More orthogonal double covers of complete graphs by Hamiltonian paths

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Abstract

An orthogonal double cover (ODC) of the complete graph K_n by a graph G is a collection $\mathcal{G}$ of n spanning subgraphs of K_n, all isomorphic to G, such that any two members of $\mathcal{G}$ share exactly one edge and every edge of K_n is contained in exactly two members of $\mathcal{G}$. In the 1980's Hering posed the problem to decide the existence of an ODC for the case that G is an almost-Hamiltonian cycle, i.e. a cycle of length $n-1$. It is known that the existence of an ODC of K_n by a Hamiltonian path implies the existence of ODCs of K_{4n} and K_{16n}, respectively, by almost-Hamiltonian cycles. Horton and Nonay introduced two-colorable ODCs and showed: If for $n \geq 3$ and a prime power $q \geq 5$ there are an ODC of K_n by a Hamiltonian path and a two-colorable ODC of K_q by a Hamiltonian path, then there is an ODC of K_{qn} by a Hamiltonian path. In [12], two-colorable ODCs of K_n and K_{2n}, respectively, by Hamiltonian paths were constructed for all odd square numbers $n \geq 9$. Here we continue this work and construct cyclic two-colorable ODCs of K_n and K_{2n}, respectively, by Hamiltonian paths for all n of the form $n = 4k^2 + 1$ or $n = (k^2 + 1)/2$ with some integer k.

Keywords: orthogonal double cover, ODC, graph decomposition, self-orthogonal decomposition

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1 Introduction

In this paper, we construct a new large class of solutions to the well-studied problem of finding an orthogonal double cover of the complete graph K_n by Hamiltonian paths (or, equivalently, a self-orthogonal factorization of $2K_n$ into Hamiltonian paths).

An *orthogonal double cover (ODC)* of K_n by some graph G is a collection $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ of spanning subgraphs of K_n, all isomorphic to G, such that every edge of K_n is contained in exactly two members of $\mathcal{G}$ and such that every two distinct members of $\mathcal{G}$ share exactly one edge. By this definition, the existence of an ODC of K_n by G immediately implies that G has exactly $n-1$ edges.

The first question in this context was to decide the existence of an ODC of K_n by the graph G consisting of a cycle of length $n-1$ and an isolated vertex [9]. In this case we speak of an ODC of K_n by an almost-Hamiltonian cycle. It is conjectured that the answer is affirmative for all $n \geq 4$. Although there has been some progress on this problem [9, 10, 8, 11, 2, 3, 13, 6, 12, 4], it is far from being solved completely.

Here, we focus on ODCs of K_n by P_n, the path on n vertices. These are of interest also for the original problem because of the following implication.

Theorem 1 [8, 14] *If for some n there is an ODC of K_n by P_n, then there are ODCs of K_{4n} and of K_{16n}, respectively, by almost-Hamiltonian cycles.*

Fortunately, there is a multiplication theorem for ODCs by Hamiltonian paths which can be applied recursively. To formulate it, we need the notion of a two-colorable ODC. An ODC $\mathcal{G}$ of K_n by P_n is called *two-colorable* if it is possible to assign to each path in $\mathcal{G}$ a proper edge coloring, using just the colors red and blue for every path, in such a way that every edge of the underlying K_n receives the same color in the both paths containing it.

Theorem 2 [11] *Let $n \geq 3$ be an integer, and let $q \geq 5$ be a prime power. If there exists an ODC of K_n by P_n as well as a two-colorable ODC of K_q by P_q, then there is an ODC of K_{qn} by P_{qn}.*

In this article we deal with *cyclic* ODCs, exclusively. Therefore, throughout we suppose that $V(K_n) = \mathbb{Z}_n$. Let G be a graph on the same vertex set. For $i \in \mathbb{Z}_n$ let $G + i$ be the graph defined by $V(G + i) = \mathbb{Z}_n$ and $(x + i,\ y + i) \in E(G + i) \iff (x, y) \in E(G)$. If $G + \mathbb{Z}_n := \{G + i \mid i \in \mathbb{Z}_n\}$ is an ODC of K_n (by G), then it is called a cyclic ODC.

The *length* of an edge $e = (x, y) \in E(G)$ is defined to be the set $\ell(e) := \{x-y,\ y-x\}$, where the *distance* of two distinct edges $e_1 = (x, y) \in E(G)$ and $e_2 = (x + z,\ y + z) \in E(G)$ of the same length is the set $d(e_1, e_2) := \{z, -z\}$. Furthermore, we put $\mathbb{Z}_n^* := \mathbb{Z}_n \setminus \{0\}$.

Lemma 3 (cf. [5]) *Let G be a graph with $V(G) = \mathbb{Z}_n$ and $|E(G)| = n - 1$. The set $G + \mathbb{Z}_n$ is a (cyclic) ODC of K_n by G if and only if the following conditions are satisfied:*

(1) For every $x \in \mathbb{Z}_n$ with $2x \neq 0$, there are exactly two edges of length $\{x, -x\}$ in $E(G)$.

(2) The union of all distances $d(e_1, e_2)$ with $e_1, e_2 \in E(G)$, $e_1 \neq e_2$, $\ell(e_1) = \ell(e_2)$ is equal to $\{x \in \mathbb{Z}_n \mid 2x \neq 0\}$.

To decide whether a cyclic ODC of K_n by P_n is two-colorable is easily done just looking at one of its members. Let G be a Hamiltonian path G on the vertex set $\mathbb{Z}_n$ such that $G+\mathbb{Z}_n$ is an ODC of K_n. We call G two-colorable if there is a proper edge-coloring of G with two colors in which edges of the same length have the same color.

Lemma 4 *Let G be a Hamiltonian path on the vertex set $\mathbb{Z}_n$ such that $G+\mathbb{Z}_n$ is an ODC of K_n. If G is two-colorable, then $G + \mathbb{Z}_n$ is two-colorable.*

For a more detailed motivation of and introduction into the subject we refer to [5] and to the references given there, especially to [1, 7, 12].

2 The main result

In [12], two-colorable ODCs of K_n by P_n were constructed for the case that $n = m^2$ or $n = 2m^2$ for some odd $m \geq 3$. These ODCs are group-generated but not cyclic.

Here we use a similar construction to obtain the following result.

Theorem 5 *Let $k \geq 2$ be an integer, and let $n = k^2 + 1$.*

(a) There is a two-colorable cyclic ODC of K_n by P_n.

(b) If n is even, then there is a two-colorable cyclic ODC of $K_{n/2}$ by $P_{n/2}$.

(c) If n is odd, then there is a two-colorable cyclic ODC of K_{2n} by P_{2n}.

Together with Theorem 1 and Theorem 2, this gives large new classes of ODCs by Hamiltonian paths and almost-Hamiltonian cycles.

3 Proof of Theorem 5

Let $m \geq 5$ be an odd integer such that -1 is quadratic modulo m, i.e. all prime factors of m are congruent 1 modulo 4. In the sequel, we operate in $\mathbb{Z}_m$, where $i \in \mathbb{Z}_m$ satisfies the equation $i^2 = -1$.

Consider the graph G on the the vertex set $\mathbb{Z}_m$ with $(x, y) \in E(G)$ if and only if $y \in \{ix, -ix\}$. Clearly, G consists of $(m-1)/4$ disjoint cycles, each of the form $(x,\ ix,\ -x,\ -ix,\ x)$ for some $x \in \mathbb{Z}_m^*$, and the isolated vertex 0. The cycle $(x,\ ix,\ -x,\ -ix,\ x)$ contains two disjoint edges of length $\{\pm(i-1)x\}$ whose distance is $\{\pm(i+1)x\}$ and two disjoint edges of length $\{\pm(i+1)x\}$ whose distance is $\{\pm(i-1)x\}$. These lengths and distances do not occur in any other of the four-cycles in G because of $(i-1)(i+1) = -2 \neq 0$. Hence, by Lemma 3:

Lemma 6 *$G + \mathbb{Z}_m$ is a cyclic ODC of K_m.*

Let G^* denote the graph obtained by omitting the isolated 0 in G, and let H be a one-factor in G^*. Furthermore, for an arbitrary graph F on the vertex set $\mathbb{Z}_m$ and some $x \in \mathbb{Z}_m^*$ let xF and $F - x$ be the graphs on the vertex set $\mathbb{Z}_m$ defined by $(xy, xz) \in E(xF) \iff (y, z) \in E(F)$ and $(y-x,\ z-x) \in E(F-x) \iff (y, z) \in E(F)$, respectively. Clearly, $\{H, iH\}$ is a one-factorization of G^* in which edges of the same length lie in the same factor. Moreover, it is evident that in F and $F - x$ exactly the same lengths and distances occur. Using Lemmas 3 and 6, we therefore obtain:

Lemma 7 *$\big(H \cup (iH - 1)\big) + \mathbb{Z}_m$ is a cyclic ODC of K_m.*

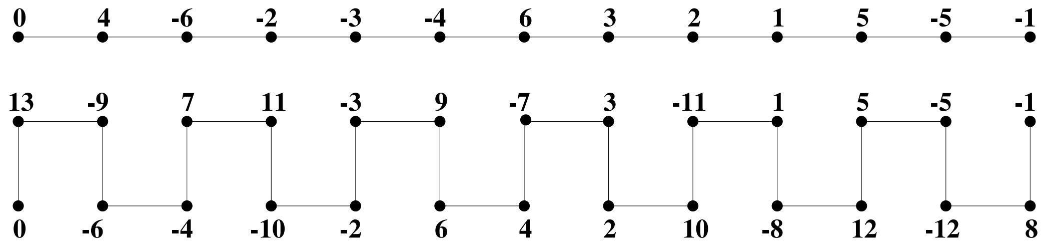

Figure 1: P (top) and P' (bottom) for $m = 13$.

By the choice of H, it is immediately clear that the components of $H \cup (iH - 1)$ are a path joining the vertices 0 and -1 and a number of cycles of even length. Moreover, also by the choice of H, if $H \cup (iH - 1)$ has just one component (i.e. if it is a Hamiltonian path on $\mathbb{Z}_m$), then it is two-colorable. Using Lemma 4, this implies:

Lemma 8 *If $H \cup (iH - 1)$ is a path, then $\big(H \cup (iH - 1)\big) + \mathbb{Z}_m$ is a two-colorable cyclic ODC of K_m by P_m.*

Obviously, the components of the graph $H \cup (H - 1)$ are also a path connecting 0 to -1 and cycles of even length. Although in this graph half of the lengths occur four times while the other half of the lengths do not occur, it can be used to find ODCs if it has just one component:

Lemma 9 *If $H \cup (H - 1)$ is a path, then there is a two-colorable cyclic ODC of K_{2m} by P_{2m}.*

Proof. Let $H \cup (H - 1) =: P$ be a path. We will use this path to generate a Hamiltonian path P' on the vertex set $\mathbb{Z}_{2m}$ such that $P' + \mathbb{Z}_{2m}$ is a two-colorable ODC. It will be convenient for us to represent $\mathbb{Z}_{2m}$ as $\mathbb{Z}_m \times \mathbb{Z}_2$.

To transform P into P', we first replace every $v \in V(P)$ by $(v, 1)$, the resulting path is denoted by P^*. Now P' is obtained from P^* by replacing all edges $\big((u, 1), (v, 1)\big) \in E(P^*)$ that satisfy $(u, v) \in E(H)$ by the path $\big((u, 1), (ui, 0), (vi, 0), (v, 1)\big)$ (i.e. every second edge of P^* is replaced), and by finally introducing one new edge which joins $(0, 1)$ to $(0, 0)$. Figure 1 illustrates this construction for $m = 13$ and for $E(H)$ containing all edges from $E(G)$ of lengths $\{\pm 1\}$, $\{\pm 3\}$, and $\{\pm 4\}$, respectively.

By the above construction, P' is a Hamiltonian path on the vertex set $\mathbb{Z}_{2m}$. There is exactly one edge of length $\{(0,1)\}$ in $E(P')$, namely $\big((0,0),(0,1)\big)$. Furthermore, $\big((u,0),(v,0)\big) \in E(P')$ holds for all $(u,v) \in E(iH)$, and $\big((u,1),(v,1)\big) \in E(P')$ holds for all $(u,v) \in E(H-1)$. Since $\big(iH \cup (H-1)\big) + \mathbb{Z}_m$ is a cyclic ODC by Lemma 7, Lemma 3 implies that $\{(x,0),(-x,0)\}$ for every $x \in \mathbb{Z}_m^*$ is the length of two edges in P' and the distance of two edges of the same length in P'. Moreover, by construction, for every $(u,v) \in E(G)$ either $\big((u,0),(v,1)\big) \in E(P')$ or $\big((u,1),(v,0)\big) \in E(P')$. Hence, by Lemmas 6 and 3, $\{(x,1),(-x,1)\}$ for every $x \in \mathbb{Z}_m^*$ is the length of two edges in P' and the distance of two edges of the same length in P'. Consequently, $P' + \mathbb{Z}_{2m}$ is an ODC.

Finally, P' is two-colorable since its edges alternately are of length $(*,0)$ and $(*,1)$. By Lemma 4, $P' + \mathbb{Z}_{2m}$ is two-colorable as well. ■

In the sequel, for special values of m we will provide choices of H such that $H \cup (iH-1)$ is a path and choices of H such that $H \cup (H-1)$ is a path.

Assume that $m = i^2+1$ for some even $i > 1$ (this equation, of course, is supposed to hold not only in $\mathbb{Z}_m$ but also in $\mathbb{Z}$). In this case, it is easy to check that G^* is the collection of the $(m-1)/4$ disjoint four-cycles

$$(ai+b,\ bi-a,\ -ai-b,\ -bi+a,\ ai+b), \qquad 0 \le a < b < i-a.$$

Now we fix the one-factor H_1 of G^* by

$$\begin{array}{lcl} (ai+b,\ bi-a),(-ai-b,\ -bi+a) \in E(H_1) & \Longleftrightarrow & b \text{ is odd,} \\ (bi-a,\ -ai-b),(-bi+a,\ ai+b) \in E(H_1) & \Longleftrightarrow & b \text{ is even} \end{array}$$

for $0 \le a < b < i-a$, and the one-factor H_2 of G^* is given by

$$(ai+b,\ bi-a),(-ai-b,\ -bi+a) \in E(H_2)$$

for $0 \le a < b < i-a$.

Lemma 10 *Let $i > 1$ be even, and let $m = i^2+1$.*

(a) $H_1 \cup (iH_1 - 1)$ is a path.

(b) $H_2 \cup (H_2 - 1)$ is a path.

Proof. (a) Let $d = 1, 2, \ldots, i/2$ and $-d < c < d$ such that $d-c$ is even. By the choice of H_1, if d is odd, then the following path of length 4 is contained in $H_1 \cup (iH_1 - 1)$:

$$\big(ci-d,\ -(d-1)i-(c+1),\ -(c+1)i+(d-1),\ -di-(c+2),\ (c+2)i-d\big),$$

and if d is even, then $H_1 \cup (iH_1 - 1)$ contains:

$$\bigl(ci - d,\ di + c,\ -(c+1)i + (d-1),\ (d-1)i + (c+1),\ (c+2)i - d\bigr).$$

Hence, $H_1 \cup (iH_1 - 1)$ contains a path of length $4(d-1)$ with the end-vertices $-(d-2)i - d$ and $di - d$.

If d is odd and $\neq i/2$, then $iH - 1$ contains an edge joining $di - d$ to $-(d-1)i - (d+1)$. If d is even and $\neq i/2$, then the following path of length 3 is in $H_1 \cup (iH_1 - 1)$:

$$\bigl(di - d,\ di + d,\ -(d+1)i + (d-1),\ -(d-1)i - (d+1)\bigr).$$

Consequently, there is a path P_1 with the end-vertices $i-1$ and $\frac{i}{2}(i-1)$ contained in $H_1 \cup (iH_1 - 1)$ whose length is $(m-5)/2$ if $i/2$ is odd and $(m-5)/2 - 1$ if $i/2$ is even.

Analogously, one observes that $H_1 \cup (iH_1 - 1)$ contains another path P_2 with the end-vertices $-i$ and $-\frac{i}{2}(i-1) - 1$ such that P_1 and P_2 are disjoint, where the length of P_2 is $(m-5)/2$ if $i/2$ is odd and $(m-5)/2 + 1$ if $i/2$ is even. (To find P_2, proceed like we did in finding P_1, where at the beginning consider the elements $-ci + d - 1$ instead of $ci - d$.)

Now th claim follows from the fact that the following edges are in $E\bigl(H_1 \cup (iH_1 - 1)\bigr)$:

$$(0, -i-1),\ (-i-1, i-1),\ \left(\frac{i}{2}(i-1), -\frac{i}{2}(i-1) - 1\right),\ (-i, -1).$$

(b) We define a subset S of $\mathbb{Z}_m$ by

$$S := \left\{ (i-1)a + (i+1)b \;\middle|\; \begin{array}{ll} & (1 \le a \le i/2 \ \text{ and } \ -i/2 \le b < i/2) \\ \text{or} & (a = 0 \ \text{ and } \ 1 \le b < i/2) \end{array} \right\}.$$

Partition S into the subsets $S_{c,1},\ S_{c,2}$ $(c = 1, 2, \ldots, i/2 - 1)$ and $S_{i/2}$, where

$$\begin{aligned} S_{c,1} &= \{(i-1)c + (i+1)b \mid -i/2 \le b \le 0\}, \\ S_{c,2} &= \{(i-1)a + (i+1)c \mid 0 \le a \le i/2\}, \\ S_{i/2} &= \left\{\tfrac{i}{2}(i-1) + (i+1)b \mid -i/2 \le b \le 0\right\}. \end{aligned}$$

Furthermore, put $S_c := S_{c,1} \cup S_{c,2}$ for $c = 1, 2, \ldots, i/2 - 1$.

Next, we introduce a linear order on S. For $x = (i-1)a_1 + (i+1)b_1$ and $y = (i-1)a_2 + (i+1)b_2$ with a_1, b_1 and a_2, b_2 like in the definition of S, we put $x \prec y$ whenever:

1. $x \in S_c$ and $y \in S_d$ with $1 < c < d \leq i/2$, or
2. $x \in S_{c,1}$ and $y \in S_{c,2}$ for some $1 \leq c \leq i/2 - 1$, or
3. $x, y \in S_{c,1}$ for some $1 \leq c \leq i/2 - 1$ and $b_1 > b_2$, or
4. $x, y \in S_{i/2}$ and $b_1 > b_2$, or
5. $x, y \in S_{c,2}$ for some $1 \leq c \leq i/2 - 1$ and $a_1 > a_2$.

We use the notation $x \prec\cdot\, y$ to indicate that $x \prec y$ such that x and y are consecutive in the linear order.

Now the assertion is implied by the following observations which are easy to verify: If we walk along the path connecting 0 and -1 in $H_2 \cup (H_2 - 1)$, then we hit the elements of S in the order defined above. Moreover, there is an edge connecting 0 and the first element of S, there is another edge connecting the last element of S and -1. Let $x, y \in S$ with $x \prec\cdot\, y$. There is a path of length two in $H_2 \cup (H_2 - 1)$ whose end-vertices are x and y if $x, y \in S_{c,1}$ for some c or if $x, y \in S_{c,2}$ for some c or if $x, y \in S_{i/2}$. Finally, there is an edge connecting x and y if $x \in S_{c,1}$, $y \in S_{c,2}$ for some c or if $x \in S_c$ and $y \in S_{c+1}$ for some c. ■

Assume now that $2m = i^2 + 1$ for some odd $i > 1$. In this case, G^* is the collection of the $(m-1)/4$ four-cycles

$$\left(\tfrac{ai+b}{2},\ \tfrac{bi-a}{2},\ \tfrac{-ai-b}{2},\ \tfrac{-bi+a}{2},\ \tfrac{ai+b}{2}\right), \qquad 0 \leq a < b < i - a;\ b - a \ \text{ is even.}$$

The one-factors H_3 and H_4 of G^* are defined by

$$\left(\tfrac{ai+b}{2},\ \tfrac{bi-a}{2}\right), \left(\tfrac{-ai-b}{2},\ \tfrac{-bi+a}{2}\right) \in E(H_3) \iff \begin{array}{r} b \equiv 2 \pmod 4 \\ \text{or} \ \ i - b \equiv 2 \pmod 4, \end{array}$$

$$\left(\tfrac{bi-a}{2},\ \tfrac{-ai-b}{2}\right), \left(\tfrac{-bi+a}{2},\ \tfrac{ai+b}{2}\right) \in E(H_3) \iff \begin{array}{r} b \not\equiv 2 \pmod 4 \\ \text{and} \ \ i - b \not\equiv 2 \pmod 4, \end{array}$$

for $0 \leq a < b < i - a$ with even $b - a$ and

$$H_4 = \begin{cases} H_3 & \text{if} \quad i \equiv 1 \pmod 4, \\ iH_3 & \text{if} \quad i \equiv 3 \pmod 4, \end{cases}$$

respectively. The choice of these one-factors for $i = 7$, i.e. for $m = 25$, is displayed in Figure 2.

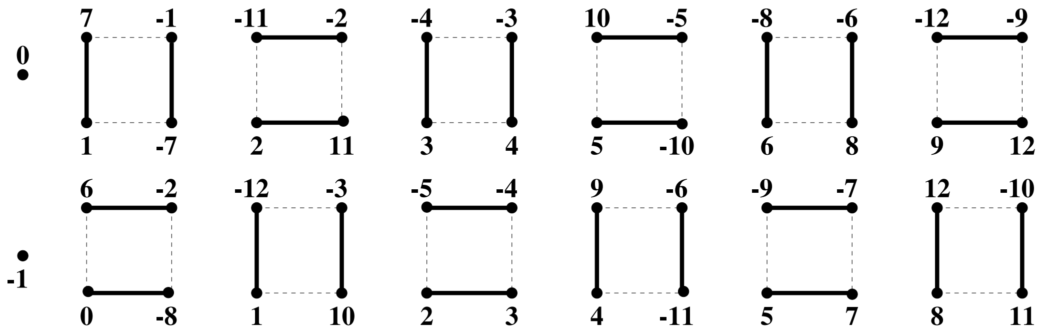

Figure 2: $m = 25$; top: H_3 (bold) and $H_4 = iH_3$ (dashed); bottom: $H_3 - 1$ (dashed) and $H_4 - 1 = iH_3 - 1$ (bold).

Lemma 11 *Let $i > 1$ be odd, and let $2m = i^2 + 1$.*

(a) $H_3 \cup (iH_3 - 1)$ is a path.

(b) $H_4 \cup (H_4 - 1)$ is a path.

Proof. The lemma can be proved similar to Lemma 10. ∎

Finally, observe that Theorem 5 (a) is implied by Lemmas 8 and 10 (a) with $m = n$, $i = k$ and Lemmas 9 and 11 (b) with $m = n/2$, $i = k$. Theorem 5 (b) follows by Lemmas 8 and 11 (a) with $m = n/2$, $i = k$, and Theorem 5 (c) by Lemmas 9 and 10 (b) with $m = n$, $i = k$.

4 A more general conjecture

Let us now formulate an observation we made starting with the ODCs constructed in the preceding section. We will avoid introducing more formalism than necessary, we will rather try to formulate our observation as it appeared to us.

Look at the graph G on the vertex set $\mathbb{Z}_m$ which was introduced in the preceding section for any odd $m \geq 5$ with the property that -1 is quadratic in $\mathbb{Z}_m$. Use the integers $0, \pm 1, \pm 2, \ldots, \pm(m-1)/2$ to represent $\mathbb{Z}_m$. Draw the four-cycles in G in such a way that for each of them the vertex in the lower left corner is a positive integer, and this integer is the smaller one of the two positive integer occurring on the four-cycle. Order the four-cycles (from left to right) according to the rule that their lower left corners come

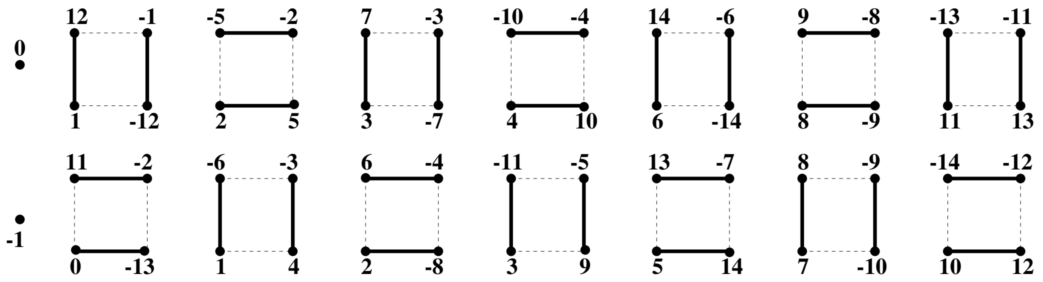

Figure 3: $m = 29$; top: H (bold) and iH (dashed); bottom: $iH - 1$ (bold) and $H - 1$ (dashed).

in increasing order. (One naturally comes up with this kind of picture when drawing an example by hand.) As an example, this order is given for $m = 29$ in Figure 3.

Now it can easily be verified that the one-factors H_1 and H_3 of G^* defined in the preceding section for $m = i^2 + 1$ with even i and $m = (i^2 + 1)/2$ with odd i, respectively, are built up as follows: From the four-cycles (in the order described above) take alternately the two vertical edges and the two horizontal edges into the one-factor, starting with the two vertical edges $(1, i), (-1, -i)$ from the first four-cycle $(1, i, -1, -i, 1)$.

In general, let us denote the one-factor obtained this way by H (also for those m which are not of the form $i^2 + 1$ or $(i^2 + 1)/2$). By Lemmas 10 (a) and 11 (a), we know that for the particular cases $m = i^2 + 1$ and $m = (i^2 + 1)/2$ the graph $H \cup (iH - 1)$ is a path (to be precise, a Hamiltonian path on the vertex set $\mathbb{Z}_m$). We investigated whether this statement remains true if m is not of this particular form. (Actually, without much hope for an affirmative answer because we work in $\mathbb{Z}_m$ and the order of the four-cycles looks somewhat artificial there.) To our surprise, although checking many instances m, we did not find a single counterexample. Therefore, we here formulate the following conjecture:

Conjecture 12 *Let $m \geq 5$ be an odd integer such that -1 is quadratic modulo m. Then $H \cup (iH - 1)$ is a path.*

Note that, by Lemma 8, this would imply the existence of a two-colorable cyclic ODC of K_m by P_m. In the light of Theorem 2, such solutions are especially valuable if m is a prime. For this reason, we checked the structure of $H \cup (iH-1)$ for primes systematically by computer. For primes up to $100,000$ the result always was a path which gives as a corollary:

Theorem 13 *For every prime number p with $p \equiv 1 \pmod 4$ and $p < 10^5$ there is a two-colorable cyclic ODC of K_p by P_p.*

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