beta - 1\}$ so that we obtain the $K + \beta - k$ equations

$$\sum_{\nu=0}^{N} \Delta^k s_{N_0+\mu-\nu} \varphi_n(t+\nu) = \Delta^k \tilde{p}_{n+\beta}(t + N_0 + \mu) \qquad (|t| < 1) \tag{5.5}$$

($\mu \in \{1, \ldots, K+\beta-k\}$). Since $\tilde{p}_{n+\beta}$ is a polynomial in $t$ with the main term $\frac{1}{(n+\beta)!}t^{n+\beta}$ when $n+\beta \geq 0$, the right-hand side of (5.5) is a polynomial with the main term $\frac{1}{(n+\beta-k)!}t^{n+\beta-k}$ when $k \leq n+\beta$, and the zero polynomial when $k = n+\beta+1$. The vectors $\Delta^k \underline{s}_\mu$, $\mu \in \{1, \ldots, K+\beta-k\}$, are linearly independent, since a linear dependence of differences would imply the linear dependence of the original vectors.

It remains to show that, for fixed $k$, there exist no further linearly independent vectors $\underline{\xi}$ satisfying (5.1) with a polynomial $Q = Q_L$ and $L \leq n+\beta-k$. If there would exist such a vector then by $n+\beta-k+1$ differentiations of (5.1) we would obtain $K+\beta-k+1$ linearly independent vectors $\underline{\xi}$ satisfying

$$\sum_{\nu=0}^{N} \xi_\nu \varphi_{k-\beta-1}(t+\nu) = 0 \qquad (|t| < 1).$$

But this is impossible since by Theorem 4.7 with $n = k-\beta-1$ there exist at most $K+\beta-k$ such vectors in both cases $k \geq 1$ and $k = 0$ ■

**Remark 5.4**

**1.** The right-hand side of (5.5) is vanishing when $k > n+\beta$, and in this case an equation of the form (5.1) is only possible with the zero polynomial $Q = 0$. In the case $k = n+\beta+1$ Proposition 5.3 is applicable for $-\beta-1 \leq n \leq K-2$ and yields the following extension of Theorem 4.7 that the $K-n-1$ vectors $\Delta^{n+\beta+1}\underline{s}_\mu$, ($\mu \in \{1, \ldots, K-n-1\}$), form a basis for the solutions of (4.1) in the case $-\beta-1 \leq n \leq K-2$. By differentiation of (4.1) with $n = -\beta-1$ it follows that the $K+\beta$ vectors $\underline{s}_\mu$ remain a basis also for $n < -\beta-1$ which already follows from Theorem 5.1 and Corollary 5.2.2.

**2.** In the case $k = K+\beta-1$ Proposition 5.3 is applicable for $n \geq K-2$ and yields, up to a constant factor, only one equation (5.1), namely (5.5) with $\mu = 1$.

The vectors $\Delta^k \underline{s}_\mu$ in Proposition 5.3 can be determined explicitly where we recall (4.3):

**Proposition 5.5** *For $0 \leq k \leq K+\beta-1$ the vectors $\Delta^k \underline{s}_\mu$ ($\mu = 1, \ldots, K+\beta-k$) with $\underline{s}_\mu$ ($\mu = 1, \ldots, N-N_0$) from (5.3) can be written as*

$$\Delta^k \underline{s}_\mu = (\sigma_{N_0+k+\mu-\nu})_{\nu=0}^N \tag{5.6}$$

*where the coordinates $\sigma_j$ are defined for $j < 0$ by $\sigma_j = 0$, and for $j \geq 0$ and fixed $k$ by*

$$\frac{(1-w)^{k-\beta}}{p(w)} = \sum_{j=0}^\infty \sigma_j w^j. \tag{5.7}$$

**Proof.** According to $s_j = 0$ for $j < 0$ we can write the vectors (5.3) in the short form

$$\underline{s}_\mu = (s_{N_0+\mu-\nu})_{\nu=0}^N.$$

Using the formula

$$\Delta^k \underline{s}_\mu = \sum_{\ell=0}^k \binom{k}{\ell} (-1)^\ell \underline{s}_{k+\mu-\ell}$$

and the notation

$$\sigma_j = \sum_{\ell=0}^j \binom{k}{\ell} (-1)^\ell s_{j-\ell} \tag{5.8}$$

we obtain (5.6), and (5.8) together with (5.2) prove (5.7) ■

The vectors (5.6) read in detail

$$\left.\begin{array}{lcllllllll}
\Delta^k \underline{s}_1 & = & ( & \sigma_{N_0+k+1} & \sigma_{N_0+k} & \cdots & \sigma_0 & 0 & \cdots & 0 \; ) \\
\Delta^k \underline{s}_2 & = & ( & \sigma_{N_0+k+2} & \sigma_{N_0+k+1} & \cdots & & \sigma_0 & \cdots & 0 \; ) \\
\vdots & & & & & \ddots & & & \ddots & \\
\Delta^k \underline{s}_{K+\beta-k} & = & ( & \sigma_N & \sigma_{N-1} & \cdots & & & & \sigma_0 \; )
\end{array}\right\}, \tag{5.9}$$

i.e. $\xi_\nu = \sigma_{N_0+k+\mu-\nu}$ are the coefficients of (5.1). Note that the coefficients $\sigma_j$ depend on $k \in \{0,\ldots,K+\beta-1\}$ according to (5.7), cf. (5.8). In particular, for $k=0$ we have $\sigma_j = s_j$ comparing (5.7) and (5.2).

Finally in this section we consider (5.1) with a given non-zero polynomial $Q$. The case $Q \equiv 0$ was already treated in the previous section. Again we recall Convention 4.3.

**Proposition 5.6** *Let $Q$ be a fixed polynomial of degree $L$. Then in the case $L-\beta \le n \le K-1$ equation* (5.1) *has exactly $K-n$ linearly independent solutions $\underline{\xi} = (\xi_\nu)_{\nu=0}^N$. In the cases $n < L-\beta$ and $n \ge K+L$, respectively, it has no solution. In the remaining case $K \le n < K+L$ it is solvable only for special $Q$, but not for all.*

**Proof.** Let be $L-\beta \le n \le K-1$ so that $0 \le n+\beta-L$ and $n+\beta \le K+\beta-1$. Then for $k = n+\beta-L, n+\beta-L+1, \ldots, n+\beta$ the equations (5.5) show that the monomials $Q = t^\ell$ ($\ell = L, L-1, \ldots, 0$) can recursively be represented in the form (5.1). Hence, also an arbitrary polynomial $Q$ of degree $L$ can be represented in this way, and since for the representation of the last term with $\ell = 0$, i.e. $k = n+\beta$, there exist exactly $K-n$ possibilities according to Proposition 5.3, there exist exactly $K-n$ linearly independent vectors $\underline{\xi}$ satisfying (5.1).

For $n < L-\beta$ there exist no solutions $\underline{\xi}$ of (5.1) according to Corollary 5.2.2. In the case $n \ge K+L$ we choose $k = K+\beta$, and Proposition 5.3 yields that there are no solutions of (5.1).

The case $K \le n < K+L$ is possible only for $L \ge 1$. Here we choose $k = n+\beta-L$, so that Proposition 5.3 yields $K+\beta-(n+\beta-L) = K-n+L \le L$ equations (5.1) with certain polynomials $Q$ of degree $L$ and linearly independent coefficient vectors. But a polynomial of degree $L$ has $L+1$ coefficients, so that not all such polynomials can be represented in the form (5.1) ■

**Remark 5.7** From the complete system of linearly independent solutions $\underline{\xi}$ of the inhomogeneous equation (5.1) with fixed $Q \not\equiv 0$ one obtains the general solution of this equation as weighted arithmetic average with arbitrary (real or complex) coefficients.

# 6. Partitions of unity

The most important special case of (5.1) is the case $Q = 1$, i.e. the equation

$$\sum_{\nu=0}^{N} \xi_\nu \varphi_n(t+\nu) = 1 \qquad (|t| < 1) \tag{6.1}$$

with $n \in \mathbb{Z}$, because all other cases arise from (5.1) by differentiation or integration up to a constant factor, and (6.1) still includes the equation (1.24). In the case $k = n+\beta \ge 0$ all equations (5.5) have the form (6.1). Hence, Proposition 5.3 and Proposition 5.5 both with this $k$, as well as Proposition 5.6 with $L = 0$ immediately imply:

**Theorem 6.1** *In the cases $n < -\beta$ and $n \ge K$, respectively, there exists no vector $\underline{\xi}$ satisfying* (6.1). *In the remaining case $-\beta \le n \le K-1$, the equation* (6.1) *has exactly $K-n$ linearly independent solutions $\underline{\xi}$ with the basis $\underline{\xi}_\mu = \Delta^{n+\beta}\underline{s}_\mu$ ($\mu = 1,\ldots,K-n$), and with $\underline{s}_\mu$ from* (5.3). *The vectors of this basis can be written as*

$$\Delta^{n+\beta}\underline{s}_\mu = (\sigma_{N-K+n+\mu-\nu})_{\nu=0}^N \tag{6.2}$$

*where the coordinates $\sigma_j$ are defined for $j < 0$ by $\sigma_j = 0$, and for $j \ge 0$ and fixed $n$ by*

$$\frac{(1-w)^n}{p(w)} = \sum_{j=0}^{\infty} \sigma_j w^j. \tag{6.3}$$

Formula (6.2) shows that $\xi_\nu = \sigma_{N-K+n+\mu-\nu}$ ($\mu = 1,\ldots,K-n$) are the coefficients of (6.1), and the expansion (6.3) shows that they are independent of $\beta$. Concerning (6.1) cf. also Remark 5.7 with $Q = 1$.

**Remark 6.2**

**1.** According to Theorem 6.1 there exist partitions of unity if and only if $K+\beta \geq 1$, i.e. if $P$ is not a minimal characteristic polynomial. For $n = K-1$ equation (6.1) has exactly one solution $\underline{\xi} = (\xi_\nu)_{\nu=0}^N$ which, in view of (6.2) for $\mu = 1$, is given by

$$\xi_\nu = \sigma_{N-\nu} \tag{6.4}$$

where the coefficients $\sigma_j$ are defined by (6.3) with $n = K-1$.

In particular, in the special case $K = 0$ we have $n = -1$, $p = 1$, and $\beta \geq 1$. Hence expansion (6.3) yields $\sigma_j = 1$ for $j \in \mathbb{N}_0$, so that $\xi_\nu = 1$ $(\nu = 0, 1, \ldots, N)$ according to (6.4), cf. Proposition 1.15.

In the second special case $K = 1$, i.e. $n = 0$, the polynomial (4.2) reads $p(w) = \varrho_0 + (1-\varrho_0)w$ with $\varrho_0 \notin \{0, 1\}$. Hence, (6.3) and (6.4) yield

$$\xi_\nu = \frac{1}{\varrho_0}\left(\frac{\varrho_0 - 1}{\varrho_0}\right)^{N-\nu}. \tag{6.5}$$

**2.** If we choose $p = R$ as in Example 3.7 and

$$P(w) = \frac{R(-w)}{R(0)} = 1 - w + w^2,$$

i.e. $N = K = 2$, $N_0 = \beta = 0$, $P_0 = 1$, then Theorem 6.1 implies that equation (6.1) has the linearly independent solutions $\underline{s}_1 = (-3, 3, 0)$ and $\underline{s}_2 = (0, -3, 3)$ in the case $n = 0$, and the unique solution $\Delta\underline{s}_1 = \underline{s}_2 - \underline{s}_1 = (3, -6, 3)$ in the case $n = 1$.

The next result improves our knowledge about the coefficients in (6.1) where we again recall Convention 4.3.

**Theorem 6.3** *The equality* (6.1) *is valid if and only if first the integer* $n$ *satisfies the inequalities*

$$-\beta \leq n \leq K-1 \tag{6.6}$$

*and second there exists a polynomial* $q$ *of degree at most* $K-n-1$ *with* $q(1) = 1$ *such that*

$$(1-w)^n \frac{q(w)}{p(w)} = \sum_{\nu=0}^{N} \xi_{N-\nu} w^\nu + \mathcal{O}\left(w^{N+1}\right) \qquad (w \to 0). \tag{6.7}$$

**Proof.** If $\underline{\xi}$ satisfies (6.1) then by Theorem 6.1 it follows (6.6) and that $\underline{\xi}$ is representable as weighted arithmetic average of the $K-n$ linearly independent vectors $\Delta^{n+\beta}\underline{s}_1, \ldots, \Delta^{n+\beta}\underline{s}_{K-n}$. Hence, there exist uniquely determined constants $C_0, \ldots, C_{K-n-1}$ with

$$C_0 + \ldots + C_{K-n-1} = 1 \tag{6.8}$$

such that

$$\underline{\xi} = \sum_{\mu=0}^{K-n-1} C_\mu \Delta^{n+\beta} \underline{s}_{K-n-\mu}. \tag{6.9}$$

In view of Proposition 5.5 with $k = n+\beta$ equation (6.9) is equivalent to

$$\sum_{\nu=0}^{N} \xi_{N-\nu} w^\nu = \sum_{\mu=0}^{K-n-1} C_\mu \sum_{\nu=0}^{N-\mu} \sigma_\nu w^{\mu+\nu} \tag{6.10}$$

where the coefficients $\sigma_\nu$ are given by (6.3). The expansion (6.3) implies that

$$\sum_{\nu=0}^{N-\mu} \sigma_\nu w^{\mu+\nu} = \frac{w^\mu (1-w)^n}{p(w)} + \mathcal{O}\left(w^{N+1}\right) \qquad (w \to 0), \tag{6.11}$$

and hence (6.10) can be written in the form (6.7) with

$$q(w) = \sum_{\mu=0}^{K-n-1} C_\mu w^\mu, \tag{6.12}$$

and (6.8) yields $q(1) = 1$. Hence, the both conditions (6.6) and (6.7) are necessary for the validity of (6.1).

Conversely, assume that $n$ satisfies (6.6) and that the polynomial (6.12) with $q(1) = 1$ satisfies (6.7). Using (6.12) and the expansion (6.3) we obtain

$$\begin{aligned}(1-w)^n \frac{q(w)}{p(w)} &= \sum_{\mu=0}^{K-n-1} C_\mu \frac{w^\mu (1-w)^n}{p(w)} \\ &= \sum_{\mu=0}^{K-n-1} C_\mu \sum_{\nu=0}^{N-\mu} \sigma_\nu w^{\mu+\nu} + \mathcal{O}\left(w^{N+1}\right) \qquad (w \to 0).\end{aligned}$$

Comparison with (6.7) implies (6.10) which is equivalent to (6.9) according to Proposition 5.5 with $k = n+\beta$. This means in view of $q(1) = 1$ that the vector $\underline{\xi}$ is representable as weighted arithmetic average of the $K-n$ vectors $\Delta^{n+\beta}\underline{s}_1, \ldots, \Delta^{n+\beta}\underline{s}_{K-n}$. The latter are solutions of (6.1) according to Theorem 6.1. Hence, the vector $\underline{\xi}$ satisfies (6.1), too, so that the both conditions (6.6) and (6.7) are also sufficient for the validity of (6.1) ■

The condition (6.7) of Theorem 6.3 can be applied in two different ways: Either by given $n$ and $q$ it determines the coefficients $\xi_\nu$ of (6.1), or by given numbers $\xi_\nu$ it answers the question whether they can appear as coefficients in (6.1). Since the left-hand side of (6.7) is always a proper fraction the numbers $\xi_\nu$ are exponential polynomials in $\nu$ as in Corollary 5.2.3. The simplest special case $\xi_\nu = c$ in (6.1) yields a generalization of Proposition 1.15:

**Proposition 6.4** *The equality*

$$c \sum_{\nu=0}^{N} \varphi_n(t+\nu) = 1 \qquad (|t| < 1). \tag{6.13}$$

*is valid if and only if* $n = -1$, $c = 1$ *and* $\beta \geq 1$.

**Proof.** According to Theorem 6.3 we have to check (6.6) and (6.7). The asymptotic relation (6.7) with $\xi_\nu = c$ ($\nu = 0, \ldots, N$) can be written in the form

$$(1-w)^n \frac{q(w)}{p(w)} = \frac{c}{1-w} + \mathcal{O}\left(w^{N+1}\right) \qquad (w \to 0). \tag{6.14}$$

We show that this relation is equivalent to

$$p(w) = \frac{1}{c}(1-w)^{n+1} q(w). \tag{6.15}$$

In the case $n \geq 0$ the relation (6.14) implies

$$(1-w)^{n+1} q(w) = c p(w) + \mathcal{O}\left(w^{N+1}\right) \qquad (w \to 0)$$

where $\deg q \leq K-n-1$ so that both polynomials in this relation have a degree at most $K \leq N$, and it follows (6.15). In the case $n < 0$ the relation (6.14) implies

$$(1-w) q(w) = c(1-w)^{-n} p(w) + \mathcal{O}\left(w^{N+1}\right) \qquad (w \to 0).$$

In view of $-n \leq \beta$ and (4.3) both polynomials in this relation have a degree at most $K+\beta \leq N$ so that it follows (6.15), too. In view of $p(1) = q(1) = 1$ equation (6.15) is satisfied if and only if $n = -1$ and $c = 1$ where

$$q(w) = p(w). \tag{6.16}$$

Since for $n = -1$ the inequalities (6.6) reduce to $\beta \geq 1$ the proof is finished ■

Under the conditions of Proposition 6.4 it holds (6.16) so that comparison of (4.2) with (6.12) yields $C_\mu = \varrho_\mu$ ($\mu = 0, \ldots, K$) for the coefficients in (6.9). Analogously, Theorem 6.3 yields in a straightforward way:

**Example 6.5** In the case $\xi_\nu = c\,\zeta^\nu$, $\zeta \neq 1$, the equality (6.1) is valid if and only if $n = 0$, $p(\zeta) = 0$ and $c = \frac{\zeta - 1}{\zeta^{N+1}}$, cf. the special case (6.5) with $\zeta = \frac{\varrho_0}{\varrho_0 - 1}$.

The differentiation of (6.1) with the coefficients of Example 6.5 yields a certain connection to [10: Theorem 4.1]. In order to study a more complicated example we come back to the equality (2.12) in the case $k = 2$ and $j = 0$ where we again recall Convention 4.3.

**Proposition 6.6** *The equality*

$$c \sum_{j=0}^{[\frac{N}{2}]} \varphi_n(t + 2j) = 1 \qquad (|t| < 1) \tag{6.17}$$

*is valid if and only if either $K \leq N-1$, $n = -1$, $p(-1) = 0$ and $c = 2$ or $K = N$ even, $n = 0$, $p(-1) = -1$ and $c = \frac{2}{1+\overline{p}'(1)}$.*

**Proof.** According to Theorem 6.3 we have to check (6.6) and (6.7) now with $\xi_\nu = \frac{c}{2}(1+(-1)^\nu)$ $(\nu = 0, \dots, N)$ where $c \neq 0$ in view of (6.17). Owing to

$$\sum_{\nu=0}^{N} \xi_{N-\nu} w^\nu = \frac{cw^r}{1-w^2} + \mathcal{O}\left(w^{N+1}\right) \qquad (w \to 0), \tag{6.18}$$

where $r \in \{0, 1\}$ such that $r \equiv N \bmod 2$, it follows as in the proof of Proposition 6.4 that, for the actual $\xi_\nu$, the asymptotic relation (6.7) is equivalent to

$$(1-w)^{n+1}(1+w)q(w) = cw^r p(w) \tag{6.19}$$

so far as $K \leq N - 1$. This is satisfied if and only if $n = -1$, $p(-1) = 0$ and $c = 2$ in view of $p(1) = q(1) = 1$.

In the case $K = N$ relation (4.3) implies $N_0 = \beta = 0$ so that (6.6) specializes to $0 \leq n \leq K-1$, and (6.7) is equivalent to

$$(1-w)^{n+1}(1+w)q(w) = cw^r p(w) + dw^{K+1} \tag{6.20}$$

with a certain constant $d$. For $w = 1$ it follows $d = -c$ in view of $p(1) = 1$, hence for $w = -1$ we obtain $p(-1) = -1$ in view of $(-1)^r = (-1)^K$. According to $N_0 = \beta = 0$ the representation (1.23) reduces to

$$P(w) = \frac{p(w^2)}{p(w)} \tag{6.21}$$

so that by Lemma 3.1 the zeros of p are closed under $w \mapsto w^2$. According to Remark 3.5.3 the degree $K$ is even, so that $r = 0$ in (6.20), and it holds

$$\lim_{w \to 1} \frac{p(w) - w^{K+1}}{1-w} = K + 1 - p'(1) = 1 + \overline{p}'(1) \neq 0.$$

Hence, (6.20) with $d = -c$ implies both $n = 0$ and $c = \frac{2}{1+\overline{p}'(1)}$ in view of $q(1) = 1$. Conversely, in the case $K = N$ even, where $P$ has the form (6.21), equation (6.20) is satisfied if $n = 0$, $p(-1) = -1$ and $c = \frac{2}{1+\overline{p}'(1)}$ where

$$q(w) = c\,\frac{p(w) - w^{K+1}}{1-w^2}$$

is a polynomial of degree $K - 1$ and $q(1) = 1$ ■

Note that (2.5) with $k = 2^r$ $(r \in \mathbb{N}_0)$ is sufficient, but not necessary for $p(-1) = 0$. An example for the case $N = K = 4$ is given by the cyclic polynomial $p(w) = R(w)$ from Example 3.9 with $c = \frac{1}{12}(3 - i\sqrt{15})$ and $q(w) = c(-1 + bw + w^2 + w^3)$.

# 7. Linear combinations yielding special distributions

Next we consider linear combinations of shifts of an eigenfunction where the sums can be expressed by means of periodic distributions. As preparation we introduce the infinite vector

$$\Psi(t) = (\varphi_n(t), \varphi_n(t+1), \varphi_n(t+2), \ldots)^{\mathrm{T}} \tag{7.1}$$

and the infinite two-slanted matrix

$$\boldsymbol{A} = (c_{2j-k}) \qquad (j, k \geq 0) \tag{7.2}$$

with $c_j = 0$ for $j \notin \{0, \ldots, N\}$, so that the solution $\varphi = \varphi_n$ of (1.1) with $\lambda = 2^n$ $(n \in \mathbb{Z})$ is equivalent to the solution of

$$2^n \Psi\left(\frac{t}{2}\right) = \boldsymbol{A}\Psi(t) \qquad (t < 1), \tag{7.3}$$

both equations subject to **(i)**, **(ii)** and **(iii)**, cf. [13], [4] and the literature quoted there. We also consider the (finite) vector

$$\psi(t) = (\varphi_n(t), \varphi_n(t+1), \ldots, \varphi_n(t+N))^{\mathrm{T}} \tag{7.4}$$

and the matrix

$$A = (c_{2j-k}) \qquad (0 \leq j, k \leq N). \tag{7.5}$$

Introducing suitable block matrices $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{O}$ where $\boldsymbol{O}$ is a zero matrix, we can split $\Psi$ and $\boldsymbol{A}$ into

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \Psi(t+N+1) \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} A & \boldsymbol{B} \\ \boldsymbol{O} & \boldsymbol{C} \end{pmatrix}.$$

and (7.3) can be written as

$$2^n \psi\left(\frac{t}{2}\right) = A\psi(t) + \boldsymbol{B}\Psi(t+N+1), \qquad 2^n \Psi\left(\frac{t}{2} + N + 1\right) = \boldsymbol{C}\Psi(t+N+1). \tag{7.6}$$

It suffices to restrict $t$ to $|t| < 1$ where the components of $\Psi(t+N+1)$ are known polynomials. Hence we only have to deal with the first equation in (7.6).

The case that the characteristic polynomial $P$ has symmetric zeros can be excluded by application of Lemma 1.10 and Theorem 1.7, and changing from $P$ to $\tilde{P}$. This has the following consequence:

**Lemma 7.1** *If $P$ does not have symmetric zeros, then all eigenvalues of $A$ are non-vanishing.*

**Proof.** Let $A$ have the eigenvalue $\mu = 0$, and let $x = (x_0, x_1, \ldots, x_N)$ be the corresponding left eigenvector. According to [4: Theorem 2.3/(i)], $x$ can be extended to a left eigenvector $\boldsymbol{x} = (x_0, x_1, \ldots)$ of $\boldsymbol{A}$, i.e.

$$\sum_{j=0}^{\infty} x_j c_{2j-k} = 0$$

for $k \in \mathbb{N}_0$, or for $k = 2\ell$ and $k = 2\ell + 1$, respectively, both

$$\sum_{j=0}^{[\frac{N}{2}]} c_{2j} x_{j+\ell} = 0 \quad \text{and} \quad \sum_{j=1}^{[\frac{N+1}{2}]} c_{2j-1} x_{j+\ell} = 0.$$

The last equations are linear difference equations with constant coefficients which have a common non-trivial solution only in the case that the polynomials

$$\sum_{j=0}^{[\frac{N}{2}]} c_{2j} w^j = 0, \qquad \sum_{j=1}^{[\frac{N+1}{2}]} c_{2j-1} w^j = 0$$

have a common zero $\zeta \neq 0$. But then $\pm\sqrt{\zeta}$ must be zeros of $P(w)$, which is a contradiction to the hypothesis ■

Let $A = T^{-1}JT$ be Jordan's normal form of the matrix (7.5) and $\mu_0, \mu_1, \ldots, \mu_N$ the algebraic eigenvalues of $A$, i.e. the diagonal elements of $J$. The $i^{th}$ row of $T$ we denote by $T_i$. By means of 1-periodic distributions $Q_i^+(\cdot)$ and $Q_i^-(\cdot)$ we define

$$Q_i(t) = \begin{cases} Q_i^+\left(\frac{\ln t}{\ln 2}\right) & \text{for} \quad 0 < t < 1, \\ 0 & \text{for} \quad t = 0, \\ Q_i^-\left(\frac{\ln |t|}{\ln 2}\right) & \text{for} \quad -1 < t < 0, \end{cases} \tag{7.7}$$

for $i = 0, 1, \ldots, N$. The definition at $t = 0$ shall indicate that the distribution $Q_i$ does not contain a part which is concentrated in 0.

**Proposition 7.2** *Let $A$ be non-singular and diagonalizable, $-n \in \mathbb{N}$, $\alpha_i = n - \frac{\ln \mu_i}{\ln 2}$ and $-\alpha_i \notin \mathbb{N}$ for $i = 0, 1, \ldots, N$. Then there exist distributions* (7.7) *such that*

$$T_i\psi(t) = |t|^{\alpha_i} Q_i(t) \qquad (|t| < 1). \tag{7.8}$$

**Proof.** According to $-n \in \mathbb{N}$ we have $\Psi = \underline{0}$ in (7.6) and these equations reduce to

$$2^n \psi\left(\frac{t}{2}\right) = A\psi(t) \qquad (|t| < 1). \tag{7.9}$$

The diagonalizability of $A$ means $J = \operatorname{diag}(\mu_0, \mu_1, \ldots, \mu_N)$, so that (7.9) is equivalent to

$$2^n T_i \psi\left(\frac{t}{2}\right) = \mu_i T_i \psi(t) \qquad (|t| < 1) \tag{7.10}$$

for $i = 0, \ldots, N$. Obviously, the general solution of this equation has the structure (7.8) ■

**Remark 7.3**

**1.** With the notation $T = (\tau_{ij})$ we can write (7.8) in the form

$$\sum_{j=0}^{N} \tau_{ij}\varphi_n(t + j) = |t|^{\alpha_j} Q_i(t) \qquad (|t| < 1),$$

$i = 0, \ldots, N$, and we have obtained linear combinations of shifted eigenfunctions of (1.1) which are expressed by means of periodic distributions.

**2.** With the notation

$$\pi(t) = \left(|t|^{\alpha_0} Q_0(t), \ldots, |t|^{\alpha_N} Q_N(t)\right)^{\mathrm{T}} \tag{7.11}$$

we can gather up (7.8) in the form $T\psi(t) = \pi(t)$, so that

$$\psi(t) = T^{-1}\pi(t) \qquad (|t| < 1). \tag{7.12}$$

**3.** Denoting by $T_i^{-1}$ the $i^{th}$ row of $T^{-1}$, the distributions $Q_i$ must satisfy the compatibility conditions

$$T_i^{-1}\pi(t) = T_{i+1}^{-1}\pi(t-1) \qquad (0 < t < 1)$$

for $i = 0, \ldots, N-1$, whereas $T_0^{-1}\pi(t-1) = T_N^{-1}\pi(t) = 0$ for $0 < t < 1$.

The hypotheses of Proposition 7.2 can be weakened by means of

**Lemma 7.4** *A special solution of the inhomogeneous equation*

$$2^n \varphi\left(\frac{t}{2}\right) = \mu\varphi(t) + t^k$$

*with $k \in \mathbb{N}_0$ reads, for $\mu \neq 2^{n-k}$,*

$$\varphi(t) = \frac{2^k}{2^n - \mu 2^k} t^k,$$

*and for $\mu = 2^{n-k}$*

$$\varphi(t) = -\frac{2^{k-n}}{\ln 2}\, t^k \ln |t|.$$

In the case $n \geq 0$ we have to solve the equations

$$2^n T_i \psi\left(\frac{t}{2}\right) = \mu_i T_i \psi(t) + T_i \boldsymbol{B} \Psi(t + N + 1) \tag{7.13}$$

instead of (7.10). Since the inhomogeneous term is a polynomial, the general solution can easily be constructed by means of (7.8) and Lemma 7.4.

In the general case that $A$ is not diagonalizable let $J_j$ be a $q \times q$-Jordan block of $J$ with upper unities belonging to the eigenvalue $\mu_j$. Then for $-n \in \mathbb{N}$ besides of (7.10) with $i = j$ we have to solve the equations

$$2^n T_i \psi\left(\frac{t}{2}\right) = \mu_j T_i \psi(t) + T_{i+1} \psi(t) \tag{7.14}$$

for $i = j-1, \ldots, j-q-1$. The first of these equations has the solution

$$T_{j-1}\psi(t) = |t|^{\alpha_j}\left(Q_{j-1}(t) - \frac{1}{\mu_j \ln 2} Q_j(t) \ln|t|\right)$$

with $Q_i$ as in (7.7), and the solutions of the following equations have a similar structure, however with higher powers of $\ln|t|$, cf. [16: Lemma 3.9]. The corresponding inhomogeneous equations for $n \geq 0$ can be treated analogously.

In the case that $-\alpha_i = q_i$ ($q_i \in \mathbb{N}$) the general solution of (7.10) reads

$$T_i \psi(t) = t^{-q_i} Q_i(t) + a_i \delta^{(q_i - 1)}(t)$$

with an arbitrary constant $a_i$. Again, for $n \geq 0$ the corresponding inhomogeneous equation (7.13) can be solved by means of Lemma 7.4, but the solution of (7.14) remains an open problem by appearing $\delta$-distributions.

# 8. Supplements

In this section we give four supplements to the foregoing results.

**8.1. Infinite products.** The first result generalizes Proposition 1.2 concerning the rationality of the infinite product (1.7). Obviously, $S(0) = 1$ is a necessary condition for the convergence of (1.7) at $w = 0$.

**Proposition 8.1** *The function* (1.7) *with* $S(0) = 1$ *is rational if and only if both* $Q$ *is a rational function with* $Q(0) = 1$ *and there exists an integer* $\alpha \in \mathbb{Z}$ *such that*

$$\lim_{w \to 1} (1 - w)^\alpha S(w) = c \tag{8.1}$$

*with* $c \neq 0$. *If* $S$ *satisfies* (8.1) *then* $Q(1) = 2^\alpha$.

**Proof.** If $S$ is a rational function then it holds (8.1) with a certain $\alpha \in \mathbb{Z}$. Moreover, (1.7) can be written as

$$S(w) = S(w^2) Q(w) \tag{8.2}$$

so that $Q$ is a rational function with $Q(0) = 1$ since $S(0) = 1$.

Conversely, let $Q$ be a rational function with $Q(0) = 1$ and let $S$, defined by (1.7), satisfy (8.1) with an integer $\alpha$ and a constant $c \neq 0$. Then equality (8.2) implies

$$Q(1) = \lim_{w \to 1} (1 + w)^\alpha \frac{(1 - w)^\alpha S(w)}{(1 - w^2)^\alpha S(w^2)} = 2^\alpha,$$

and owing to $Q(0) = 1$ the rational function $Q$ can be written in the form

$$Q(w) = \frac{2^\alpha P(w)}{\tilde{P}(w)} \tag{8.3}$$

where $P$ and $\tilde{P}$ are some characteristic polynomials. Substituting (8.3) into (8.2) we get the basic functional equation (1.19) with $R = \frac{1}{S}$, and Proposition 1.5 yields the rationality of $R$, and hence also of $S$ ■

From (8.2) and (1.21) with $R = \frac{1}{S}$ we immediately obtain the generalization of Proposition 1.2:

**Corollary 8.2** *The function* (1.7) *with* $S(0) = 1$ *is rational if and only if* $Q$ *is of the form*

$$Q(w) = (1+w)^{\alpha}\frac{R_1(w^2)}{R_1(w)} \tag{8.4}$$

*where* $R_1$ *is an arbitrary rational function with* $R_1(0)R_1(1) \neq 0$, *and* $\alpha \in \mathbb{Z}$.

**8.2. Cyclic polynomials.** In addition to the three examples from Section 3 we now present some results concerning the more complicated case $K = 5$, $L = 31$ which were found by means of the DERIVE system. Let $\zeta = e^{\frac{2\pi i}{31}}$ then there are the six cycles of length 5:

$$\begin{array}{llllllllllll} 1) & \zeta, & \zeta^2, & \zeta^4, & \zeta^8, & \zeta^{16}, & 2) & \zeta^{30}, & \zeta^{29}, & \zeta^{27}, & \zeta^{23}, & \zeta^{15}, \\ 3) & \zeta^7, & \zeta^{14}, & \zeta^{28}, & \zeta^{25}, & \zeta^{19}, & 4) & \zeta^{24}, & \zeta^{17}, & \zeta^3, & \zeta^6, & \zeta^{12}, \\ 5) & \zeta^5, & \zeta^{10}, & \zeta^{20}, & \zeta^9, & \zeta^{18}, & 6) & \zeta^{26}, & \zeta^{21}, & \zeta^{11}, & \zeta^{22}, & \zeta^{13}. \end{array}$$

According to Remark 3.5.2 the corresponding cyclic polynomials must have the form

$$R_n(w) = \frac{i}{\varrho_n}\left(1 + (a_n + ib_n)w + (c_n + id_n)w^2 + (-c_n + id_n)w^3 + (-a_n + ib_n)w^4 - w^5\right)$$

$(n = 1, \ldots, 6)$ with

$$\varrho_n = -2(b_n + d_n) \tag{8.5}$$

owing to $R_n(1) = 1$. The zeros of $R_{2n}$ are the conjugates of the zeros of $R_{2n-1}$ so that

$$a_{2n} = a_{2n-1}, \quad b_{2n} = -b_{2n-1}, \quad c_{2n} = c_{2n-1}, \quad d_{2n} = -d_{2n-1}, \quad (n = 1, 2, 3),$$

and according to (8.5) also $\varrho_{2n} = -\varrho_{2n-1}$. Hence $R_{2n}(w) = \overline{R}_{2n-1}(w)$, and it suffices to consider odd $n$ only. Moreover, the coefficients satisfy the relations

$$c_n = a_n - \frac{1}{2}$$

and

$$d_1 = b_3 - b_5, \quad d_3 = b_5 - b_1, \quad d_5 = b_1 - b_3. \tag{8.6}$$

The coefficients $a_n$ are the zeros of

$$2a^3 - a^2 - 5a + 2 = 0.$$

With the notation $\alpha = \frac{1}{3}\arctan(\frac{3}{2}\sqrt{3})$ they read

$$\begin{array}{lcllcr} a_1 & = & \frac{1}{6} - \frac{\sqrt{31}}{3}\sin\left(\alpha + \frac{\pi}{3}\right) & \approx & -1.541936, \\ a_3 & = & \frac{1}{6} + \frac{\sqrt{31}}{3}\sin\alpha & \approx & 0.393401, \\ a_5 & = & \frac{1}{6} + \frac{\sqrt{31}}{3}\cos\left(\alpha + \frac{\pi}{6}\right) & \approx & 1.648535. \end{array}$$

The coefficients $b_n$ are the zeros of

$$2b^3 - \sqrt{31}\,\left(b^2 - 1\right) = 0.$$

With the notation $\beta = \frac{1}{3}\arcsin(\frac{23}{31})$ they read

$$\begin{array}{lcllr} b_1 & = & \frac{\sqrt{31}}{6}\left(1 + 2\cos\left(\beta + \frac{\pi}{6}\right)\right) & \approx & 2.217995, \\ b_3 & = & \frac{\sqrt{31}}{6}\left(1 - 2\sin\left(\beta + \frac{\pi}{3}\right)\right) & \approx & -0.872561, \\ b_5 & = & \frac{\sqrt{31}}{6}\left(1 + 2\sin\beta\right) & \approx & 1.438448. \end{array}$$

Though we can calculate $d_n$ and $\varrho_n$ from (8.6) and (8.5), respectively, it is possible to calculate them directly from

$$d^3 - \frac{31}{4}d - \sqrt{31} = 0$$

and

$$\varrho^3 + \sqrt{31}\,\varrho^2 - 31\varrho + \sqrt{31} = 0,$$

namely with $\gamma = \frac{1}{3}\arctan\left(\frac{12\sqrt{3}}{23}\right)$

$$\begin{aligned} d_1 &= -\sqrt{\tfrac{31}{3}}\cos\left(\gamma + \tfrac{\pi}{6}\right) &\approx& -2.311010, \\ d_3 &= -\sqrt{\tfrac{31}{3}}\sin\gamma &\approx& -0.779547, \\ d_5 &= \sqrt{\tfrac{31}{3}}\sin\left(\gamma + \tfrac{\pi}{3}\right) &\approx& 3.090557, \end{aligned}$$

and with $\beta$ as before

$$\begin{aligned} \varrho_1 &= \tfrac{\sqrt{31}}{3}\left(-1 + 4\sin\beta\right) &\approx& 0.186029, \\ \varrho_3 &= \tfrac{\sqrt{31}}{3}\left(-1 + 4\cos\left(\beta + \tfrac{\pi}{6}\right)\right) &\approx& 3.304217, \\ \varrho_5 &= \tfrac{\sqrt{31}}{3}\left(-1 - 4\sin\left(\beta + \tfrac{\pi}{3}\right)\right) &\approx& -9.058010. \end{aligned}$$

There exist further relations as

$$\begin{aligned} a_1^2 + b_1^2 &= 4 + 2a_5, \qquad & 8\left(c_1^2 + d_1^2\right) &= 87 - \varrho_3^2, \\ a_3^2 + b_3^2 &= 4 + 2a_1, \qquad & 8\left(c_3^2 + d_3^2\right) &= 87 - \varrho_5^2, \\ a_5^2 + b_5^2 &= 4 + 2a_3, \qquad & 8\left(c_5^2 + d_5^2\right) &= 87 - \varrho_1^2, \end{aligned}$$

where the values $w = \varrho_n^2$ are the zeros of

$$w^3 - 31\left(3w^2 - 29w + 1\right) = 0,$$

and also similar relations as before for the coefficients of the real polynomials $\varrho_{2n-1}^2 R_{2n-1}R_{2n}$ $(n = 1, 2, 3)$. In view of

$$\frac{1}{R_1(w)} = 31\frac{1-w}{1-w^{31}}R_2(w)R_3(w)R_4(w)R_5(w)R_6(w)$$

these relations could be useful to simplify the calculations of the coefficients $s_\nu$ from (3.19) for $R = R_1$ in (3.18), but we do not intend to present them, too.

**8.3. Expansions of rational functions.** Let $s_\nu$ $(\nu \in \mathbb{N}_0)$ be a sequence with the (formal) generating function

$$S(w) = \sum_{\nu=0}^{\infty} s_\nu w^\nu, \tag{8.7}$$

let be $s_\nu = 0$ for $\nu < 0$ and

$$\underline{S}_n = (s_n, s_{n+1}, s_{n+2}, \ldots) \tag{8.8}$$

for $n \in \mathbb{Z}$. A weaker version of the following proposition concerning the rationality of $S$ follows of our results in Section 4, but we shall give here a proof without these results.

**Proposition 8.3** *Under the foregoing notations assume that*

$$\sum_{n=\ell}^{L+\ell} C_n \underline{S}_n = 0 \tag{8.9}$$

*with* $C_\ell C_{L+\ell} \neq 0$ $(L \in \mathbb{N}, \ell \in \mathbb{Z}, \ell > -L)$*. Then* (8.7) *is a rational function of the form*

$$S(w) = \frac{q_{L+\ell-1}(w)}{q(w)} \tag{8.10}$$

*where the numerator is a polynomial of degree at most* $L + \ell - 1$ *and*

$$q(w) = \sum_{\nu=0}^{L} C_{L+\ell-\nu} w^\nu. \tag{8.11}$$

*Conversely, if $S$ from* (8.7) *has the form* (8.10)*, then it holds not only* (8.9) *but $L+1$ arbitrary sequences* (8.8) *with $n \geq \ell$ are linearly dependent.*

**Proof.** From (8.7) we obtain

$$\sum_{\nu=0}^{\infty} s_{n+\nu} w^{\nu} = \begin{cases} w^{-n} S(w) & \text{for} \quad n \leq 0, \\ w^{-n} \left( S(w) - \sum\limits_{\nu=0}^{n-1} s_{\nu} w^{\nu} \right) & \text{for} \quad n > 0. \end{cases} \tag{8.12}$$

Replacing (8.8) into (8.9) yields

$$\sum_{n=\ell}^{L+\ell} C_n s_{n+\nu} = 0 \qquad (\nu \in \mathbb{N}_0). \tag{8.13}$$

By multiplication with $w^{\nu}$, summing over $\nu$ and considering (8.12) we get

$$S(w) \sum_{n=\ell}^{L+\ell} \frac{C_n}{w^n} = \sum_{n=m}^{L+\ell} \frac{C_n}{w^n} \sum_{\nu=0}^{n-1} s_{\nu} w^{\nu}$$

or

$$S(w) = \frac{1}{q(w)} \sum_{n=m}^{L+\ell} C_n \sum_{\nu=0}^{n-1} s_{\nu} w^{L+\ell+\nu-n}$$

with (8.11) and $m = \max(1, \ell)$.

Conversely, if (8.7) has the form (8.10), then by means of (8.12) it holds

$$q(w) \sum_{\nu=0}^{\infty} s_{n+\nu} w^{\nu} = \frac{1}{w^n} \left( q_{L+\ell-1}(w) - q(w) \sum_{\nu=0}^{n-1} s_{\nu} w^{\nu} \right) \tag{8.14}$$

where the last sum over $\nu$ vanishes for $n < 0$. Since the left-hand side of (8.14) is regular at $w = 0$, the right-hand side must be a polynomial $g$ depending on $n$ with

$$\deg g \leq \begin{cases} L + \ell - n - 1 & \text{for} \quad n \leq 0, \\ L - 1 & \text{for} \quad n > 0. \end{cases}$$

In view of $n \geq \ell$ it is $\deg g \leq L - 1$ in any case. But $L + 1$ polynomials of degree at most $L - 1$ are always linearly dependent. By means of (8.14) this implies that also $L + 1$ sequences of (8.8) with $n \geq \ell$ are linearly dependent ■

**Remark 8.4** The sequence $s_{\nu}$ from (8.7) with (8.10) is a solution of the difference equation (8.13) with the initial values $s_{\ell}, \ldots, s_{L+\ell-1}$ where we recall that $L + \ell - 1 \geq 0$ and, in the case $\ell < 0$, that $s_{\nu} = 0$ for $\nu < 0$. Every shift $s_{n+k}$ with $k \in \mathbb{N}$ is also a solution. By means of (8.8) the second assertion of Proposition 8.3 expresses the well known fact that $L + 1$ solutions of a linear homogeneous difference equation of order $L$ are linearly dependent. As in Corollary 5.2.3 the coefficients $s_{\nu}$ are exponential polynomials in $\nu$.

**8.4. Eigenvalues of two-slanted matrices.** The considerations of Section 7 can be rounded off by means of a theoretical result concerning the eigenvalues of a two-slanted matrix giving a new formulation for the main part of [ 5 : Theorem 1 ].

**Theorem 8.5** *Let the characteristic polynomial $P$ have the factorization*

$$P = q\tilde{P}, \tag{8.15}$$

*where $q$ is a polynomial with properties as in Lemma* 3.1 *and $\tilde{P}$ a rest polynomial. Let $\mathcal{M}$ be the zero set of the polynomial $p$ in* (3.1)*, let $\zeta_1, \ldots, \zeta_m$ be a cycle of $\mathcal{M}$ and $P(\zeta_j) \neq 0$ $(j = 1, \ldots, m)$, then all $m$ roots $\mu$ of*

$$\mu^m = \frac{1}{2^m} \prod_{j=1}^{m} P(\zeta_j) \tag{8.16}$$

*are eigenvalues of the two-slanted matrix $A$ belonging to $P$.*

**Remark 8.6**

**1.** According to (8.15) we have $P(\omega) = 0$ for all zeros $\omega \in \Omega$ of $q$. Hence, the factor $q$ in (8.15) can be determined in the following way. Consider the set $\Omega_0$ of all zeros of $P$ being roots of unity but not roots of another zero of $P$. Form the closure $\overline{\Omega}_0$ with respect to the mapping $w \mapsto w^2$. If this closure contains several cycles, consider also the separate orbits $\overline{\Omega}_\nu$ ($\nu \in \mathbb{N}$) consisting of the subsets closed under $w \mapsto w^2$ with one single cycle only. Let $\mathcal{M}_\nu = \overline{\Omega}_\nu \setminus \Omega_0$ with $\nu \in \mathbb{N}$. A set $\mathcal{M}_\nu$ can be taken as the set $\mathcal{M}$ in Theorem 8.5, if and only if all absent roots of $\mathcal{M}_\nu$ belong to $\Omega_0$. If this comes true we denote the set of these absent roots by $\Omega_\nu$, and $\mathcal{M} = \bigcup \mathcal{M}_\nu$ can also be taken in Theorem 8.5 where $\Omega = \bigcup \Omega_\nu$.

**2.** If all elements $\omega \in \Omega_\nu$ are zeros of $P$ with a multiplicity at least $r \geq 2$, then according to [5: Theorem 1] even the $rm$ numbers $\frac{\mu}{2^j}$ $(j = 0, \ldots, r-1)$ with the $m$ roots $\mu$ of (8.16) are eigenvalues of $A$. In the case $m = 1$ and therefore $\zeta_1 = 1$ equation $P(1) = 1$ yields $\mu = \frac{1}{2}$ and the well known result that $\frac{1}{2^j}$ $(j = 2, \ldots, r)$ are also eigenvalues, cf. [13].

**3.** In the excluded case that $P$ has $2k$ symmetric zeros, $A$ has $k$ zero vectors, cf. [4: Remark 4.2(2)].

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