# On the Asymptotics of Nonlinear Difference Equations

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**Abstract**. Solutions of nonlinear difference equations of second order are investigated with respect to their asymptotic behaviour. In particular, seven conjectures of Kulenović and Ladas concerning rational difference equations are verified.
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## 1 Introduction

The book Kulenović and Ladas [4] contains a large number of open problems and conjectures concerning the dynamics of the rational difference equations

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}} \tag{1.1}$$

($n \in \mathbb{N}_0$) with non-negative parameters and of more general equations. The problems concerning the asymptotic behaviour of the solutions $x_n$ of (1.1) can be solved by constructing two bounds $y_n$, $z_n$ with

$$y_n \leq x_n \leq z_n \tag{1.2}$$

for suitable great $n$. This construction can be realized in the following way (cf. [2]): Choose an asymptotic scale $\varphi_k(n)$ ($k \in \mathbb{N}_0$), i.e. a sequence of positive functions with $\varphi_{k+1}(n) = o(\varphi_k(n))$ for $n \to \infty$, such that all shifts $\varphi_k(n+1)$, $\varphi_k(n-1)$ and all products $\varphi_l \varphi_m$ possess asymptotic expansions with respect to this scale. In the case $\alpha \neq 0$ also the constant function 1 must possess such an expansion. Then make the ansatz

$$x_{nK} = \sum_{k=0}^{K} c_k \varphi_k(n) \tag{1.3}$$

with a fixed $K \geq 1$, determine the coefficients out of

$$x_{n+1}(A + Bx_n + Cx_{n-1}) - \alpha - \beta x_n - \gamma x_{n-1} = O(\varphi_L(n)) \tag{1.4}$$

as $n \to \infty$ with $x_n = x_{nK}$ and $L$ as great as possible, and put

$$y_n = x_{n,K-1} + a\varphi_K(n), \quad z_n = x_{n,K-1} + b\varphi_K(n) \tag{1.5}$$

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with $a < c_K < b$. Simple examples for possible scales are $\varphi_k = \frac{1}{n^k}$ and $\varphi_k = t^{kn}$ with $0 < t < 1$. After having found the bounds $y_n$, $z_n$ it remains to show the existence of a solution $x_n$ of (1.1) with (1.2) which shall be done in Section 2.

If we have no idea how to choose the scale $\varphi_k$, we can try the following possibility (cf. [2]). Replace (1.1) by a differential equation which approximates (1.1) asymptotically as $n \to \infty$ and which can be solved explicitly. Then take this solution (or an asymptotic approximation of it) as $x_{n,K-1}$ in (1.5). In the simplest case the approximating differential equation can be obtained by substituting into (1.1) the first terms of the Taylor expansions for $x_{n+1}$ and $x_{n-1}$. However, this requires that the derivatives with respect to $n$ (considered as continuous variable) have a smaller order than the functions as it comes true by the functions $\frac{1}{n^k}$, but not by the functions $t^{kn}$. For more complicated possibilities cf. [2].

If asymptotically two-periodic solutions are sought, then put $u_n = x_{2n-1}$, $v_n = x_{2n}$ and replace (1.1) by the system

$$u_{n+1} = \frac{\alpha + \beta v_n + \gamma u_n}{A + B v_n + C u_n}, \quad v_{n+1} = \frac{\alpha + \beta u_{n+1} + \gamma v_n}{A + B u_{n+1} + C v_n}, \tag{1.6}$$

to which the foregoing procedures can be transferred.

In the following we deal on the one hand with a generalization of (1.1), and on the other hand with the special cases

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}, \tag{1.7}$$

$$x_{n+1} = \beta + \frac{x_{n-1}}{x_n}, \tag{1.8}$$

$$x_{n+1} = \frac{1 + x_{n-1}}{x_n}, \tag{1.9}$$

$$x_{n+1} = \frac{\alpha + x_{n-1}}{1 + x_n} \tag{1.10}$$

with $\alpha > 0$. In particular, we verify the following conjectures:

**Conjecture ([4]: 4.8.2).** *Show that* (1.7) *has a solution which converges to zero.*

**Conjecture ([4]: 4.8.3).** *Show that* (1.8) *has a solution which remains above the equilibrium* $\overline{x} = \beta + 1$ *for all* $n \geq -1$.

**Conjecture ([4]: 5.4.6).** *Show that* (1.9) *has a nontrivial positive solution which decreases monotonically to the equilibrium of the equation.*

**Conjecture ([4]: 6.10.3).** *Show that* (1.10) *has a positive and monotonically decreasing solution.*

We also deal with asymptotically periodic solutions of (1.7) and we give a partial answer to the Open Problem [4: 4.8.4], which among other things demands to investigate the global character of the solution of (1.7) in dependence on their initial values $x_{-1}$, $x_0$.

Finally, we verify three conjectures of [4] concerning bounded solutions of (1.1), and we refer to a further conjecture of [4] concerning the rational difference equation

$$x_{n+1} = \frac{p + x_{n-2}}{x_n} \tag{1.11}$$

($n \in \mathbb{N}_0$) which is not of the type (1.1). We shall verify the conjecture for $p = 0$, whereas for $p > 0$ we shall replace it by another.

For some calculations we have used the DERIVE system.

# 2 The inclusion theorem

In order to verify the inequalities (1.2) we consider the equation

$$x_{n-1} = f(x_n, x_{n+1}) \tag{2.1}$$

which can be either the solution of (1.1) with respect to $x_{n-1}$ (in the case $\gamma + C > 0$) or an equation with an arbitrary $f$ (such that (2.1) is uniquely solvable with respect to $x_{n+1}$).

**Theorem 1.** *Let the function $f$ be continuous and non-decreasing in both arguments, and let be $y_n < z_n$ for $n \geq n_0$ as well as*

$$y_{n-1} \leq f(y_n, y_{n+1}), \quad f(z_n, z_{n+1}) \leq z_{n-1} \tag{2.2}$$

*for $n > n_0$. Then there exists a solution of* (2.1) *with* (1.2) *for $n \geq n_0$.*

*Proof.* Choosing an arbitrary integer $N > n_0$, then all initial values $x_{N+1}$, $x_N$ with (1.2) for $n = N+1$ and $n = N$ can be continued by means of (2.1) to the left. The inequalities (2.2) and the monotony of $f$ yield the validity of (1.2) for all $n$ with $n_0 \leq n \leq N+1$. Let $A_N$ be the non-empty set of all pairs $(x_{n_0}, x_{n_0+1})$ such that the solutions $x_n$ of (2.1) satisfy (1.2) for $n_0 \leq n \leq N+1$. The continuity of $f$ implies that $A_N$ is a closed set, and the monotony of $f$ that $A_N \supset A_{N+1}$. Hence, there exists a non-empty set $A = \bigcap_{N=n_0+1}^{\infty} A_N$ of pairs $(x_{n_0}, x_{n_0+1})$ such that all attached solutions $x_n$ of (2.1) satisfy (1.2) for all $n \geq n_0$ ∎

As the proof shows, the continuity and the monotony of $f$ are only necessary for such arguments which satisfy (1.2) for $n > n_0$.

Theorem 1 can be modified in different ways (cf. [1, 2, 3, 5]), but we do not need here such modifications. Instead of that we come back to the special cases (1.7)-(1.10) of (1.1).

**Example 1.** For the example (1.7) the inversion (2.1) yields the function $f(x_n, x_{n+1}) = (1+x_n)x_{n+1}$, which satisfies the assumptions of Theorem 1 for positive arguments. Writing $x_n = x$ and using the approximations $x_{n+1} \approx x + x'$, $x_{n-1} \approx x - x'$, we replace (1.7) by the differential equation

$$(2 + x)x' + x^2 = 0$$

with the solution

$$x = \frac{2}{n + \ln x + C}\,.$$

In the case $x \to 0$ as $n \to \infty$ we find $x \sim \frac{2}{n}$ and therefore iteratively, choosing $C = -\ln 2$, the asymptotic approximations

$$x^{[0]} = \frac{2}{n}\,, \quad x^{[1]} = \frac{2}{n - \ln n}\,, \quad x^{[2]} = \frac{2}{n - \ln n + \frac{1}{n}\ln n}\,.$$

Taking into account that $x^{[2]} = \frac{2}{n} + \frac{2}{n^2}\ln n + \frac{2}{n^3}\ln^2 n + O\left(\frac{1}{n^3}\ln n\right)$ we make the ansatz

$$y_n = \frac{2}{n} + \frac{2}{n^2}\ln n + \frac{a}{n^3}\ln^2 n\,, \quad z_n = \frac{2}{n} + \frac{2}{n^2}\ln n + \frac{b}{n^3}\ln^2 n$$

with $a < 2 < b$, cf. (1.5) with $K = 2$. Then we find the asymptotic relation

$$y_{n+1}(1 + y_n) - y_{n-1} \sim \frac{2}{n^4}(2 - a)\ln^2 n\,,$$

and an analogous relation with $z$ and $b$ instead of $y$ and $a$, respectively. These relations show that the inequalities (2.2) are satisfied for sufficiently great $n$.

Hence, Theorem 1 can be applied and it yields, in particular, the existence of a solution of (1.7) converging to zero, i.e. it verifies the corresponding conjecture from [4].

The next three examples are special cases of

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{A + x_n}\,. \tag{2.3}$$

The inversion (2.1) yields the function $f(x_n, x_{n+1}) = (A + x_n)(x_{n+1} - \beta) + A\beta - \alpha$, which is continuous and increasing for $x_n > 0$ and $x_{n+1} > \beta$. An equilibrium $\overline{x}$ of (2.3) is a solution of $\overline{x}^2 + (A - \beta - 1)\overline{x} = \alpha$, here we need the non-negative equilibrium

$$\overline{x} = \frac{1}{2}\left(\beta + 1 - A + \sqrt{(\beta + 1 - A)^2 + 4\alpha}\right)\,. \tag{2.4}$$

Making with an unknown $t \in (0, 1)$ the ansatz

$$x_n = \overline{x} + t^n + ct^{2n} + o(t^{2n}) \tag{2.5}$$

as $n \to \infty$, we find, according to (1.4),

$$\overline{x} = \frac{1+\beta t - At^2}{(1+t)t}\,, \quad c = \frac{(1+t)t^3}{(1-t)(1+t+t^2+(A+\beta)t^3)}\,, \tag{2.6}$$

provided that the first equation has a solution $t \in (0,1)$. In this case the ansatz

$$y_n = \overline{x} + t^n + at^{2n}\,, \qquad z_n = \overline{x} + t^n + bt^{2n}$$

leads to the asymptotic representation

$$f(y_n, y_{n+1}) - y_{n-1} \sim \left(1 - \frac{a}{c}\right) t^{2n+1}$$

and an analogous one with $z$ and $b$ instead of $y$ and $a$, respectively. These representations show that the inequalities (2.2) are satisfied for sufficiently great $n$, since $c > 0$, and Theorem 1 yields the existence of a solution of (2.3) with the asymptotic behaviour (2.5) which will verify the corresponding conjectures from [4]. However, it remains to prove that $t \in (0,1)$.

**Example 2.** Choosing in (2.3) $\alpha = A = 0$ we get example (1.8). The equations (2.4) and (2.6) specialize to

$$\overline{x} = \beta + 1 = \frac{1+\beta t}{(1+t)t}\,, \quad c = \frac{t^3}{(1-t)(1+t-t^2)}\,,$$

and one solution of the first equation is $t = \frac{1}{2(\beta+1)}\left(\sqrt{4\beta+5} - 1\right)$, which satisfies $t \in (0,1)$ even for $\beta > -1$. Hence, there exists a solution of (1.8) with (2.5), i.e. in particular, a solution of (1.8) with $x_n > \overline{x} = \beta + 1$ when $\beta > -1$ and $n \geq n_0$. But there exists also such a solution when $n \geq -1$, namely $x_{n+n_0+1}$.

**Example 3.** Choosing in (2.3) $\alpha = 1$ and $\beta = A = 0$ we get example (1.9). The equations (2.4) and (2.6) specialize to

$$\overline{x} = \frac{1}{2}\left(1 + \sqrt{5}\right) = \frac{1}{(1+t)t}\,, \quad c = \frac{(1+t)t^3}{1-t^3}\,,$$

and $t = \frac{1}{2}\left(\sqrt{2\sqrt{5}-1} - 1\right) \approx 0.4317$ is the solution of the first equation contained in $(0,1)$. Hence, there exists a solution of (1.9) with (2.5). This asymptotic relation shows that $x_n$ is eventually monotonically decreasing to $\overline{x}$, and a suitable shift of $x_n$ is decreasing for all $n \geq -1$.

**Example 4.** Choosing in (2.3) $\beta = 0$ and $A = 1$ we get example (1.10). The equations (2.4) and (2.6) specialize to

$$\overline{x} = \sqrt{\alpha} = \frac{1-t}{t}\,, \quad c = \frac{(1+t)t^3}{1-t^4}\,,$$

and the first equation implies $t = \frac{1}{\sqrt{\alpha}+1} \in (0,1)$. Hence, there exists a solution of (1.10) with (2.5). The validity of the corresponding conjecture of [4] follows as in the foregoing examples.

## 3 Asymptotically two-periodic solutions

Equation (1.7) possesses the two-periodic solution $x_{2n-1} = 0$, $x_{2n} = p$ with an arbitrary constant $p$. Looking for an asymptotically two-periodic solution, we put $u_n = x_{2n-1}$ and $v_n = x_{2n}$ as before and make the ansatz

$$u_n = \sum_{\nu=1}^{\infty} a_\nu c^\nu t^{\nu n}\,, \quad v_n = \sum_{\nu=0}^{\infty} b_\nu c^\nu t^{\nu n} \tag{3.1}$$

with $b_0 = p$ and arbitrary $c$, since (1.6) is an autonomous equation. We choose $c > 0$. In the case (1.6) the equations (1.5) specialize to

$$(1 + v_n)u_{n+1} = u_n\,, \quad (1 + u_{n+1})v_{n+1} = v_n\,. \tag{3.2}$$

Substitution of (3.1) into these equations and comparing the coefficients yields $t = \frac{1}{p+1}$, $a_1 = b_1$ undetermined, and

$$a_\nu = \frac{1}{(p+1)^{\nu-1} - 1} \sum_{\mu=1}^{\nu-1} b_\mu a_{\nu-\mu} (p+1)^{\mu-1}\,, \quad b_\nu = \frac{1}{(p+1)^\nu - 1} \sum_{\mu=0}^{\nu-1} b_\mu a_{\nu-\mu} \tag{3.3}$$

for $\nu \geq 2$. In view of the presence of the arbitrary constant $c$ we can choose $a_1 = b_1 = 1$. The next coefficients read

$$a_2 = \frac{1}{2}\,, \quad b_2 = \frac{2}{p(p+2)}\,, \quad a_3 = \frac{3p+4}{p^2(p+2)^2}\,, \quad b_3 = \frac{p^2+9p+12}{p^2(p+2)^2(p^2+3p+3)}\,.$$

For positive $p$ it is $0 < t < 1$, and the coefficients $a_\nu$, $b_\nu$ are also positive. It can easily be proved by induction that the further coefficients allow the estimates

$$a_\nu \leq \frac{1}{p^{\nu-1}}\,, \quad b_\nu \leq \frac{1}{p^{\nu-1}}$$

for all $\nu \geq 1$. This means that the series (3.1) are not only asymptotic ones as $n \to \infty$, but that they even converge for $t^n < \frac{p}{c}$, i.e. for suitable great $n$.

**Remark.** 1. By positive initial values $u_0$, $v_0$ it follows from (3.2) that all solutions are also positive and decreasing, hence converging to a non-negative limit. At least one limit equals zero (cf. [4]).

2. By elimination it can be shown that both solutions of (3.2) are also solutions of the rational difference equation

$$w_{n+1} = \frac{w_n + w_n^2}{w_{n-1} + w_n^2} w_n$$

which is not of the type (1.1).

## 4 Dependence on the initial values

Next, we want to study the solution of (1.7) in dependence on their initial values $x_{-1}$, $x_0$.

**Proposition 1.** *For $n \in \mathbb{N}_0$ and positive $x_{-1}$, $x_0$ the solution of (1.6) satisfies the estimates*

$$x_{2n} \leq x_0 t^n \,, \quad x_{2n-1} \geq p + (x_{-1} - p)t^n \tag{4.1}$$

*with $t = \frac{1}{\sqrt{x_{-1}+1}}$, $p = \sqrt{x_{-1}+1} - 1$ when*

$$x_0 \leq \frac{1}{2}\left(\sqrt{y_{-1}+1} - 1\right) \,, \tag{4.2}$$

*and the estimates*

$$x_{2n+1} \leq x_1 t^n \,, \quad x_{2n} \geq p + (x_0 - p)t^n \tag{4.3}$$

*with $t = \frac{1}{\sqrt{x_0+1}}$, $p = \sqrt{x_0+1} - 1$ when*

$$x_1 \leq \frac{1}{2}\left(\sqrt{x_0+1} - 1\right) \,. \tag{4.4}$$

*Proof.* We use the foregoing notations $u_n = x_{2n-1}$, $v_n = x_{2n}$ for which the estimates (4.1) read

$$v_n \leq v_0 t^n \,, \quad u_n \geq p + (u_0 - p)t^n \,. \tag{4.5}$$

Since these estimates are valid for $n = 0$ we shall prove them by induction. Hence, according to (3.2), we have to show

$$\frac{v_0 t^n}{1 + p + (u_0 - p)t^{n+1}} \leq v_0 t^{n+1} \,, \quad \frac{p + (u_0 - p)t^n}{1 + v_0 t^n} \geq p + (u_0 - p)t^{n+1}$$

for $n \in \mathbb{N}_0$, i.e. (for $t > 0$, $v_0 > 0$ and $0 < p < u_0$)

$$1 \leq (1 + p)t + (u_0 - p)t^{n+2} \,, \quad (u_0 - p)(1 - t) \geq v_0\left(p + (u_0 - p)t^{n+1}\right) \,.$$

The optimal solution of the first inequality for $n \in N_0$ is $t = \frac{1}{p+1}$, so that $0 < t < 1$. The second inequality is valid, if it is valid for $n = 0$, i.e. if

$$(u_0 - p)p \geq v_0(p^2 + u_0) \,. \tag{4.6}$$

For $p = \sqrt{u_0+1} - 1$ this inequality turns over into

$$v_0 \leq \frac{1}{2}\left(\sqrt{u_0+1} - 1\right) \,. \tag{4.7}$$

Hence, (4.7) implies (4.5), i.e. in view of $u_0 = x_{-1}$ and $v_0 = x_0$, (4.2) implies (4.1).

Writing $\eta_n = x_{2n}$, $\xi_n = x_{2n+1}$ then (1.6) is equivalent to

$$(1+\xi_n)\,\eta_{n+1} = \eta_n\,, \quad (1+\eta_{n+1})\xi_{n+1} = \xi_n\,.$$

For $u_n = \eta_n$ and $v_n = \xi_n$ these equations coincide with (3.2) so that (4.5) turns over into

$$\xi_n \le \xi_0 t^n\,, \quad \eta_n \ge p + (\eta_0 - p)t^n \tag{4.8}$$

with $t = \frac{1}{p+1}$, $p = \sqrt{\eta_0 + 1} - 1$, and (4.8) is valid for $n \in N_0$ when

$$0 < \xi_0 \le \frac{1}{2}\left(\sqrt{\eta_0 + 1} - 1\right)\,.$$

According to $\eta_n = x_{2n}$, $\xi_n = x_{2n+1}$ this means that (4.3) is valid when (4.4) ■

**Remark.** 1. In view of (1.7) condition (4.4) can be written as

$$x_{-1} \le \frac{1}{2}(x_0+1)\left(\sqrt{x_0+1} - 1\right) \tag{4.9}$$

and (4.2) by inversion as

$$4x_0(x_0+1) \le x_{-1}\,. \tag{4.10}$$

Hence, by positive initial values, Proposition 1 implies $x_{2n} \to 0$, $\lim\limits_{n\to\infty} x_{2n-1} \ge \sqrt{x_{-1}+1} - 1 > 0$ when (4.9), and $x_{2n-1} \to 0$, $\lim\limits_{n\to\infty} x_{2n} \ge \sqrt{x_0+1} - 1 > 0$ when (4.10).

2. The choice of $p$ in the proof of Proposition 1 is optimal, since the domain (4.6) in the first quadrant of the $(u, v)$-plane has the *envelope*

$$(v+1)p^2 - up + uv = 0\,, \quad 2(v+1)p - u = 0\,,$$

so that

$$p = \frac{u}{2(v+1)}\,, \quad u = 4v(v+1)\,,$$

i.e.

$$p = 2v\,, \quad v = \frac{1}{2}\left(\sqrt{u+1} - 1\right)\,.$$

# 5 Asymptotically three-priodic solutions

Looking for a three-periodic solution of (1.7) generated by $x_{-1} = p$, $x_0 = q$, $x_1 = r$, we have to solve the equations

$$p = (1+q)r\,, \quad q = (1+r)p\,, \quad r = (1+p)q\,. \tag{5.1}$$

Not all solutions of (5.1) can be positive, because every positive solutions of (1.7) converges to a two-periodic solution (cf. [4]). The non-trivial solutions of (5.1) are solutions of the polynomial equation

$$z^3 + 3z^2 = 3\,,$$

and if $p = z$ is one solution then

$$q = \frac{3}{z^2 - 3}\,, \quad r = \frac{3(z+1)}{z^2-3}\,.$$

Hence, e.g.

$$\begin{aligned} p &= \quad 2\cos\left(\frac{\pi}{9}\right) - 1 &\approx& \quad 0.879385\,,\\ q &= -2\sin\left(\frac{\pi}{18}\right) - 1 &\approx& -1.347296\,,\\ r &= -2\cos\left(\frac{2\pi}{9}\right) - 1 &\approx& -2.532089\,. \end{aligned}$$

For the first terms of an asymptotically three-periodic solutions we expect, as in Section 3, the structure

$$x_{3n-1} = p + at^n\,, \quad x_{3n} = q + bt^n\,, \quad x_{3n+1} = r + ct^n \tag{5.2}$$

up to an $O(t^{2n})$ where the coefficients must satisfy the equations

$$\begin{aligned} p + at^n &= (1 + q + bt^n)(r + ct^n)\,,\\ q + bt^n &= (1 + r + ct^n)(p + at^{n+1})\,,\\ r + ct^n &= (1 + p + at^{n+1})(q + bt^{n+}) \end{aligned}$$

again up to an $O(t^{2n})$, i.e. besides of (5.1),

$$(1+q)c + rb = a\,, \quad (1+r)ta + pc = b\,, \quad (1+p)tb + qta = c\,. \tag{5.3}$$

This homogeneous system has a non-trivial solution, if its determinant

$$\begin{vmatrix} -1 & r & 1+q \\ (1+r)t & -1 & p \\ qt & (1+p)t & -1 \end{vmatrix} = t^2 + 9t - 1 \tag{5.4}$$

vanishes. Since it must be $|t| < 1$ we expect the existence of an asymptotically three-periodic solution with the asymptotic approximations (5.2) and

$$t = \frac{1}{2}\left(-9 + \sqrt{85}\right) \approx 0.109772\,.$$

The corresponding solution of (5.3) reads up to a constant factor

$$a = 11z^2 + 5z - 14\,, \quad b = -z^2(t+5) - 2z + t + 6\,, \quad c = z^2(2-t) + z(1-t) - 2\,.$$

Now, we could proceed as in Section 3, but we resign from doing this. Note that the existence of a second zero of (5.4) with $t < -1$ indicates that the three-periodic solution $p$, $q$, $r$ is unstable.

## 6 Bounded solutions

Next, we verify a generalization of three conjectures concerning bounded solutions.

**Conjecture ([4]: 11.4.1).** *Assume that all coefficients of* (1.1) *are positive. Show that every positive solution is bounded.*

Even in the case that all coefficients of (1.1) are non-negative an analogous conjecture comes true if there exists a constant $M$ satisfying

$$\alpha \leq MA\,, \quad \beta \leq MB\,, \quad \gamma \leq MC\,,$$

because then every non-negative solution of (1.1) satisfies $x_n \leq M$ for $n \in \mathbb{N}$. If all coefficients in the denominator of (1.1) are positive whereas the coefficients in the numerator can remain non-negative, then we can choose

$$M = \max\left(\frac{\alpha}{A}, \frac{\beta}{B}, \frac{\gamma}{C}\right)\,.$$

This means in particular, that the preceding conjecture comes true.

The case $\gamma = 0$ was already treated in [4: Theorem 9.2.2]. The case $\beta = 0$ verifies Conjecture [4: 9.5.2], and the case $\alpha = 0$ Conjecture [4: 9.5.3].

## 7 Global behaviour

Finally, we refer to

**Conjecture ([4]: 11.4.11).** *Show that the difference equation* (1.11) *has the following trichotomy character:*

(i) *When* $p > 1$ *every positive solution converges to the positive equilibrium.*

(ii) *When* $p = 1$ *every positive solution converges to a period-five solution.*

(iii) *When* $p < 1$ *there exist positive unbounded solutions.*

In the elementary case $p = 0$ the conjecture turns out to be true. Otherwise for $p > 0$ we only can replace it by another one.

Preliminarily, we make the ansatz

$$x_n = \sum_{j=0}^{\infty} c_j a^j z^{nj} \tag{7.1}$$

with an arbitrary $a$ and put it into equation (1.11) in the form

$$x_n = x_{n+3}x_{n+2} - p\,. \tag{7.2}$$

Comparing coefficients we obtain

$$c_0 = c_0^2 - p\,, \quad ac_1(1 - c_0(z^3 + z^2)) = 0 \tag{7.3}$$

and for $k \geq 2$ the recursions

$$c_k = \frac{z^{2k}}{1 - c_0 z^{2k}(z^k + 1)} \sum_{j=1}^{k-1} c_j c_{k-j} z^j\,, \tag{7.4}$$

provided that the denominator is different from zero. The first equation of (7.3) means that $c_0$ is an equilibrium of (7.2), we choose the solution

$$c_0 = \frac{1}{2}\left(1 + \sqrt{1 + 4p}\right)\,. \tag{7.5}$$

As a function of $p$ it is strictly increasing with $c_0 \geq \frac{1}{2}$ for $p \geq -\frac{1}{4}$. The second equation yields either $ac_1 = 0$ which leads to the stationary solution $y_n = c_0$, or it leaves $ac_1$ undetermined. Without loss of generality we choose $c_1 = 1$, and it remains to study the solutions of the equation

$$z^3 + z^2 = \frac{1}{c_0} \tag{7.6}$$

for $c_0 \geq \frac{1}{2}$, which is the characteristic equation of the linearized equation associated with (7.2). The solution $z = 1$ of (7.6) with $c_0 = \frac{1}{2}$ is useless since then all denominators in (7.4) vanish. For $c_0 > \frac{1}{2}$ there exists always a positive solution with $z < 1$. For $c_0 = \frac{27}{4}$, i.e. for $p = \frac{621}{16}$, there exists also the twofold negative solution $-\frac{2}{3}$, and for $c_0 > \frac{27}{4}$ there exist two different solutions with $-1 < z < 0$. For $\frac{1}{2} < c_0 < \frac{27}{4}$ there exist two conjugate complex solutions to which we come back later on. In particular, for $c_0 = \frac{1}{2}\left(\sqrt{5} + 1\right)$, i.e. for $p = 1$, the solutions of (7.6) are

$$z_1 = e^{\frac{4\pi i}{5}}\,, \quad z_2 = e^{\frac{6\pi i}{5}}\,, z_3 = \frac{1}{2}\left(\sqrt{5} - 1\right)\,. \tag{7.7}$$

In order to construct further solutions of (7.2) we extend the ansatz (7.1) to

$$x_n = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk} a^j z^{nj} b^k w^{nk} \tag{7.8}$$

with $w \neq z$. The recursions for the coefficients are the two-dimensional generalizations of (7.4). It turns out that $w$ must be also a solution of (7.6), that $c_{j0} = c_j$, and replacing $z$ by $w$, we obtain $c_{0k}$ from $c_k$. More generally, $c_{jk}$ arises from $c_{kj}$ by exchanging $z$ and $w$. Hence $c_{jk} = \overline{c}_{kj}$ when $w = \overline{z}$. Some special cases are

$$c_{20} = \frac{z^5}{1 - c_0 z^4(z^2 + 1)}\,, \quad c_{11} = \frac{z^2 w^2 (z + w)}{1 - c_0 z^2 w^2 (zw + 1)}\,, \quad c_{02} = \frac{w^5}{1 - c_0 w^4 (w^2 + 1)}\,,$$

$$c_{30} = \frac{c_{20} z^2 (z + 1)}{1 - c_0 z^6 (z^3 + 1)}\,, \quad c_{21} = \frac{z^4 w^2 (c_{20}(z^2 + w) + c_{11} z (w + 1))}{1 - c_0 z^4 w^2 (z^2 w + 1)}\,.$$

The most general ansatz for a solution of (7.2) reads

$$x_n = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{jkl} a^j z^{nj} b^k w^{nk} c^l t^{nl} \tag{7.9}$$

with three different solutions $z$, $w$, $t$ of (7.6), where for a twofold solution $z = w$, which appears only for $p = \frac{621}{16}$, we have to replace $w^n$ by $nz^n$. There are analogous recursions, symmetries and relations as before, in particular $c_{ij0} = c_{ij}$. In the case $p \neq \frac{621}{16}$ the recursions for $c_{jkl}$ contain the denominator

$$D = 1 - c_0 z^{2j} w^{2k} t^{2l} (z^j w^k t^l + 1) \tag{7.10}$$

which has the following property:

**Lemma.** *Let $z$, $w$, $t$ be three pairwise different solutions of (7.6), let be $c_0 > \frac{1}{2}\left(\sqrt{5}+1\right)$ and $j + k + l \geq 2$ ($j, k, l \in \mathbb{N}_0$). Then $D$ from* (7.10) *is different from zero.*

**Proof.** For $j + k + l = 1$ it is $D = 0$ in view of (7.6). If the solutions $z$, $w$, $t$ are real, then they have absolute values less then 1, and the powers of these values diminish. Hence, $D > 0$ for $j + k + l \geq 2$.

Now, let $z$ be complex and $w = \bar{z}$, and assume that $D = 0$. For fixed $j$, $k$, $l$ we introduce the notation

$$z^j w^k t^l = \rho e^{i\vartheta}\,.$$

The assumption $D = 0$ implies

$$\frac{1}{c_0} = \rho^3 \cos 3\vartheta + \rho^2 \cos 2\vartheta\,, \quad \rho^3 \sin 3\vartheta + \rho^2 \sin 2\vartheta = 0\,,$$

and elimination of $\vartheta$ yields

$$c_0 = \frac{1}{2\rho^4}\left(1 + \sqrt{1 + 4\rho^2}\right)\,. \tag{7.11}$$

Since the right-hand side of (7.11) is strictly decreasing, there exists exactly one $\rho$ satisfying (7.11) for given $c_0$, namely $\rho = |z|$. For $c_0 > \frac{1}{2}(\sqrt{5}+1)$ it is $\rho < 1$, and the powers of $|z|$, $|w|$ and $t$ again diminish, so that $D \neq 0$ ■

The lemma implies that all coefficients $c_{jkl}$ exist for $c_0 > \frac{1}{2}(\sqrt{5}+1)$, i.e. for $p > 1$, where $|z| < 1$ for all solutions of (7.6). However, for $\frac{1}{2} < c_0 \leq \frac{1}{2}(\sqrt{5}+1)$, i.e. for $-\frac{1}{4} < p \leq 1$, we have $|z| = |w| \geq 1$, $t < 1$, so that $D = 0$ is possible. E.g. for $z = z_1$, $w = z_2$ from (7.7) it is $zw = 1$ and therefore $D = 0$ in (7.10) for $j = 2$, $k = 1$, $l = 0$, but then the numerator in $c_{21}$ also vanishes, and $c_{21} = c_{210}$ exists nevertheless.

In the case $p = 0$ it can easily be seen that

$$x_n = e^{az^n + b\bar{z}^n + ct^n}\,,$$

where $z$ is a complex and $t$ the real solution of (7.6) with $c_0 = 1$, is the general complex solution of (7.2) when $a$, $b$, $c$ are arbitrary, and the general positive solution when $c$ is real and $b = \overline{a}$ (cf. [4: Section 3.3]). For $a \neq 0$ it is indeed unbounded as conjectured in (iii), and obviously, it can be expanded into the form (7.9) with $c_{jkl} = \frac{1}{j!k!l!}$.

After these preparations we make the following new

**Conjecture.** *The coefficients $c_{jkl}$ exist also for $0 < p \leq 1$, for $0 < p$ the series* (7.10) *(including its modification for $p = \frac{621}{16}$) converges for all $n \in \mathbb{Z}$, and the parameters $a$, $b$, $c$ can be determined uniquely out of given positive initial values $x_{-2}$, $x_{-1}$, $x_0$.*

If this conjecture comes true, then (7.10) is the general positive solution of (1.11) and, in view of the behaviour of the solutions of (7.6) described before, the sub-conjectures (i) and (ii) are valid, and we can expect that (iii) is also valid. For $p \geq 1$ the series (7.9) are simultaneously asymptotic expansions as $n \to \infty$.

In the case $p = 1$ we can modify the ansatz (7.9) for the solutions (7.7) of (7.6) in the following way. With the notations $z = z_1$, $t = z_3$ it is $w = z_2 = \overline{z}$ so that $z^j w^k = z^{j+4k}$, and in view of $z^5 = 1$, we can replace (7.9) by

$$x_n = \sum_{m=0}^{4} \sum_{l=0}^{\infty} b_{ml} z^{nm} c^l t^{nl} \tag{7.12}$$

with

$$b_{ml} = \sum_{j+4k \equiv m \mod 5} c_{jkl} a^j b^k \,. \tag{7.13}$$

The special case of (7.12) with $c = 0$, i.e.

$$x_n = \sum_{m=0}^{4} b_{m0} z^{nm} \,, \tag{7.14}$$

yields the 5-periodic solution of (7.2) with $p = 1$ generated by

$$x_0 = r \,, \quad x_1 = s \,, \quad x_2 = \frac{r+1}{rs-1} \,, \quad x_3 = rs - 1 \,, \quad x_4 = \frac{s+1}{rs-1} \,. \tag{7.15}$$

Here $r$, $s$ are arbitrary positive parameters satisfying $rs > 1$, if we look for positive $x_n$.

Since (7.14) is a discrete Fourier-transform we easily find by inversion

$$b_{m0} = \frac{1}{5} \sum_{k=0}^{4} x_m z^{-mk}$$

with $x_m$ from (7.15). The coefficients contain the arbitrary parameters $r$, $s$ instead of $a$, $b$ in (7.13), they determine the further coefficients $b_{ml}$ in (7.12) recursively. For $r = s =$

$\frac{1}{2}(\sqrt{5}+1)$ the 5-periodic solution degenerates to the equilibrium, to which the solution (7.9) converges in the case $a = b = 0$. For the initial values $x_{-2} = x_0$, $x_{-1} = \frac{1}{x_0}$ the solution of (7.2) with $p = 1$ continuous to the left by

$$x_{2-5n} = x_{1-5n} = 0\,, \quad x_{-5n} = x_{-1-5n} = x_{-2-5n} = -1 \quad (n \in \mathbb{N})\,.$$

For $p < 0$ it is not possible to choose the initial values for the solutions of (7.2) arbitrarly, cf. [4]. Moreover, for $-\frac{1}{4} < p < 0$ besides of (7.5) also the second equilibrium $c_0 = \frac{1}{2}(1 - \sqrt{1+4p})$ is positive and must be taken into consideration.

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