On Orthogonal Double Covers of Graphs

Hans-Dietrich O.F. Gronau, Martin Grüttmüller, Sven Hartmann,
Uwe Leck and Volker Leck
Universität Rostock, Germany (sven.hartmann@mathematik.uni-rostock.de)

Abstract. An orthogonal double cover (ODC) is a collection of n spanning subgraphs (pages) of the complete graph such that they cover every edge of the complete graph twice and the intersection of any two of them contains exactly one edge. If all the pages are isomorphic to some graph G, we speak of an ODC by G. ODCs have been studied for almost 25 years, and existence results have been derived for many graph classes. We present an overview of the current state of research along with some new results and generalizations. As will be obvious, progress made in the last 10 years is in many ways related to the work of Ron Mullin. So it is natural and with pleasure that we dedicate this article to Ron, on the occasion of his 65th birthday.

Dedicated to Ron Mullin on the occasion of his 65-th birthday

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1. Introduction

Let K_n be the complete graph on an n-element vertex set V. A collection $\mathcal{O} = \{G_i : i \in V\}$ of n spanning subgraphs (called *pages*) of K_n is an *orthogonal double cover* (briefly ODC) of K_n if it has the following properties:

1. *Double cover property*
   Every edge of K_n belongs to exactly two of the pages.

2. *Orthogonality property*
   Any two distinct pages intersect in exactly one edge.

If any two pages of $\mathcal{O}$ are isomorphic, i.e. $G_i \cong G$ for all $i \in V$, then $\mathcal{O}$ is an *ODC of K_n by G*. Note that double cover and orthogonality property force every page of an ODC to contain exactly $n-1$ edges.

As a first example we present an ODC of K_5 by $C_4 \cup E_1$ in Figure 1. (Throughout the paper we will make use of the usual notations C_n for the cycle of length n, P_n for the path on n vertices, E_n for the empty graph on n vertices, S_n for the star with n edges on $n+1$ vertices, and $K_{m,n}$ for the complete bipartite graph with independent sets of sizes m and n.)

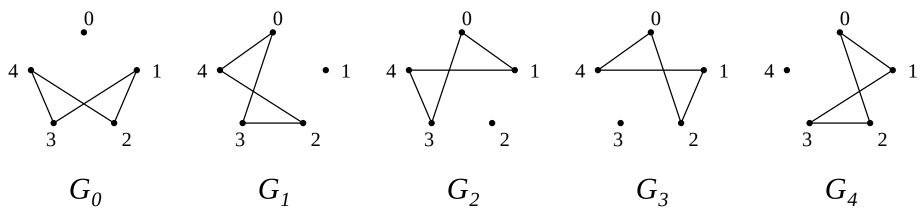

Figure 1. An ODC of K_5 by $C_4 \cup E_1$

The concept of orthogonal double covers originates in problems concerning database constraints, statistical combinatorics, and design theory. In the early 1980s Demetrovics, Füredi, and Katona [18, 17] investigated Armstrong representations of minimum size for key and functional dependencies. In our terminology, Armstrong representations of size n are equivalent to ODCs of K_n whose pages consist of distinct cliques. More information on this topic is given in Section 3.2. Another important origin is the question for ODCs the pages of which are cycles. This problem was posed first by Hering and Rosenfeld [50] in 1979. Their question arose from statistical design of experiments. Section 3.3 presents results in this regard. A variation of the problem was suggested in 1991 by Chung and West [14] for pages having maximum degree 2, i.e. pages consisting of cycles and/or a path. They came across this problem when studying intersection graphs. Finally, ODCs are related to several graph decomposition problems [2, 46].

Although the original questions of Demetrovics, Füredi and Katona and Chung and West were completely settled, the story is not quite over. During the last two decades there has been a steadily growing interest in the concept of ODCs.

The principal question is the following: Given a graph G, decide whether there is an ODC of K_n by G. While this problem is far from being solved in general, it gave rise to a large number of results on ODCs by particular graphs such as trees [32, 54, 55] or graphs with maximum degree two [10, 14, 28, 34], in particular paths [4, 43, 47, 51] and cycles [3, 7, 28, 29]. These results are discussed in Section 3.

There are two major ways to generate ODCs: direct constructions and recursive constructions. Direct constructions often exploit properties of finite algebraic structures. For example note that in Figure 1 all pages are generated from page G_0 by rotating the edges, i.e. by mapping edge (a, b) in G_0 to edge $(a+i, b+i)$ in G_i (with calculations modulo n). We study such cyclically generated ODCs and more generally group-generated ODCs in detail in Sections 2.1 and 2.3. Figure 2 shows an ODC of K_6 by $C_5 \cup E_1$ which is not generated cyclically. In fact, there is no group-generated ODC by $C_5 \cup E_1$ at all. Nevertheless, also this ODC has special properties, it is nearly-cyclic, see Section 2.6. Recursive constructions exploit for instance pairwise balanced designs as explained in Section 2.2.

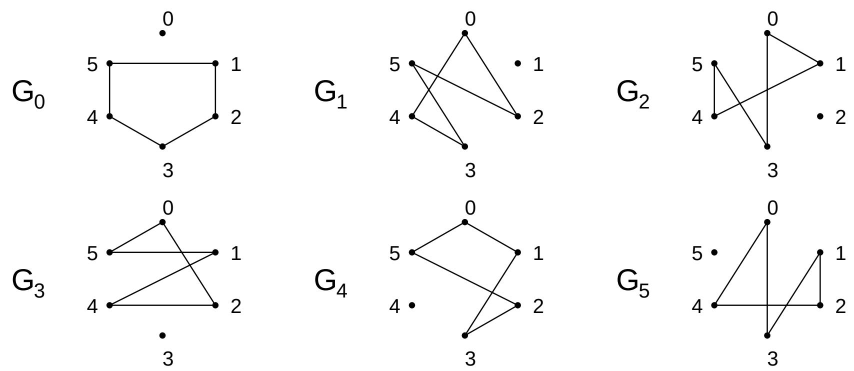

Figure 2. An ODC of K_6 by $C_5 \cup E_1$

There are graphs G which do not admit an ODC by G. The smallest graph of this kind is the P_4 which can be verified by the following argument due to Harborth [35]. Assume, on the contrary, that there is an ODC of K_4 by P_4. The first and last edge of any P_4 form a 1-factor in K_4. But K_4 contains exactly three 1-factors. Hence, two of the 4 pages share at least two edges, in contradiction to the orthogonality condition.

Let us also mention that there are cases that correspond to difficult combinatorial problems. For example, if all pages of an ODC consist of one clique and the proper number of isolated vertices, then the cliques form a biplane. See Section 3.2 for more information.

In their 1979 paper [50], which initiated the study of orthogonal double covers, Hering and Rosenfeld actually asked for the directed analogue of ODCs. For every positive integer n, let $\vec{D}_n$ denote the complete (symmetric) digraph on n vertices. A *factorization* of $\vec{D}_n$ is a collection $\mathcal{O}$ of mutually arc-disjoint, spanning subdigraphs of $\vec{D}_n$ such that every arc of $\vec{D}_n$ occurs in exactly one of the subdigraphs in $\mathcal{O}$. We call such a factorization an *orthogonal directed cover* of $\vec{D}_n$ if the union of any two of the subdigraphs in $\mathcal{O}$ contains exactly one pair of oppositely oriented arcs. For convenience, we shall again use the abbreviation *ODC* to refer to an orthogonal directed cover. Similarly, the subdigraphs in $\mathcal{O}$ are said to be the *pages* of $\mathcal{O}$ as well.

It is easy to check that an orthogonal directed cover of $\vec{D}_n$ consists of n pages, each containing precisely $n-1$ arcs. If all pages are isomorphic to some digraph $\vec{G}$, then we speak of an *ODC of $\vec{D}_n$ by $\vec{G}$*. Figure 3 exhibits an ODC of $\vec{D}_4$ by $\vec{C}_3 \cup E_1$.

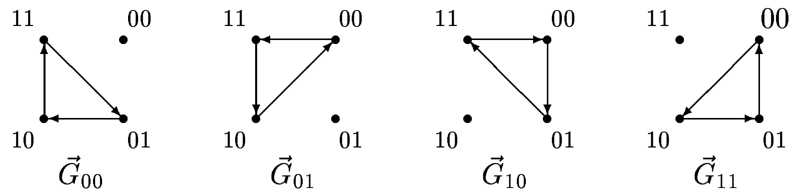

Figure 3. An ODC of $\vec{D}_4$ by $\vec{C}_3 \cup E_1$

Given an orthogonal double cover of the complete graph K_n, we may derive an orthogonal directed cover of $\vec{D}_n$: For every edge e of K_n, we simply simply assign opposite orientations to e in the two pages containing e. Vice versa, when deleting all orientations of arcs in an orthogonal directed cover of $\vec{D}_n$, we obtain an orthogonal double cover of K_n.

In general, when constructing an orthogonal directed cover from an orthogonal double cover by some graph G, the resulting pages are no longer mutually isomorphic. Conversely, however, starting with an orthogonal directed cover by some digraph $\vec{G}$, we always derive an orthogonal double cover whose pages are all isomorphic. In this sense, the concept of an orthogonal directed cover is more restrictive than the one of an orthogonal double cover.

As an example, consider the dipath $\vec{P}_3$. It is not difficult to check that there is no orthogonal directed cover of $\vec{D}_3$ by $\vec{P}_3$, although there exists an orthogonal double cover of K_3 by the underlying (undirected) path P_3.

The aim of this paper is to survey the current state of methods and results on ODCs and common generalizations. In addition we present a number of new ideas and results.

2. Construction methods

Various techniques for constructing ODCs have been developed. In this section we will explain the most important and general concepts while special constructions will be introduced as their need arises.

2.1. Group-generated ODCs

Here we focus on ODCs with large automorphism groups, since they are usually easier to find than others.

Let an additive group Γ of order n be given, and let the vertices of K_n be denoted by the elements of Γ. We call an ODC $\mathcal{O}$ of K_n *group-generated* by Γ, if we derive the page G_i of $\mathcal{O}$ from the page G_0 by adding the element $i \in \Gamma$ to each vertex of G_0, e.g. $E(G_i) = \{(a+i, b+i) \mid (a,b) \in E(G_0)\}$. This concept also applies to orthogonal directed covers. Figure 1 presents an ODC generated by $\mathbb{Z}_5$, the example in Figure 3 is generated by $\mathbb{Z}_2 \times \mathbb{Z}_2$.

A group-generated ODC is determined completely already by one arbitrary page and the generating group. Throughout this paper, let $G + e$ denote the graph that arises from a graph G on the elements of Γ by adding e to each vertex. The collection $\{G + e \mid e \in \Gamma\}$ is called the *orbit* of G under Γ and we denote it by $G + \Gamma$.

Although an ODC can be generated by any group, we restrict our attention to abelian groups. Often, we are especially interested in ODCs generated by cyclic groups. We call them *cyclic*.

We now characterize the pages of a group-generated ODC, first in the case of an orthogonal directed cover, later for an orthogonal double cover.

Let Γ be an additive group of order n and let further $\vec{G}$ be a simple digraph with $n-1$ arcs on the elements of Γ. We define the *length* $\ell(a,b)$ of an arc (a,b) to be $b-a$. The length of an arc is invariant under translation, e.g. $\ell(a,b) = \ell(a+e, b+e)$ for all $e \in \Gamma$. Hence, all pages of $\vec{G} + \Gamma$ contain the same lengths as $\vec{G}$. Thus, in order to make $\vec{G} + \Gamma$ a directed cover, the digraph $\vec{G}$ has to contain each possible length exactly once.

Let $e_1 = (a, b)$ and $e_2 = (s, t)$ be two arcs with inverse lengths. We define the *distance* $\mathrm{dist}(e_1, e_2)$ of the arc e_1 from e_2 to be $t - a = s - b$, that is, the distance is the translation (with respect to Γ) that moves e_1 onto the reverse of e_2. Now, let $d := \mathrm{dist}(e_1, e_2)$ and $e_1 \in E(\vec{G})$. Then $e_2 \in E(\vec{G} + d)$, and hence, in order for $\vec{G} + \Gamma$ to respond to the orthogonality, all the distances in $\vec{G}$ must be distinct and thus consist of all non-zero elements from Γ. Note that if $\ell(e)$ is of order two, then $\ell(e) = -\ell(e) = \mathrm{dist}(e, e)$.

In conclusion, we call $\vec{G}$ an *ODC-generating digraph* (or *ODC-generator*) with respect to the group Γ if the following conditions are satisfied:

1. *Length condition*
   For every non-zero element e of Γ, there is exactly one arc of length e in $\vec{G}$.

2. *Distance condition*
   The set of the distances of all pairs of arcs of inverse lengths in $\vec{G}$ consists of all non-zero elements from Γ.

Now, the collection $\vec{G} + \Gamma$ is an ODC, if and only if $\vec{G}$ is an ODC-generating digraph.

With respect to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the graph G_{00} in Figure 3 has the lengths $\ell(01, 10) = 11$, $\ell(10, 11) = 01$ and $\ell(11, 01) = 10$. Since all these lengths have order two, the distance of each arc equals its length. Thus, the set of lengths as well as the set of distances include all nonzero-elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $G_{00} + (\mathbb{Z}_2 \times \mathbb{Z}_2)$ is an ODC (the one presented in Figure 3).

We now consider group-generated orthogonal double covers. Let $G + \Gamma$ be an ODC. As mentioned earlier, we derive an orthogonal directed cover from $G + \Gamma$ by assigning each edge of K_n opposite directions in the two pages which contain it. Since $G + \Gamma$ is group-generated we end up with a group-generated orthogonal directed cover. Hence, the observations in the directed case apply.

We define the *length* $\ell(\{a, b\})$ of an edge $\{a, b\}$ to be the set $\{a - b, b - a\}$. Let $e_1 = \{a, b\}$ and $e_2 = \{s, t\}$ be two edges with the same lengths. W.l.o.g., let $a - t = b - s$. The *distance* $\mathrm{dist}(e_1, e_2)$ of the edges e_1 and e_2 is the set $\{t - a, a - t\}$, that is, the distance is the set of the two translations (with respect to Γ) that move one edge onto the other.

We call G an *ODC-generating graph* (or *ODC-generator*) with respect to the group Γ if the following conditions are satisfied:

1. *Length condition*
   For every element e of Γ of order greater than two, there are exactly

two edges of length $\{e, -e\}$ in G. For every element e of order two, there is exactly one edge of length $\{e\}$ in G.

2. *Distance condition*
The union of the distances of all pairs of distinct edges of the same lengths in G consists of all elements from Γ of order greater than two.

Again, $G + \Gamma$ is an ODC, if and only if G is an ODC-generating graph.

The graph G_0 in Figure 1 is an ODC-generator with respect to $\mathbb{Z}_5$, since $\ell(1,2) = \ell(3,4) = \{\pm 1\}$, $\ell(1,3) = \ell(2,4) = \{\pm 2\}$ and $\mathrm{dist}(\{1,2\},\{3,4\}) = \{\pm 2\}$, $\mathrm{dist}(\{1,3\},\{2,4\}) = \{\pm 1\}$. The ODC presented in the figure is $G_0 + \mathbb{Z}_5$.

The edges the length of which is an element of order 2 play a special role in an ODC-generating graph. This is, because each such edge moves onto itself under translation by its length.

The existence of a group-generated ODC has been proven for many graph classes (see Section 3). The following nonexistence result was first proved essentially by Ganter, Gronau and Mullin [28].

THEOREM 2.1 ([28, 58]). *Let Γ be an abelian group of order n, where $n \equiv 2 \bmod 4$. There is no ODC-generating graph G with respect to Γ, whose vertices are all of even degree.*

In the proof of Theorem 2.1, only the length condition for an ODC-generating graph is used. It shows that this already does not respond to the assumptions of the theorem.

Since we explore ODCs by cycles in Sections 3.3 and 3.5, the following consequence of Theorem 2.1 is especially important.

COROLLARY 2.2. *There is no ODC-generating graph with respect to an abelian group of order n consisting of disjoint cycles, whenever $n \equiv 2 \bmod 4$.*

2.2. PBD-CLOSURE

In this section we describe a recursive construction method to obtain ODCs of K_n using pairwise balanced designs. This approach has been successfully applied in several papers, particularly in [27, 28, 34, 29, 32, 37, 54].

A *pairwise balanced design* (PBD$[n, K]$) of order n with block sizes from K is a pair $(V, \mathcal{B})$, where V is a finite set (the *point set*) of cardinality n and $\mathcal{B}$ is a family of subsets of V called *blocks* such that every 2-subset of V is contained in exactly one block of $\mathcal{B}$, and $|B| \in K$ for every block $B \in \mathcal{B}$. A set S of positive integers is said to be *PBD-closed* if the existence of a PBD$[n, S]$ implies that n belongs to S. Further, let K be a set of positive integers and let $B(K) = \{n \mid \exists \text{PBD}[n, K]\}$. Then $B(K)$ is a PBD-closed set called the *PBD-closure* of K. For a rigorous treatment of pairwise balanced designs and related topics, the reader should consult e.g. [6, 15].

The following result is among the most interesting general theorems in design theory. It is a powerful tool for investigations of combinatorial structures, since a finite number of known examples of a certain set of objects can establish the existence of the entire set of these objects.

THEOREM 2.3 ([66]). *Let K be a non-empty PBD-closed set different from $\{1\}$. Then K is* eventually periodic *with period $\beta(K) = \gcd\{k(k-1) \mid k \in K\}$; that is, there exists a constant $c_0(K)$ such that for every $k \in K$, $\{n \mid n \geq c_0(K), n \equiv k \mod \beta(K)\} \subseteq K$.*

The main construction idea for ODCs is established in the following theorem.

THEOREM 2.4 ([27, 32]). *Let $(V, \mathcal{B})$ be a PBD$[n, K]$, and for each block $B = \{b_1, b_2, \ldots, b_{|B|}\} \in \mathcal{B}$, let $\mathcal{O}^B = \{G^B_{b_1}, G^B_{b_2}, \ldots, G^B_{b_{|B|}}\}$ be an ODC of the complete graph $K_{|B|}$ on B. Let $H_x = \bigcup_{B \in \mathcal{B}} G^B_x$ be the graph obtained by amalgamating, at x, all graphs having index x. Then $\mathcal{H} = \{H_x : x \in V\}$ is an ODC of K_n.*

Proof. To verify the double cover property, note that for every edge $\{u, w\} \in E(K_n)$ there is a unique block $B \in \mathcal{B}$ which contains both u and w. Thus, there are exactly two pages in $\mathcal{O}^B$, say G^B_i and G^B_j, which contain $\{u, w\}$. Therefore, the edge $\{u, w\}$ belongs to H_i and H_j. It is also easily seen that the orthogonality condition is satisfied. Every edge belonging to both H_x and H_y is contained in G^B_x and G^B_y, where B is the unique block containing x and y. Clearly, we have $|E(G^B_x) \cap E(G^B_y)| = 1$ which implies $|E(H_x) \cap E(H_y)| = 1$. □

Note that the graphs H_x are in general neither mutually isomorphic nor of special structure. But a careful and judicious use of the ingredient ODCs may ensure mutually isomorphic pages in the ODC obtained. Consider the following example where all ingredients in Theorem 2.4 are ODCs by comets M_m (m-stars with all edges replaced by a path of length two). If in the ingredient ODCs every page is indexed by its central vertex, then it is easily checked that the page H_x is also a comet with central vertex x. As an immediate consequence of Theorem 2.4 we have:

PROPOSITION 2.5. *Let $n = 2m + 1$ be an odd integer. Then there exists an ODC of K_n by a comet M_m.*

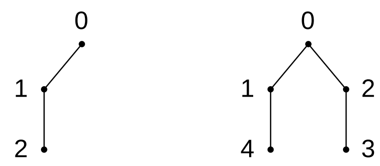

Figure 4. ODC-generators M_1 and M_2 with respect to $\mathbb{Z}_3$ and $\mathbb{Z}_5$

Proof. ODC-generators for ODCs of K_3 and K_5 by M_1 and M_2, respectively, are illustrated in Figure 4. These two ODCs are sufficient to establish the existence of all the ODCs by comets, since the PBD-closure of the set $K = \{3, 5\}$ contains all positive integers congruent to 1 modulo 2. $\square$

2.3. ODCs from rings and quasigroups

Direct constructions often exploit properties of finite groups, rings or fields. For convenience, we present the ideas for the directed case only.

Let R be a finite ring with unity 1 and let Γ denote its additive group. Given some $r \in R$, we consider the digraph $\vec{G}$ with $V(\vec{G}) = R$ and $E(\vec{G}) = \{(x, r \cdot x) \mid x \in R \setminus \{0\}\}$. According to Section 2.1, $\vec{G}$ is an ODC-generator with respect to Γ if and only if

$$\{(r-1) \cdot x \mid x \in R \setminus \{0\}\} = \{(r+1) \cdot x \mid x \in R \setminus \{0\}\} = R \setminus \{0\}$$

i.e. if and only if both, $r - 1$ and $r + 1$, are invertible.

Applying this construction to appropriate rings R and elements $r \in R$, we obtain ODCs by digraphs $\vec{G}$ of special structure. For instance, let $R = \mathbb{Z}_{k^\alpha}$ with integers $k \geq 2$, $\alpha \geq 1$, and choose $r = k$. Then $r - 1$ and $r + 1$ are invertible for $-(k-1)(k+1) = 1$, and $\vec{G}$ is an k-ary tree augmented by the arc $(k^{\alpha-1}, 0)$, see Figure 5.

THEOREM 2.6. *For all integers $k \geq 2$ and $\alpha \geq 1$, there is a cyclic ODC of $\vec{D}_{k^\alpha}$ by a k-ary tree augmented by an arc at its root.*

Let $q \geq 4$ be a prime power, and let R be the field $\mathrm{GF}(q)$. If r is a primitive k-th root of unity, then $r - 1$ and $r + 1$ are invertible. The corresponding digraph $\vec{G}$ consists of $(q-1)/k$ copies of $\vec{C}_k$ and the isolated vertex 0.

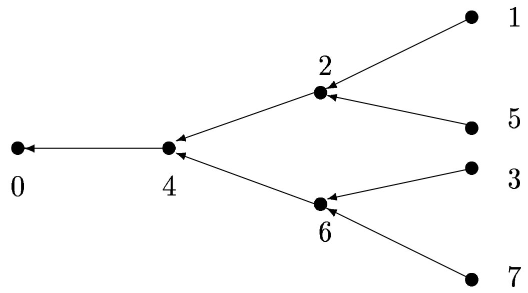

Figure 5. An almost-binary ODC-generating tree on 8 vertices

THEOREM 2.7 ([28]). *For all integers $k \geq 3$ and prime powers $q \equiv 1 \bmod k$, there is an ODC of $\vec{D}_q$ by the digraph consisting of $(q-1)/k$ disjoint copies of $\vec{C}_k$ and an isolated vertex, generated by the additive group of $GF(q)$.*

Similar results can be obtained for certain other rings. These and the ODCs given in Theorem 2.7 are often used as starting points for further construction, see [53, 42] and Section 3.5.

Another approach uses quasigroups.

THEOREM 2.8 (Ganter and Gronau [27]). *Let $\circ$ be a quasigroup operation on an n-element set P satisfying for all $x, y \in P$:*

(i) $x \circ x = x$, *and*

(ii) $(u \circ x) \circ y = u$ *has a unique solution* $u \in P$.

Then the digraphs $\vec{G}_i$ $(i \in P)$ on the vertex set P with $E(\vec{G}_i) = \{(x, x \circ i) \mid i \neq x \in P\}$ form an ODC of $\vec{D}_n$.

Several results obtained by other constructions can be restated as applications of the above theorem, using special quasigroups.

2.4. Adding ODCs

In constructing ODCs a natural approach is to try to put two given ODCs somehow together in order to obtain an ODC of a larger complete graph. More precisely, let $\mathcal{O} = \{G_1, G_2, \ldots, G_m\}$ and $\overline{\mathcal{O}} = \{G_{\overline{1}}, G_{\overline{2}}, \ldots, G_{\overline{n}}\}$ be ODCs of the complete graphs on the (disjoint) vertex sets $V = \{1, 2, \ldots, m\}$ and $\overline{V} = \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$, respectively. We are seeking graphs $H_1, H_2, \ldots, H_m$ and $H_{\overline{1}}, H_{\overline{2}}, \ldots, H_{\overline{n}}$ such that

$$\mathcal{P} = \{G_1 \cup H_1,\ G_2 \cup H_2, \ldots,\ G_m \cup H_m,\ G_{\overline{1}} \cup H_{\overline{1}},\ G_{\overline{2}} \cup H_{\overline{2}}, \ldots,\ G_{\overline{n}} \cup H_{\overline{n}}\}$$

is an ODC of K_{m+n}. Clearly, the conditions ensuring this are the following:

(1) $H_1, H_2, \ldots, H_m$ and $H_{\overline{1}}, H_{\overline{2}}, \ldots, H_{\overline{n}}$ are spanning subgraphs of the complete bipartite graph $K_{m,n}$ on the vertex set $V \cup \overline{V}$, where the independent sets are V and $\overline{V}$,

(2) $|E(H_i)| = n$ for $i = 1, 2, \ldots, m$ and $|E(H_{\overline{i}})| = m$ for $i = 1, 2, \ldots, n$,

(3) $E(H_i) \cap E(H_j) = \emptyset$ for $1 \leq i < j \leq m$ and $E(H_{\overline{i}}) \cap E(H_{\overline{j}}) = \emptyset$ for $1 \leq i < j \leq n$, and

(4) $|E(H_i) \cap E(H_{\overline{j}})| = 1$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

In other words, the graphs $H_1, H_2, \ldots, H_m$ and $H_{\overline{1}}, H_{\overline{2}}, \ldots, H_{\overline{n}}$ form an ODC of $K_{m,n}$ (see also Sections 5.2 and 5.3).

Moreover, usually we want to make sure that the pages of $\mathcal{P}$ are all isomorphic to some graph F. Obviously, F must have two (not necessarily disjoint) independent sets of size m and n, respectively, which are separated from their complements in $V(F)$ by cuts of size m and n, respectively. This rather strong condition explains that this addition construction can only be used to find ODCs by graphs with many isolated vertices or graphs (in particular, trees) with many pendant vertices.

A first application of the above method is due to Gronau, Mullin, and Rosa:

THEOREM 2.9 ([32]). *Let G be a graph on $n \notin \{2, 6\}$ vertices. Further, let G^* be the graph on $2n$ vertices obtained from G by adding n new disjoint edges, one at each vertex of G, joining this and a new vertex. If there is an ODC of K_n by G, then there is an ODC of K_{2n} by G^*.*

Proof. We use the above construction, where $\mathcal{O}$ and $\overline{\mathcal{O}}$ are both ODCs of K_n by G. Furthermore, we choose $E(H_i) = \{\{a_{ij}, \overline{b}_{ij}\} \mid j = 1, \ldots, n\}$ and $E(H_{\overline{i}}) = \{\{a_{ji}, \overline{b}_{ji}\} \mid j = 1, \ldots, n\}$, where $A = (a_{ij})$ and $B = (b_{ij})$ are two orthogonal Latin squares of order n on $\{1, 2, \ldots, n\}$. It is immediately clear that the resulting collection $\mathcal{P}$ is an ODC of K_{2n} by G^*. □

Let $\mathcal{O} = \{G_1, \ldots, G_n\}$ be an ODC of K_n by some graph G. A vertex $v \in V(G)$ is called *rotating vertex* of $\mathcal{O}$ if there are isomorphisms $\varphi_i : G \mapsto G_i$ ($i = 1, 2, \ldots, n$) such that $\{\varphi_i(v) \mid i = 1, \ldots, n\} = V(K_n)$. Note that in a group-generated ODC all vertices of G are rotating vertices of $\mathcal{O}$.

The following lemma turned out to be very useful in constructing ODCs by trees (see Section 3.4).

LEMMA 2.10 (Leck and Leck [55]). *Let F be a graph on $m+n$ vertices, and let G and G' be subgraphs of F on m and on n vertices, respectively, such that the following conditions are satisfied:*

(1) G can be obtained from F by deleting exactly n pendant vertices, all adjacent to some $v \in V(G)$, and the n edges joining these vertices and v.

(2) There exist vertices $v_1, v_2, \ldots, v_k \in V(G)$ such that G' can be obtained from F deleting exactly m pendant vertices, each adjacent to one of the vertices $v_1, \ldots, v_k$, and the m edges joining these vertices and the v_i's.

(3) There exists an ODC $\mathcal{O}$ of K_m by G such that v is a rotating vertex of $\mathcal{O}$.

(4) There exists an ODC $\overline{\mathcal{O}}$ of K_n by G' such that $v_1, \ldots, v_k$ are rotating vertices of $\overline{\mathcal{O}}$.

Then there is an ODC of K_{m+n} by F.

To show the lemma, use the above construction, and choose the H_i's and $H_{\bar{i}}$'s in the obvious way such that $G_i \cup H_i \cong G_{\bar{j}} \cup H_{\bar{j}} \cong F$. It is easy to check that the conditions for the resulting collection $\mathcal{P}$ to be an ODC are satisfied by the H_i's and $H_{\bar{i}}$'s.

As an example, an ODC of K_9 by a graph F is given in Figure 6. Consider the $m = 5$ pages on the left-hand side. The subgraphs of these pages which are induced by the vertices $1, 2, 3, 4, 5$ form an ODC $\mathcal{O}$ of K_5 by a graph G. The corresponding edges are printed bold. In all the 5 pages the vertices $\bar{1}, \bar{2}, \bar{3}, \bar{4}$ are joined to the same vertex of G, and this is a rotating vertex of $\mathcal{O}$. The corresponding thin edges form the stars $H_1, \ldots, H_5$. Similarly, the subgraphs of the $n = 4$ pages on the right-hand side of Figure 6 induced by $\bar{1}, \bar{2}, \bar{3}, \bar{4}$ form an ODC $\overline{\mathcal{O}}$ of K_4 by G'. Again, the corresponding edges are drawn bold. The thin edges in these 4 pages form the graphs $H_{\bar{1}}, \ldots, H_{\bar{4}}$ each of which is a collection of disjoint stars.

REMARK 2.11. *Lemma 2.10 can be applied recursively, due to the following simple observation: If $F, G, G', \mathcal{O}, \overline{\mathcal{O}}$ are like in the lemma and $u \in V(G) \cap V(G')$ is a rotating vertex of $\mathcal{O}$ as well as of $\overline{\mathcal{O}}$, then u is a rotating vertex of the resulting ODC of K_{m+n} by F.*

In Figure 6, for instance, the ODCs $\mathcal{O}$ and $\overline{\mathcal{O}}$ are cyclic. Hence, in the resulting ODC of K_9 the three vertices forming the only triangle in F are rotating.

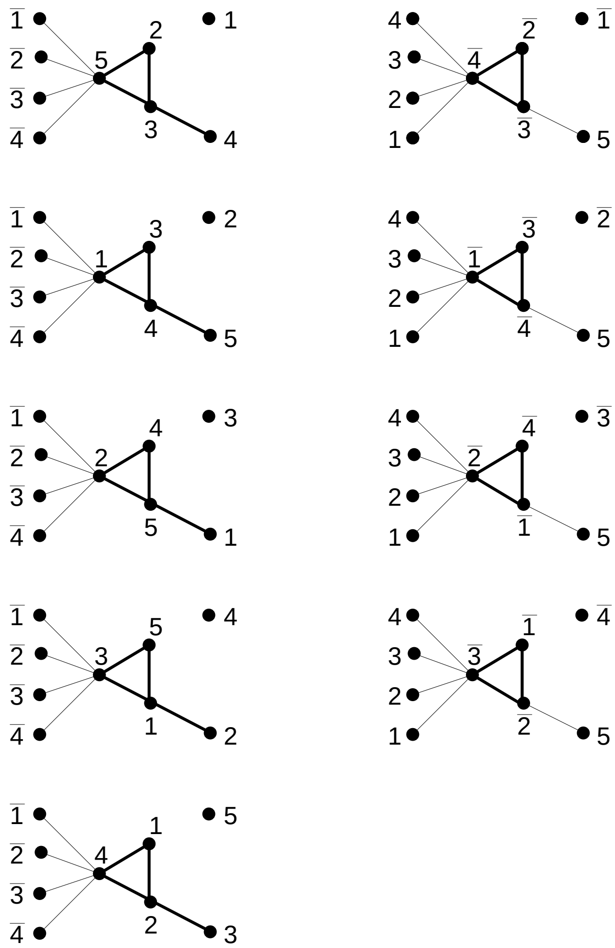

Figure 6. Adding two ODCs

Under certain circumstances it is even possible to add up group-generated ODCs to a new group-generated ODC. The following theorem covers just the very basic case of a *coset construction* where we have only stars between different cosets. More sophisticated applications of the same idea can be found as well (see [55]).

THEOREM 2.12. *Let $\mathcal{O}$ be an ODC of K_n by some graph G which is generated by a group Γ. Further let $V_1, \ldots, V_{m-1}$ be mutually disjoint sets of size n such that $V_i \cap V(G) = \emptyset$ for $i = 1, 2, \ldots, m-1$, and let G^* be obtained from G joining some $v_i \in V(G)$ to all $v \in V_i$ for $i = 1, 2, \ldots, m-1$. (Note that the v_i's are not necessarily distinct.) Then there is an ODC of K_{mn} by G^* which is generated by $\Gamma \times \mathbb{Z}_m$.*

Proof. The conditions in Section 2.1 for G^* to be an ODC-generator w.r.t. $\Gamma \times \mathbb{Z}_m$ are obviously satisfied if we label the vertices of G^* as follows: For $i = 0, 1, \ldots, m-1$ the vertices of V_i are labeled by (x, i) $(x \in \Gamma)$, where $V_0 := V(G)$. On V_0, this is done in such a way that we obtain a generator of a solution generated by Γ if we remove the second entry from all labels in G. □

2.5. Starters, halfstarters and their translates

Starters have been proven useful in the construction of a variety of combinatorial structures. We use this concept and the related notion of a halfstarter to construct ODC-generating graphs consisting of disjoint cycles and/or a path.

Let Γ be an abelian group of odd order $2n+1$. A *starter* with respect to Γ is a set of unordered pairs $S := \{\{s_i, t_i\} : 1 \leq i \leq n\}$ which satisfies:

1. $\{s_i : 1 \leq i \leq n\} \cup \{t_i : 1 \leq i \leq n\} = \Gamma \setminus \{0\}$
2. $\{\pm(s_i - t_i) : 1 \leq i \leq n\} = \Gamma \setminus \{0\}$

A *translate* $S + e$ of a starter S is the set $\{\{s_i + e, t_i + e\} : 1 \leq i \leq n\}$ for some fixed $e \in \Gamma$. The element e is called the *isolated point* of the translate.

Usually, we will view a starter or translate S as a graph on the elements of Γ. The edges of this graph are given by the unordered pairs. Hence, the graph consists of disjoint edges and exactly one isolated vertex.

EXAMPLE 2.13. *The set of unordered pairs* $\{\{1,3\},\{2,6\},\{4,5\}\}$ *is a starter with respect to* $\mathbb{Z}_7$.

For any group Γ of odd order, the set $S_p = \{\{a, -a\} : a \in \Gamma \setminus \{0\}\}$ is a starter with respect to Γ. It is called the *patterned* starter.

Now, let $S = \{\{s_i, t_i\} : 1 \leq i \leq n\}$ and $T = \{\{u_i, v_i\} : 1 \leq i \leq n\}$ be starters or (more generally) translates with respect to Γ. We may assume, w.l.o.g., that $s_i - t_i = u_i - v_i$ for all i. Then S and T are said to be *skew-orthogonal*, if $u_i - s_i = \pm(u_j - s_j)$ implies that $i = j$, and if $u_i \neq s_i$ for all i.

EXAMPLE 2.14. *The sets* $\{\{1,3\},\{2,6\},\{4,5\}\}$ *and* $\{\{1,6\},\{2,5\},\{3,4\}\}$ *are skew-orthogonal starters with respect to* $\mathbb{Z}_7$. *Note that the latter one is a patterned starter.*

Skew-orthogonal starters were first used by Stanton and Mullin [64] to construct Room squares. They are useful in the construction of ODCs because of the following fact.

PROPOSITION 2.15. *Let S and T be two skew-orthogonal starter translates with respect to some odd order group Γ. Then, the graph $S \cup T$ is an ODC-generator with respect to Γ.*

If the two starter translates share the same isolated point the resulting graph consists of disjoint cycles and exactly one isolated vertex. Otherwise, the graph includes a path and the vertices not on this path form mutually disjoint cycles. We are especially interested in the two extremal cases, i.e. the graph consists of one almost-hamiltonian cycle and an isolated vertex or the graph is a path.

EXAMPLE 2.16. *The starters in Example 2.14 yield an ODC-generating almost-hamiltonian cycle with respect to $\mathbb{Z}_7$.*

A starter translate $S = \{\{s_i, t_i\} : 1 \leq i \leq n\}$ with respect to some abelian group Γ is said to be *skew* if $\{\pm(s_i + t_i) : 1 \leq i \leq n\} = \Gamma \setminus \{0\}$, that is $s_i + t_i = \pm(s_j + t_j)$ implies $i = j$, and $s_i + t_i \neq 0$ for all i. Skew translates are an important tool in the construction of ODC-generating graphs because of the following fact, observed first in the case of starters, by Mullin and Nemeth [61].

LEMMA 2.17 ([61]). *If there exists a skew translate S with respect to some group Γ, then the translates S, $-S$ and the patterned starter S_p with respect to Γ are pairwise skew-orthogonal.*

Mullin and Nemeth [61] present a first class of skew starters, later called *Mullin-Nemeth* starters. Let q be a prime power of the form $q = 2^k t + 1$, where $t > 1$ is odd. Let ω be a primitive element in the finite field $GF(q)$. Further, define S_0 to be the multiplicative subgroup of $GF(q) \setminus \{0\}$ of order t. Then, S_0 has the cosets $S_i = \omega^i S_0$ for $1 \leq i \leq 2^k - 1$. With $\Delta := 2^{k-1}$ and $H := \bigcup_{i=0}^{\Delta-1} S_i$, the Mullin-Nemeth starter is defined by $M = \{\{x, \omega^{\Delta} x\} : x \in H\}$.

The starter in Example 2.13 is a Mullin-Nemeth starter in GF(7) with $S_0 = \{1, 2, 4\}$, $\Delta = 1$ and $\omega = 3$.

The Mullin-Nemeth construction yields a skew starter in any field $GF(q)$ when q is odd, except when $q = 9$ or when q is a Fermat prime. Chong and Chan [13] give a construction of a skew starter in $GF(q)$ when q is a Fermat prime greater than 5. This construction is generalized by Dinitz [19] to work in the ring $\mathbb{Z}_{16t^2+1}$. Lins and Schellenberg [59] give a short proof of this generalization. Thus, for all odd prime powers q, there exists an abelian group that admits a skew starter, with the exceptions $q = 3, 5,$ and 9 where no skew starter exists.

Mullin-Nemeth starters have been generalized in many ways, e.g. to quotient coset starters [20] and to Dinitz starters [19].

The concept of halfstarters, introduced by Anderson and Leonard [4] and strongly related to starters, turns out to be useful for our purposes as well.

Let Γ be an abelian group of order $4n+1$. A *halfstarter* with respect to Γ is a set of unordered pairs $H := \{\{s_i, t_i\} : 1 \leq i \leq 2n\}$, which satisfies

1. $\{s_i : 1 \leq i \leq 2n\} \cup \{t_i : 1 \leq i \leq 2n\} = \Gamma \setminus \{0\}$,
2. the set of lengths $\{\pm(s_i - t_i) : 1 \leq i \leq 2n\}$ contains exactly half the nonzero elements of Γ and each such element occurs exactly twice as a length.

As in the case of a starter, we define a *translate* $H + e$ of the halfstarter H to be the set $\{\{s_i + e, t_i + e\} : 1 \leq i \leq 2n\}$ for some fixed $e \in \Gamma$. Again, the element e is called the *isolated point* of the translate.

Usually, we will view a halfstarter as a graph on the elements of the group Γ. Similar to the starter case, the resulting graph consists of disjoint edges and exactly one isolated vertex.

EXAMPLE 2.18. *The set*

$$H = \{\{2, 10\}, \{1, 4\}, \{0, 3\}, \{5, 9\}, \{6, 11\}, \{8, 12\}\}$$

is a halfstarter translate with respect to $\mathbb{Z}_{13}$.

Two halfstarter translates H and K with respect to Γ are said to be *supplementary* if their sets of lengths are disjoint. This means that the graph $H \cup K$ fulfills the length condition for an ODC-generating graph.

Since every occurring length in a halfstarter H appears exactly twice, a halfstarter induces a set D_H of distances of edges of the same length. If, for two supplementary halfstarters (or translates) H and K, the union of D_H and D_K is $\Gamma \setminus \{0\}$, we call the halfstarters (or translates) *complementary.*

PROPOSITION 2.19. *Let* H *and* K *be complementary halfstarter translates with respect to* Γ. *Then, the graph* $H \cup K$ *is an ODC-generator with respect to* Γ.

As in the case of starters, two halfstarter translates form a graph that consists of disjoint cycles and exactly one isolated vertex or a graph that includes a path and disjoint cycles. Again, we are especially interested in the two extremal cases, i.e. the graph consists of one almost-hamiltonian cycle and an isolated vertex or the graph is a path. For example, the translate

$$K = \{\{5, 12\}, \{9, 11\}, \{4, 6\}, \{0, 1\}, \{7, 8\}, \{3, 10\}\}$$

with respect to $\mathbb{Z}_{13}$ is complementary to the translate H from Example 2.18. Their union $H \cup K$ is an ODC-generating path with respect to $\mathbb{Z}_{13}$.

An important property of a halfstarter H is the invariance of its distance set D_H under translation. Clearly, for all translates $H + e, e \in \Gamma$, the set D_{H+e} equals D_H. Hence, for two complementary halfstarters H and K, not only the graph $H \cup K$ is ODC-generating, but also $(H + e) \cup K$ for any $e \in \Gamma$.

PROPOSITION 2.20. *Let H and K be complementary halfstarter translates with respect to Γ. Then, the graph $(H + e) \cup K$ is an ODC-generator with respect to Γ for any $e \in \Gamma$.*

The next theorem presents a class of complementary halfstarters in a finite field of order $q = 4mt + 1$.

THEOREM 2.21. *Let $GF(q)$ be a finite field of order $q \equiv 1 \bmod 4$. Furthermore, let $\mathcal{S}$ denote the multiplicative subgroup of $GF(q)^*$ generated by an element e with multiplicative order $4m$. Let $\{r_1, \ldots, r_k\}$ be a coset representation of $GF(q)^*$ modulo $\mathcal{S}$. Then, the sets*

$$\begin{aligned} H_1 = \{&\{r_1, er_1\}, \{e^2 r_1, e^3 r_1\}, \ldots, \{e^{4m-2} r_1, e^{4m-1} r_1\}, \\ &\{r_2, er_2\}, \{e^2 r_2, e^3 r_2\}, \ldots, \{e^{4m-2} r_2, e^{4m-1} r_2\}, \\ &\quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &\{r_k, er_k\}, \{e^2 r_k, e^3 r_k\}, \ldots, \{e^{4m-2} r_k, e^{4m-1} r_k\}\} \end{aligned}$$

and

$$\begin{aligned} H_2 = \{&\{er_1, e^2 r_1\}, \{e^3 r_1, e^4 r_1\}, \ldots, \{e^{4m-1} r_1, r_1\}, \\ &\{er_2, e^2 r_2\}, \{e^3 r_2, e^4 r_2\}, \ldots, \{e^{4m-1} r_2, r_2\}, \\ &\quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &\{er_k, e^2 r_k\}, \{e^3 r_k, e^4 r_k\}, \ldots, \{e^{4m-1} r_k, r_k\}\} = eH_1 \end{aligned}$$

are complementary halfstarters with respect to the additive group of $GF(q)$.

2.6. Nearly-cyclic ODCs and extendability

Hering [48] introduces a class of ODCs by almost-hamiltonian cycles in which all but one page are generated cyclically from a given cycle.

Let $g \in \mathbb{Z}_n$ be an element of order n. An ODC $\mathcal{O}$ of K_{n+1} by an almost-hamiltonian cycle is called *nearly-cyclic*, if it consists of cycles $G_0, \ldots, G_{n-1}$ and G_∞ on the vertex set $\mathbb{Z}_n \cup \{\infty\}$ such that, for $i \in \mathbb{Z}_n$, $G_i = G_0 + i$ where the addition in $\mathbb{Z}_n$ is extended by $\infty + i = \infty$. Furthermore, G_∞ is the almost-hamiltonian cycle $(0, g, 2g, 3g, \ldots, (n-1)g)$ generated by the multiples of g with the isolated vertex ∞. We say, the pair (G_0, g) generates $\mathcal{O}$.

EXAMPLE 2.22. *If we choose $g = 1$ in $\mathbb{Z}_5$ and let $G_0 = (\infty, 0, 3, 4, 2)$, then the pair (G_0, g) generates a nearly-cyclic ODC of K_6 consisting of the almost-hamiltonian cycles $G_0 = (\infty, 0, 3, 4, 2)$, $G_1 = (\infty, 1, 4, 0, 3)$, $G_2 = (\infty, 2, 0, 1, 4)$, $G_3 = (\infty, 3, 1, 2, 0)$, $G_4 = (\infty, 4, 2, 3, 1)$ and $G_\infty = (0, 1, 2, 3, 4)$.*

In non-cyclic groups there are no generating elements. Hence, for those groups the graph G_∞ is not an almost-hamiltonian cycle. Therefore, we focus on cyclic groups.

Next, we describe necessary and sufficient conditions on a pair to generate a nearly-cyclic ODC.

LEMMA 2.23 (Hering [48]). *Let G be an almost-hamiltonian cycle on the elements of $\mathbb{Z}_n \cup \{\infty\}$ with the edges $\{\infty, a\}$ and $\{\infty, b\}$. Furthermore, let P_G denote the path arising from G by deleting the edges $\{\infty, a\}$ and $\{\infty, b\}$. The pair (G, g), where $g \in \mathbb{Z}_n$ is of order n, generates a nearly-cyclic ODC by an almost-hamiltonian cycle, if and only if the following conditions hold:*

1. *For all elements $z \in \mathbb{Z}_n \setminus \{0, g\}$ of order not 2, there are exactly two edges of length $\{\pm z\}$ in P_G.*
2. *If n is even, then there is exactly one edge of length $\{\frac{n}{2}\}$ in P_G.*
3. *There is exactly one edge of length $\{\pm g\}$ in P_G.*
4. *The order of the element $a - b$ is not 2, e.g. $a - b \neq \frac{n}{2}$.*
5. *The union of the distances of edges of the same length includes all elements from $\mathbb{Z}_n \setminus \{0, a-b, -a+b\}$ of order not 2.*

In the cycle $G = (\infty, 0, 3, 4, 2)$ from Example 2.22, we have that $P_G = 0, 3, 4, 2$. In $\mathbb{Z}_5$, $\ell(\{0,3\}) = \{\pm 2\}$, $\ell(\{3,4\}) = \{\pm 1\}$, $\ell(\{4,2\}) = \{\pm 2\}$ and $\operatorname{dist}(\{0,3\}, \{4,2\}) = \{\pm 1\}$. Clearly, there is no element of order 2 in $\mathbb{Z}_5$. Hence, the pair $(G, 1)$ fulfills the conditions of Lemma 2.23.

The path $0, 6, 2, 7, 1, 4, 5$ contains with respect to $\mathbb{Z}_8$ the lengths $\{\pm 1\}$ and $\{\pm 4\}$ once, and the lengths $\{\pm 2\}$ and $\{\pm 3\}$ twice. The distances of the edges of the latter lengths are $\{\pm 2\}$ and $\{\pm 1\}$. Furthermore, $0 - 5 = 3$ is of order 8. Thus, with $G = (\infty, 0, 6, 2, 7, 1, 4, 5)$, the pair $(G, 3)$ generates a nearly-cyclic ODC of K_9 by an almost-hamiltonian cycle.

We observe that the conditions on a pair (G, g) to generate a nearly-cyclic ODC in Lemma 2.23 are similar to those for an ODC-generator. Hence, it is natural to construct such a pair from a cyclic ODC-generator.

Let us consider an ODC-generator H with respect to $\mathbb{Z}_n$. By deleting one edge of H, we want to generate the path P_G of an almost-hamiltonian cycle G on the vertex set $\mathbb{Z}_n \cup \infty$. Hence, H can be a path or an almost-hamiltonian cycle.

We investigate almost-hamiltonian cycle s. Let the cycle $H = (v_1, \ldots, v_{n-1})$ be an ODC-generator with respect to $\mathbb{Z}_n$. We obtain a cycle G on the vertex set $\mathbb{Z}_n \cup \{\infty\}$ after replacing an edge $\{v_i, v_{i+1}\}$ in H by the path v_i, ∞, v_{i+1}. If there is an edge $\{v_i, v_{i+1}\}$ in H and an element $g \in \mathbb{Z}_n$ such that the pair (G, g) generates a nearly-cyclic ODC of K_{n+1} by an almost-hamiltonian cycle, then we call H *extendable* at g.

There are necessary and sufficient conditions on a cyclic ODC-generating cycle to be extendable.

LEMMA 2.24. *An ODC-generating almost-hamiltonian cycle H with respect to $\mathbb{Z}_n$ is extendable, if and only if there is an element $g \in \mathbb{Z}_n$ of order n such that the two edges of length $\{\pm g\}$ in H have also distance $\{\pm g\}$.*

The two edges of length $\{\pm g\}$ have a common vertex. Hence, they are neighboring edges on the cycle.

EXAMPLE 2.25. *The cycle $(1, 5, 2, 3, 4, 6)$ is an ODC-generator with respect to $\mathbb{Z}_7$. Furthermore, $\ell(\{2, 3\}) = \ell(\{3, 4\}) = \{\pm 1\}$. Thus, by Lemma 2.24, the cycle is extendable, and the pair $((\infty, 3, 4, 6, 1, 5, 2), 1)$ generates a nearly-cyclic ODC of K_8 by an almost-hamiltonian cycle.*

Leck [57] constructs extendable ODC-generating cycles consisting of two skew-orthogonal starters. Using a hill-climbing algorithm due to Dinitz and Stinson [21] he derives the following result.

LEMMA 2.26 ([57]). *Let $7 \leq 2n + 1 \leq 101$, $2n + 1 \neq 9$. There exists an extendable ODC-generating almost-hamiltonian cycle with respect to $\mathbb{Z}_{2n+1}$.*

COROLLARY 2.27 ([57]). *Let* $8 \leq 2n \leq 102$*,* $2n \neq 10$*. There is a nearly-cyclic ODC of* K_{2n} *by an almost-hamiltonian cycle.*

The unique ODC-generating cycle with respect to $\mathbb{Z}_5$ is not extendable. There is no ODC-generating cycle with respect to $\mathbb{Z}_9$ [49].

The cycle in Example 2.25 is extendable at any $g \in \mathbb{Z}_7 \setminus \{0\}$, i.e. it is super-extendable. More general, we call an almost-hamiltonian cycle on the elements of some odd order abelian group Γ *super-extendable*, if, for every $a \in \Gamma \setminus \{0\}$, there are exactly two edges of length $\{\pm a\}$, and these two edges are adjacent.

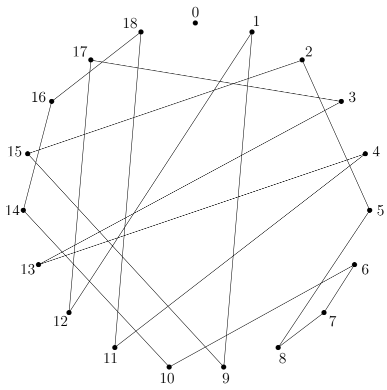

Figure 7. A super-extendable cycle with respect to $\mathbb{Z}_{19}$

A super-extendable almost-hamiltonian cycle consists of two skew-orthogonal starters. Hence, super-extendability implies the property of being ODC-generating. The resulting cycle can even be oriented alternately to supply an ODC-generating anti-directed cycle.

Since in a cyclic group $\mathbb{Z}_p$ of prime order every non-zero element generates the group, a super-extendable cycle with respect to $\mathbb{Z}_p$ is extendable at any of its edges.

For example, the cycle $(1, 4, 7, 5, 3, 12, 8, 2, 9, 10, 11, 6)$ generates an ODC with respect to $\mathbb{Z}_{13}$ and is super-extendable.

Bey, Hartmann, Leck and Leck [7] use quotient coset starters in finite fields to construct super-extendable cycles.

THEOREM 2.28 (Bey, Hartmann, Leck and Leck [7]). *Let* $GF(q)$ *be a finite field of order* $q = 2^e t + 1$ *where* t *is odd. There exists a super-extendable almost-hamiltonian cycle with respect to the additive group of* $GF(q)$ *if* $t \geq t_0(e)$*. Moreover* $t_0(1) = 3$ *and* $t_0(2) = 3$*.*

COROLLARY 2.29. *Let* $p \equiv 5$ *or* $7 \bmod 8$ *be a prime. There exists a nearly-cyclic ODC of* K_{p+1} *by an almost-hamiltonian cycle.*

3. Existence results for special graph classes

In this section, we present ODCs for several classes of graphs.

3.1. Small graphs

Using exhaustive search methods, the existence of an ODC by a graph G has been determined for all graphs with at most 10 vertices, up to two possible exceptions [56]. It turns out that most of these graphs even admit a cyclic ODC-generator.

Table I. ODCs for graphs with at most 10 vertices

vertices	no. graphs	ODCs cyclic	ODCs non-cyclic	ODCs none
2	1	1	-	-
3	1	1	-	-
4	3	2	-	1
5	6	4	1	1
6	15	13	1	1
7	41	30	7	4
8	115	95	11	9
9	345	321	17	7
10	1103	1059	36 ± 1	8 ∓ 1

All but 23 of the graphs with up to 9 vertices admit an ODC. All but 60 of them even have a cyclic ODC-generator. There are 44 graphs on 10 vertices without a cyclic ODC-generator.

3.2. Clique graphs

Figure 3 in the introduction shows an ODC of K_4 by the complete graph K_3 augmented by an isolated vertex, provided we omit the orientation of all the arcs. It is natural to ask for ODCs by other complete graphs, but this question turns out to be hard. For every $n = \binom{k}{2} + 1$, an ODC of K_n by $K_k \cup E_{n-k}$ corresponds to a biplane with block size k, that is, to a symmetric $(n, k, 2)$ block design (see [15] for definition). So far, biplanes are only known for $k = 1, 2, 3, 4, 5, 6, 9, 11, 13$. Conversely, biplanes do not exist for infinitely many values of k due to the Bruck-Ryser-Chowla theorem, cf. [15]. In particular there is no biplane with block size 7. In fact, it is widely believed that there exist only finitely many biplanes.

THEOREM 3.1. *There exists an ODC of K_n by $K_k \cup E_{n-k}$ for $n = \binom{k}{2} + 1$ and $k = 1, 2, 3, 4, 5, 6, 9, 11$ and 13.*

Every vertex-disjoint union of complete graphs is called a *clique graph*. In particular, let mK_k denote the union of m copies of K_k. Motivated by problems from combinatorial database theory, Demetrovics, Füredi and Katona [17] got interested in orthogonal double covers whose pages are clique graphs. In particular, they constructed ODCs of K_n by $mK_3 \cup E_1$ for every $m \equiv 0, 1 \bmod 4$ and $n = 3m+1$. In addition, they found an ODC of K_7 by $2K_3 \cup E_1$: A cyclic ODC-generator of this ODC consists of an isolated vertex 0 and two copies of K_3 with vertex sets $\{1, 2, 4\}$ and $\{3, 5, 6\}$. This motivated them to conjecture the existence of an ODC by $mK_3 \cup E_1$ for every m.

Gronau and Ganter [27] were the first who observed that the set of all integers $n = 3m+1$ admitting an ODC of K_n by $mK_3 \cup E_1$ is PBD-closed. Note that the PBD-closure of $\{4, 7\}$ is $\{n : n \equiv 1 \bmod 3, n \neq 10, 19\}$. For $n = 19$ there exists a cyclic ODC whose ODC-generator consists of the isolated vertex 0 and six copies of K_3 whose vertex sets are $\{1, 7, 11\}$, $\{2, 14, 3\}$, $\{4, 9, 6\}$, $\{5, 16, 17\}$, $\{8, 18, 12\}$ and $\{10, 13, 15\}$. For $n = 10$, however, Rausche [62] excluded an ODC by $3K_3 \cup E_1$.

THEOREM 3.2 ([27]). *There exists an ODC of K_n with $n = 3m + 1$ by $mK_3 \cup E_1$ if and only if $m \neq 3$, that is, $n \neq 10$.*

Surprisingly, one cannot find more than seven mutually orthogonal subgraphs of K_{10}. The same result was also obtained by Bennett and Wu [5] who analyzed all nearly-resolvable $(10, 3, 2)$ block designs.

In [17], Demetrovics, Füredi and Katona actually conjectured the existence of an ODC of K_n by clique graphs for every $n \geq 9$. The preceding results verify the conjecture for $n = 2, 11, 56$ and for all $n \equiv 1 \bmod 3$ apart from $n = 10$. Conversely, for $n = 3$ and 6 there does not even exist a clique graph with n vertices and $n-1$ edges, which is a necessary property of the pages in an ODC of K_n. For $n = 5$ the single clique graph is $K_3 \cup K_2$. Suppose there is an ODC of K_5 by $K_3 \cup K_2$. It must contain two pages which have their copy of K_2 in common. This, however, forces the two pages under inspection to share a second edge. Consequently, there is no suitable ODC of K_5. For $n = 8$ the only clique graphs are $2K_3 \cup K_2$ and $K_4 \cup K_2 \cup E_2$. A complete computer search [27] showed that there is no ODC of K_8 by clique graphs, not even if we allow non-isomorphic pages.

Eventually, the conjecture of Demetrovics, Füredi and Katona was settled by Bennett and Wu [5] and, independently, by Gronau and Mullin [30].

THEOREM 3.3 ([5, 30]). *There exists an ODC of K_n by a clique graph if and only if $n \neq 3, 5, 6, 8$.*

The proof is much more laborious than the previous one. Below we sketch the approach of Gronau and Mullin [30]. For $n = 9$ there is a cyclic ODC by $K_4 \cup 2K_2 \cup E_1$, whose ODC-generator consists of the isolated vertex 0 and the cliques with vertex sets $\{1, 2, 4, 5\}$, $\{3, 7\}$ and $\{6, 8\}$. Starting with this solution and the ODC of K_{11} arising from the biplane with block size 5, suitable ODCs can be found for every $n \leq 142$, $n \neq 3, 5, 6, 8, 10$ by PBD-closure or direct constructions. Put $S_q := \{n \leq q : n \neq 3, 5, 6, 8, 10\}$. Exploiting group divisible designs we obtain the following result:

LEMMA 3.4. *Let $q \geq 13$ be a prime power. If there are ODCs of K_n by clique graphs for all $n \in S_{11q-1}$, then there are such ODCs for all $n \in S_{q(q+1)}$, too.*

Due to Bertrand's postulate (see [24]) on primes, this lemma enables us to complete the proof of Theorem 3.3 for all n apart from $n = 10$. However there is a cyclic ODC of K_{10} by $K_4 \cup 3K_2$ whose ODC-generator consists of the cliques with vertex sets $\{0, 1, 3, 5\}$, $\{2, 9\}$, $\{4, 8\}$ and $\{6, 7\}$.

The direct constructions used in [30] for $n \leq 142$ yield suitable ODCs whose pages consist of one large clique, copies of K_2, and many isolated vertices. Of course, ODCs whose pages contain small cliques only would be of interest as well. Although this question seems to be a difficult one in general, a first result was given by Gronau, Mullin and Schellenberg.

THEOREM 3.5 ([34]). *Let $n = 6m + 2$ with $n \neq 8$. Then there exists an ODC of K_n by $2mK_3 \cup K_2$.*

Proof. Due to Brouwer, Hanani and Schrijver [9] there is a group divisible design of type 2^s and with blocks of size 4 whenever $s \equiv 1 \bmod 3$, $s \geq 7$. Therefore we may apply the PBD-construction using the well-known ODC of K_4 by $K_3 \cup E_1$ and the trivial ODC of K_2 by K_2 as ingredients. This gives the claimed ODC for every $n = 2s \equiv 2 \bmod 6$ with the exception $n = 8$. □

Further results on ODCs by clique graphs are presented in Section 4 below.

3.3. Cycle families

Clearly, the triangles in Theorem 3.2 can also be looked at as cycles. This leads to the existence problem for ODCs whose pages consist of vertex disjoint C_k's. This was first posed by Hering [49]. Recall that a necessary condition is $n \equiv 1 \bmod k$.

Granville, Gronau and Mullin proved the following results.

THEOREM 3.6 ([29]). *There exists an ODC of K_n with $n = 4m + 1$ by $mC_4 \cup E_1$ for all $m \geq 1$.*

Proof. Theorem 2.7 shows the existence of the required ODCs for $n = 5, 9, 13, 17, 25, 29$. The generating set for PBDs of order $\{n \equiv 1 \bmod 4\}$ is $\{5, 9, 13, 17, 25, 29, 33\}$, see [15]. Hence, it suffices to give an ODC-generating graph for the case $n = 33$ to establish the theorem. Here we take an isolated vertex 0 and 8 copies of C_4, namely $(1, 2, 4, 3)$, $(5, 8, 12, 19)$, $(6, 28, 20, 16)$, $(7, 31, 15, 25)$, $(9, 22, 10, 30)$, $(11, 27, 24, 17)$, $(13, 21, 26, 32)$, $(14, 23, 18, 29)$. □

Note that this construction also works for cyclically oriented C_k's. Moreover, the PBD-construction yields similar results for larger k. Unfortunately, the sizes of the generating sets increase rapidly.

However, using Theorem 2.3 together with ODCs from Theorem 2.7 for prime powers we have:

THEOREM 3.7 ([29]). *Let $k \geq 3$ be an integer. Then there exists a constant n_k such that there is an ODC of K_n by $mC_k \cup E_1$ for all $n = km + 1 > n_k$.*

The important case of ODCs by almost-hamiltonian cycles, i.e. ODCs of K_n by $C_{n-1} \cup E_1$, is treated in Section 3.5.

In [28] the existence of ODCs of K_n by pages consisting of short cycles was investigated. The following result is a typical one. (For definition of *idempotent ODC* see Section 4.1.)

THEOREM 3.8 ([28]). *An idempotent ODC of K_n by pages consisting of an isolated vertex and vertex disjoint cycles of length 3, 4, or 5 exists if and only if $n \geq 4$, $n \neq 8$.*

An interesting case is $n = 10$. There is no solution if every page is isomorphic to $G = 3C_3 \cup E_1$ or if every page is isomorphic to $G' = C_5 \cup C_4 \cup E_1$. But there exists an ODC with all pages isomorphic to either G or G'.

Theorem 3.8 solves a problem of Chung and West [14] who conjectured that for all $n \geq 4$ there is an ODC of K_n whose pages have all maximum degree 2. (A solution for $n = 8$ can be derived from Theorem 3.20.) A simple proof of this conjecture, due to Bryant and Khodkar [10], uses mutually orthogonal Latin squares.

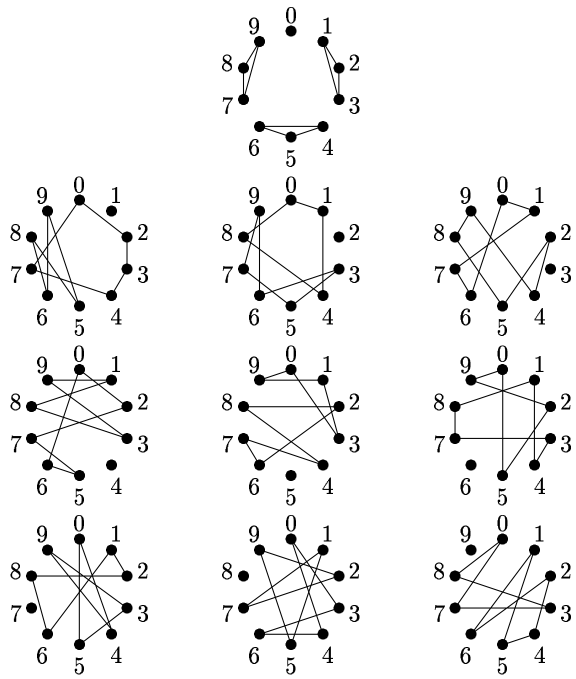

Figure 8. An ODC of K_{10} whose pages are cycle families

3.4. Trees

Since every page of an ODC of K_n has exactly $n-1$ edges it is natural to study ODCs by trees. As mentioned already in the introduction, one tree not admitting an ODC is the P_4, the path with four vertices. In fact, this makes the P_4 exceptional, at least within a class of small trees:

THEOREM 3.9. *Let $T \neq P_4$ be a tree on n vertices, where $2 \leq n \leq 14$. Then there is an ODC of K_n by T.*

This was found by computer and generalizes previous results published in [32] ($n \leq 10$) and [54] ($n \leq 13$). It turns out that exactly 5418 of the 5445 trees in question even admit cyclic solutions (see Table II). 19 of the remaining 27 trees are of diameter three, they are characterized by Theorem 3.13 below and admit non-cyclic solutions according to Theorem 3.12. Another 5 trees without cyclic solutions are caterpillars of diameter four, the $(2,2,2)$-, $(2,2,3)$-, $(3,3,2)$-, $(4,2,4)$-, and $(3,3,4)$-trees, where by $(p_1, p_2, \ldots, p_t)$-*tree* ($p_1, p_t \neq 0$) we mean the caterpillar consisting of a path $x_1, x_2, \ldots, x_t$ and exactly p_i pendant

Table II. ODCs by trees

vertices	trees	ODCs cyclic	ODCs non-cyclic
2	1	1	–
3	1	1	–
4	2	1	–
5	3	2	1
6	6	6	–
7	11	8	3
8	23	20	3
9	47	45	2
10	106	103	3
11	235	230	5
12	551	550	1
13	1301	1294	7
14	3159	3157	2

vertices adjacent to x_i ($i = 1, 2, \dots, t$) (see Figure 9). The 3 remaining trees not admitting a cyclic ODC are the path P_7, the graph Y_3 consisting of three copies of the star S_3 on four vertices glued together at a pendant vertex, and the $(1, 0, 0, 3)$-tree.

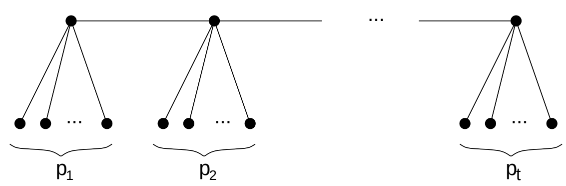

Figure 9. The $(p_1, p_2, \dots, p_t)$-tree

Theorem 3.9 supports the following conjecture of Gronau, Mullin, and Rosa:

CONJECTURE 3.10 ([32]). *Let T be an arbitrary tree on $n \geq 2$ vertices, different from P_4. Then there exists an ODC of K_n by T.*

This has been confirmed for certain classes of trees of small diameter. Clearly, every assignment of $\mathbb{Z}_n$ to the vertices of a star with n vertices results in an ODC-generator. Thus, we have the following fact about trees of diameter two:

PROPOSITION 3.11. *There exists a cyclic ODC of K_n by S_{n-1}, the star with $n-1$ edges.*

Using the construction described in Section 2.4 to add two ODCs by S_{p+1} and S_{q-1}, respectively, solves the problem for all trees of diameter three:

THEOREM 3.12. *[Gronau, Mullin and Rosa [32]] Let T be the (p,q)-tree, where $pq > 1$, and let $n = p+q+2$. Then there is an ODC of K_n by T.*

For this class of trees it is also known which of its members allow a cyclic solution. As pointed out in Section 2.4 for instance, these are often appropriate starting points for recursive constructions.

THEOREM 3.13 (Leck and Leck [54]). *Let T be the (p,q)-tree on $n = p+q+2$ vertices. There is a cyclic ODC of K_n by T if and only if n and pq are not relatively prime.*

It is easy to find an ODC-generating (p,q)-tree ($\gcd(n,pq) \geq 2$) w.r.t. $\mathbb{Z}_n$. Let $d \geq 2$ be a common divisor of n and p, and put $t := n/d$. Furthermore, let x and y be the two non-pendant vertices. The following two conditions are necessary and sufficient for such an ODC-generator:

(1) x and y lie in the same coset modulo $t \cdot \mathbb{Z}_n$.

(2) If two pendant vertices u and v are contained in the same coset modulo $t \cdot \mathbb{Z}_n$, then they are both adjacent to x, or they are both adjacent to y.

A similar statement holds for a special class of trees of diameter four.

LEMMA 3.14 ([55]). *Let T be the (p,q,r)-tree, where $q \leq p-r$, and let $n = p+q+r+3$. Then there is a cyclic ODC of K_n by T.*

The corresponding ODC-generator is given in [55]. Note that Lemma 3.14 does not describe all (p,q,r)-trees which have a cyclic solution. For instance, ODC-generating $(2p,1,2p)$-trees w.r.t. the corresponding cyclic group were given by Gronau, Mullin, and Rosa [32]. An ODC-generating $(2p+1,1,2p+1)$-tree is obtained if we assign the elements of $\mathbb{Z}_{4p+6}$ to the vertices such that its edges are $\{0,2p+3\}, \{2p+3,2p+4\}, \{2p+3,1\}$, $\{0,2i+1\}$ ($i \in \mathbb{Z}_{4p+6} \setminus \{0,p+1,2p+3,3p+4\}$), $\{2p+4,2i\}$ ($i \in \mathbb{Z}_{4p+6} \setminus \{0,p+2,2p+3,4p+5\}$).

Using the above lemma and Lemma 2.10, it is not hard to verify the following:

THEOREM 3.15 (Leck and Leck [55]). *Let T be the (p, q, r)-tree on $n = p + q + r + 3$ vertices. Then there exists an ODC of K_n by T.*

To address arbitrary trees of diameter four we introduce one more notation. Let $t \geq 2$ and $p_1 \geq p_2 \geq \cdots \geq p_t \geq 0$, where $p_2 > 0$. The the $(p_1, p_2, \ldots, p_t; 4)$*-tree* is the tree on $\sum_{i=1}^{t} p_i + t + 1$ vertices x, x_i, $x_{i,j}$ ($i = 1, 2, \ldots, t$; $j = 1, 2, \ldots, p_i$) with the edges $\{x, x_i\}$, $\{x_i, x_{i,j}\}$ (see Figure 10).

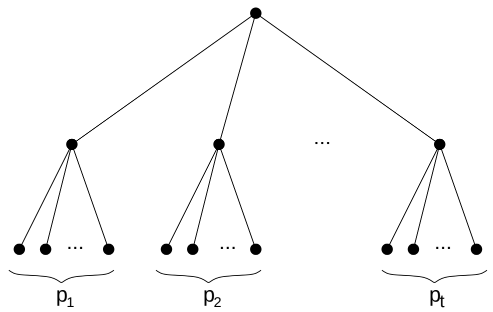

Figure 10. The $(p_1, p_2, ..., p_t; 4)$-tree

The case $t = 3$ can be settled completely essentially by an iterated application of Lemma 2.10, where one of the “summands” is an ODC of K_4 by S_3.

THEOREM 3.16 (Leck and Leck [55]). *Let T be the $(p_1, p_2, p_3; 4)$-tree on $n = p_1 + p_2 + p_3 + 4$ vertices. Then there is an ODC of K_n by T.*

A more careful use of basically the same technique yields:

THEOREM 3.17 (Leck and Leck [55]). *Let T be the $(p_1, p_2, \ldots, p_t; 4)$-tree, where $p_1 \geq 4t + 3$ and $p_i \geq t + 1$ for $i = 2, 3, \ldots, t$, and let $n = \sum_{i=1}^{t} p_i + t + 1$. Then there is an ODC of K_n by T.*

Actually, the results in [55] cover a slightly more general class of trees of diameter 4 than the one given in Theorem 3.17. Another class of diameter 4 trees not covered by the theorem are the comets for which the PBD-construction yields solutions as shown in Section 2.2. In fact, there is even a cyclic ODC by any comet, the corresponding generators were given by Gronau, Mullin, Rosa [32]. Again using PBD-closure, this can be generalized to $(k, k, \ldots, k; 4)$-trees as follows. For brevity, T_m^k denotes the $(\underbrace{k, k, \ldots, k}_{m \text{ times}}; 4)$-tree.

THEOREM 3.18. *There exists an ODC of $K_{m(k+1)+1}$ by T_m^k for all $k \geq 1$ with only finitely many possible exceptions.*

Proof. The cases $k = 2$ and $k = 3$ were investigated in [32] and [54], respectively. To prove the general case, we show that the set of integers $n = m(k+1)+1$ (k fixed) for which an ODC of K_n by T_m^k exists is PBD-closed. We take as ingredients in Theorem 2.4 a $PBD[n, \{k+2, 2k+3, 3k+4)\}]$. The PBD-closure of $\{k+2, 2k+3, 3k+4\}$ includes all integers $n \equiv 1 \pmod{k+1}$ with finitely many exceptions (see [66]). Finally, we give ODCs by T_1^k, T_2^k and T_3^k which are to be amalgamated at an end vertex of T_1^k and the central vertices of T_2^k and T_3^k, respectively. These vertices are rotating vertices of the original ODCs. The first is a $(k+1)$-star. Hence, there is a cyclic ODC by Proposition 3.11. The required ODCs by T_2^k and T_3^k are obtained from Lemma 3 and Theorem 5, respectively, in [55]. □

Similarly, solutions can be found for several other trees of diameter 4.

So far, there is not much knowledge about ODCs by trees of diameter at least 5. For diameter 5 basically the same constructions as for diameter 4 can be applied. An initial lemma providing ingredients for the adding construction is given below.

LEMMA 3.19. *For even $n \geq 12$ there is a cyclic ODC of K_n by the $(3, 0, 0, n-7)$-tree.*

Proof. The $(3, 0, 0, n-7)$-tree with edges $\{i, n/2\}$ ($i \in \{n/2+2, n/2-1, 1\}$), $\{n/2, 0\}$, $\{0, n/2-3\}$, $\{n/2-3, n-1\}$, and $\{n-1, i\}$ ($i \in \{0, 1, \ldots, n-1\} \setminus \{0, 1, n/2-3, n/2-1, n/2, n/2+2, n-1\}$) is easily verified to be a generator of a cyclic ODC. □

An example of a class of trees of larger diameter ODCs by which can be constructed is given by Theorem 2.6. Solutions for a similar class of trees can be derived applying Theorem 2.12 repeatedly (starting with a star). Finally, a class of trees studied very intensively with respect to ODCs are hamiltonian paths, see Section 3.5.

3.5. Almost-hamiltonian cycles and hamiltonian paths

The existence question for ODCs by almost-hamiltonian cycle s is one of the earliest problems in this area. It was posed by Hering and Rosenfeld [50] in 1979. Despite some efforts in the past 20 years, this question is far from being answered completely.

We give an overview of progress made towards settling the existence. Since ODCs by hamiltonian paths turn out to be a crucial tool in the construction we also focus on their existence.

Alspach, Heinrich and Rosenfeld [3] introduce a construction of an ODC-generating almost-hamiltonian cycle with respect to the additive group of a finite field. They prove the following theorem.

THEOREM 3.20 (Alspach, Heinrich and Rosenfeld [3]). *Let $GF(q)$ be a finite field of order q, $q > 3$. Furthermore, let ω be a primitive element. Then, the cycle $(1, \omega, \omega^2, \ldots, \omega^{q-2})$ is an ODC-generating almost-hamiltonian cycle with respect to the additive group of $GF(q)$.*

This construction was found, independently, also by Hering [48]. It even yields an ODC-generating dicycle. Alspach, Heinrich and Rosenfeld [3] also consider anti-directed cycles in finite fields, i.e. dicycles where no two consecutive arcs have the same orientation. They construct ODC-generating anti-directed cycles in finite fields of odd order.

We observe that the cycle of Theorem 3.20 is the union of the sets

$$H_1 = \{\{1, \omega\}, \{\omega^2, \omega^3\}, \ldots, \{\omega^{q-3}, \omega^{q-2}\}\}$$

and

$$H_2 = \{\{\omega, \omega^2\}, \{\omega^3, \omega^4\}, \ldots, \{\omega^{q-2}, 1\}\}.$$

Hence, by Theorem 2.21, when $q \equiv 1 \bmod 4$, it is the union of two complementary halfstarters. In the case $q \equiv 3 \bmod 4$, the sets H_1 and H_2 are Mullin-Nemeth starters (and thus skew) and it holds $H_1 = -H_2$. Thus, by Lemma 2.17, they are skew-orthogonal.

In $GF(5)$, 2 is a primitive element. The cycle $(1, 2, 4, 3)$ is ODC-generating with respect to $\mathbb{Z}_5$, the sets $H_1 = \{\{1, 2\}, \{3, 4\}\}$ and $H_2 = \{\{1, 3\}, \{2, 4\}\}$ are complementary halfstarters. In $GF(7)$, 3 is a primitive element. The ODC-generating cycle $(1, 3, 2, 6, 4, 5)$ with respect to $\mathbb{Z}_7$ consists of the skew-orthogonal starters $H_1 = \{\{1, 3\}, \{2, 6\}, \{4, 5\}\}$ and $H_2 = -H_1 = \{\{6, 4\}, \{5, 1\}, \{3, 2\}\}$.

Besides settling the existence for prime powers, Alspach, Heinrich and Rosenfeld presented ODCs of K_n by an almost-hamiltonian cycle for $n = 6, 10, 12, 14$ and 15.

Recall that, by Corollary 2.2, there is no ODC-generating almost-hamiltonian cycle with respect to an abelian group of order 2 mod 4.

Using the Dinitz/Stinson hill-climbing algorithm [21] to construct skew-orthogonal starters, Leck [57] establishes the following existence.

LEMMA 3.21 (Leck [57]). *Let $7 \le 2n + 1 \le 101$, $2n + 1 \ne 9$. There exists an ODC-generating almost-hamiltonian cycle with respect to the cyclic group $\mathbb{Z}_{2n+1}$ consisting of two starters.*

The ODC-generating almost-hamiltonian cycle with respect to $\mathbb{Z}_5$ obtained from Theorem 3.20 is unique [49] and, because $5 \equiv 1 \bmod 4$

consists of two complementary halfstarters. In $\mathbb{Z}_9$, there is no ODC-generating almost-hamiltonian cycle [49]. But, by Theorem 3.20, there is one with respect to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Since $9 \equiv 1 \bmod 4$, this cycle consists of two complementary halfstarters.

Heinrich and Nonay [47] give a construction of an ODC by an almost-hamiltonian cycle from an ODC of a hamiltonian path on a smaller vertex set. In fact, it even yields an ODC by an almost-hamiltonian dicycle. A generalization was given by Leck [58].

THEOREM 3.22 ([47, 58]). *If there is an ODC of K_n by a hamiltonian path, then there exist ODCs of $\vec{D}_{4n}$ and $\vec{D}_{16n}$ by almost-hamiltonian dicycles.*

Theorem 3.22 lets us focus on the existence of ODCs of K_n by hamiltonian paths. Heinrich and Nonay [47] present such ODCs for $2 \leq n \leq 20, n \neq 4$. For $n \neq 7$, the solutions are cyclic. They also introduce a "multiplication" construction, which is generalized by Horton and Nonay [51] and utilizes a class of solutions with the additional property to be 2-colorable. An ODC by a hamiltonian path is called 2-*colorable*, if the edges of each of its paths can be colored alternately with two colors, such that every edge of the complete graph is covered twice by the same color. Note that there is no 2-colorable ODC by a path P_n with $n \equiv 3 \bmod 4$. We say an ODC-generating path is 2-*colorable*, if it generates a 2-colorable ODC.

An ODC-generating path is 2-colorable, if and only if the two edges of the same length are equally colored. That means that there is an odd number of edges on the path between any two edges of the same length. The path $0, 3, 7, 2, 4, 1, 5, 6, 8, 9$, for example, is a 2-colorable ODC-generator with respect to $\mathbb{Z}_{10}$. There are $1, 3, 3$ and 3 edges between the edges of length $\{\pm 1\}, \{\pm 2\}, \{\pm 3\}$ and $\{\pm 4\}$, respectively.

THEOREM 3.23 (Horton and Nonay [51]). *Let q be a prime power, $q \geq 5$, and let $m \geq 3$. If there are an ODC by P_m and a 2-colorable ODC by P_q, then there is an ODC by the path P_{mq}.*

For $n = 5, 9, 10, 12, 13, 17, 29$, Horton and Nonay [51] give 2-colorable ODCs by P_n. For $n \neq 9$, these solutions are cyclic, the ODC by P_9 is generated by $\mathbb{Z}_3 \times \mathbb{Z}_3$. We remark that the ODCs by P_{10} and by P_{12} are not applicable to Theorem 3.23. Note that, by the repeated application of Theorem 3.23, we derive an ODC by a hamiltonian path on q^k vertices for all integers k from a single 2-colorable ODC by P_q.

Leck [57] observes that a 2-colorable ODC-generator on $4n + 1$ vertices consists of two complementary halfstarter translates. These can be constructed efficiently using hill-climbing techniques of Dinitz and Stinson [21].

LEMMA 3.24 (Leck [58]). *There is a 2-colorable ODC by the hamiltonian path P_n, $2 \leq n \leq 101$, if and only if $n \not\equiv 3 \bmod 4$ and $n \neq 4, 6$.*

Hartmann, Leck and Leck [43] use the halfstarters of Theorem 2.21 to construct 2-colorable ODC-generators in finite fields of order $4n+1$. They conjecture their construction to work in every such field and prove the following existence result.

LEMMA 3.25 ([43]). *Let p be a prime, $p \equiv 1 \bmod 4$, $5 \leq p \leq 100000$. There exists a primitive element ω in GF(p) such that the graph $H_1(\omega) \cup (H_2(\omega)+1)$ is a path. This path is ODC-generating with respect to $\mathbb{Z}_p$.*

Leck [53] gives large infinite classes of solutions.

THEOREM 3.26 (Leck [53]). *Let $n = m^2$, where $m \geq 3$ is odd. Then,*

- *(a) there is a 2-colorable ODC-generating path with respect to $\mathbb{Z}_m \times \mathbb{Z}_m$,*
- *(b) there is a 2-colorable ODC-generating path with respect to $\mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2$.*

Combining the results discussed in this section and applying Corollary 2.29 and Lemma 2.26 we derive the following existence result.

THEOREM 3.27. *Let $n \geq 4$ be an integer, and let $n = 2^e p_1^{e_1} \cdots p_k^{e_k}$, $e_i \geq 1$, be its decomposition into prime factors. Furthermore, suppose that, for odd e_i, the prime $p_i < 100000$ and $p_i \equiv 1 \bmod 4$.*

- *(a) If $2 \leq e \neq 3$, then there exist ODCs of K_n and K_{zn} by almost-hamiltonian cycle s, where $3 \leq z \leq 101$, $z \neq 4$.*
- *(b) If $e = 3$ and $e_i \geq 2$ for some i, then there exists an ODC of K_n by an almost-hamiltonian cycle.*

Moreover, let m be an integer.

- *(c) If m is a prime power and $m \geq 4$, then there is an ODC of K_m by an almost-hamiltonian cycle.*
- *(d) If $m = p+1$, where p is a prime and $p \equiv 5$ or $7 \bmod 8$, then there is an ODC of K_m by an almost-hamiltonian cycle.*
- *(e) For $4 \leq m \leq 102$, there is an ODC of K_m by an almost-hamiltonian cycle.*

4. ODCs by graph families and related results

In preceding sections we discussed ODCs by clique graphs and by cycle families. In fact, the conjecture of Demetrovics, Füredi and Katona [17] on the existence of ODCs by families of 3-cycles was one of the motivating forces behind the study of orthogonal double covers. Though the original conjecture was completely settled in [27], it gave rise to the investigation of ODCs by other graph and digraph families. Let $\vec{H}$ be a digraph with $v(\vec{H})$ vertices and $e(\vec{H})$ arcs. The vertex disjoint union of m copies of $\vec{H}$ is denoted by $m\vec{H}$, and called an *$\vec{H}$-family.*

In every page of an ODC the vertex number exceeds the arc number by one. A digraph family $m\vec{H}$ rarely meets this condition, since $m(v(\vec{H}) - e(\vec{H})) = 1$ immediately implies $m = 1$ and $v(\vec{H}) = e(\vec{H}) + 1$. In general, it will be necessary to augment $m\vec{H}$ by a suitable number of isolated vertices to satisfy the requirement under inspection. For simplicity, we still call a digraph $m\vec{H} \cup E_s$ an $\vec{H}$-family. In the subsequent sections we are going to survey various results on ODCs by graph and digraph families and discuss a couple of consequences.

4.1. Idempotent ODCs by graph and digraph families

Throughout, let $\vec{H}$ be a digraph with $v(\vec{H}) \leq e(\vec{H})$. Every page of an ODC by an $\vec{H}$-family contains at least one isolated vertex. This motivates the study of idempotent ODCs [29, 34]. Let V be the vertex set of the digraph $\vec{D}_n$. An orthogonal directed cover of $\vec{D}_n$ with pages $\vec{G}_i$, $i \in V$, is said to be *idempotent* if every page $\vec{G}_i$ contains the vertex i as an isolated vertex. Clearly, a similar concept may be introduced in the undirected case.

The property of being idempotent is of special value when we construct ODCs recursively by the help of pairwise balanced designs as shown in Section 2.2. If all the ingredient ODCs are idempotent then the resultant ODC is idempotent again. This observation was used in [29] to establish ODCs by $\vec{C}_k$-families. More generally, the set of all integers n admitting an idempotent ODC of $\vec{D}_n$ by an $\vec{H}$-family is PBD-closed for every digraph $\vec{H}$. A crucial observation to determine the spectrum of ODCs by digraph families is the following result:

THEOREM 4.1 (Hartmann [37]). *Let $\vec{H}$ be a digraph with $v(\vec{H}) \leq e(\vec{H})$. The set of all integers n such that $\vec{D}_n$ admits an idempotent ODC by an $\vec{H}$-family is eventually periodic with period $e(\vec{H})$.*

To derive the specified period, we suggested a construction of group-generated ODCs with respect to the additive group of finite fields

GF(q). This method turned out to work for all sufficiently large prime powers $q \equiv 1 \bmod e(\vec{H})$. A major tool to prove the claimed correctness was Weil's theorem [65] on upper bounds for the absolute value of certain character sums over finite fields. As a result we settled the eventual existence of ODCs by digraph families.

THEOREM 4.2 (Hartmann [37]). *Let $\vec{H}$ be a digraph with $e(\vec{H}) \leq v(\vec{H})$. There exists an idempotent ODC of $\vec{D}_n$ by an $\vec{H}$-family for almost every n satisfying the necessary condition $n \equiv 1 \bmod e(\vec{H})$.*

Clearly, the same result holds for orthogonal double covers, too: We simply omit all orientations of arcs in the orthogonal directed covers and derive the eventual existence of orthogonal double covers by H-families for every suitable graph H. As mentioned in the introduction, ODCs by clique graphs are of special interest in connection with problems from database theory. While ODCs by complete graphs K_k, i.e. k-biplanes are known so far only for finitely many values of k, Theorem 4.2 yields the existence of an ODC by the clique graph mK_k for every $k \geq 3$ and sufficiently large m.

To continue with we extend the notion of digraph families, among others to derive further results on clique graphs comprising cliques of various sizes. Let $\mathcal{H}$ be a given set of digraphs. A digraph $\vec{G}$ is called an *$\mathcal{H}$-family* if $\vec{G}$ is the vertex-disjoint union of digraphs which all are copies of some member of $\mathcal{H}$. Again we allow $\vec{G}$ to be extended by a suitable number of isolated vertices to ensure $v(\vec{G}) = e(\vec{G}) + 1$ which is necessary for the existence of an ODC by $\vec{G}$. A straightforward calculation shows that the arc number of every $\mathcal{H}$-family is a multiple of the greatest common divisor of the arc numbers $e(\vec{H})$, $\vec{H} \in \mathcal{H}$.

THEOREM 4.3 (Hartmann [41]). *Let $\mathcal{H}$ be a set of digraphs such that $v(\vec{H}) \leq e(\vec{H})$ holds for all digraphs $\vec{H} \in \mathcal{H}$. Then there exists an idempotent ODC of $\vec{D}_n$ by an $\mathcal{H}$-family for almost every n satisfying the necessary condition $n \equiv 1 \bmod \gcd\{e(\vec{H}) : \vec{H} \in \mathcal{H}\}$.*

The preceding result is useful even if $\mathcal{H}$ contains some 'bad' members, that is, digraphs $\vec{B}$ with $v(\vec{B}) > e(\vec{B})$. All we require is that there is at least one digraph $\vec{H}_0 \in \mathcal{H}$ satisfying $v(\vec{H}_0) < e(\vec{H}_0)$. For each bad member $\vec{B}$ we choose a digraph $\vec{B}' = s\vec{H}_0 \cup \vec{B}$ with sufficiently large s to ensure $v(\vec{B}') \leq e(\vec{B}')$. On replacing the bad members $\vec{B}$ in $\mathcal{H}$ by the chosen digraphs $\vec{B}'$ we obtain a new set $\mathcal{H}'$ of digraphs which satisfies the assumption of Theorem 4.3. Since every $\mathcal{H}'$-family is also an $\mathcal{H}$-family, and $\gcd\{e(\vec{H}) : \vec{H} \in \mathcal{H}\} = \gcd\{e(\vec{H}') : \vec{H}' \in \mathcal{H}'\}$ holds, we conclude the following consequence.

COROLLARY 4.4. *Let $\mathcal{H}$ be a set of digraphs, among them at least one digraph $\vec{H}_0$ with $v(\vec{H}_0) < e(\vec{H}_0)$. Then there exists an idempotent ODC of $\vec{D}_n$ by an $\mathcal{H}$-family for almost every n satisfying the necessary condition $n \equiv 1 \bmod \gcd\{e(\vec{H}) : \vec{H} \in \mathcal{H}\}$.*

The discussion above enables us to derive new results on families of dicycles and on cliques graphs. As an example, we include the following two consequences which generalize earlier observations from Section 3.3 and 3.2. Further results on orthogonal double covers by clique graphs and their application to problems from database theory are presented in [41].

COROLLARY 4.5. *Given $k \geq 3$, there exists an ODC of $\vec{D}_n$ by a $\{\vec{C}_k, \vec{C}_{k+1}\}$-family for almost every n.*

COROLLARY 4.6. *Given $k \geq 3$, there exist*

(i) an idempotent ODC of K_n by a $\{K_k, K_{k+1}\}$-family for almost every n satisfying the necessary condition $n \equiv 1 \bmod k$, and

(ii) an idempotent ODC of K_n by a $\{K_{k-1}, K_k, K_{k+1}\}$-family for almost every n.

The previous result does not cover $\{K_2, K_3\}$-families. In Section 3.2 we studied ODCs whose pages consist of several copies of K_3 and just a single copy of K_2. Evidently these ODCs are not idempotent. Whenever there exists an idempotent ODC by a $\{K_2, K_3\}$-family G, then G will not contain a copy of K_2, i.e. G actually is a K_3-family. Thus an idempotent ODC of K_n by a $\{K_2, K_3\}$-family exists only for $n \equiv 1 \bmod 3$ and $n \neq 10$. Conversely, if k exceeds 3, we find $\{K_2, K_k\}$-families which contain a copy of K_2 *and* admit an idempotent ODC. In fact, such an idempotent ODC of K_n exists for almost every n.

4.2. Rooted ODCs by graph and digraph families

Suppose we have a digraph $\vec{H}$ with $v(\vec{H}) = e(\vec{H})+1$. As soon as we take $m \geq 2$ copies of $\vec{H}$ the resultant digraph $m\vec{H}$ has too many vertices to admit an ODC. To overcome this limitation we study amalgamated digraph families. Consider a digraph $\vec{H}$ and some vertex x in $\vec{H}$. Now we take m copies of $\vec{H}$ which are mutually vertex-disjoint apart from the vertex x which lies in each of the copies and, in fact, is a fixpoint under the corresponding isomorphism. The resultant digraph is called an *amalgamated* $\vec{H}$-family with *root* x. It has $m(v(\vec{H})-1)+1$ vertices and $m \cdot e(\vec{H})$ arcs which is just the right number for an ODC supposed we have $v(\vec{H}) = e(\vec{H}) + 1$. As an example consider the digraph in Figure 11 which is an amalgamated $\vec{P}_3$-family. If $v(\vec{H}) \le e(\vec{H})$ holds, we allow the digraph family to be augmented as usual by a suitable number of isolated vertices. Figure 12 shows an amalgamated $\vec{C}_4$-family augmented by two isolated vertices.

Throughout let $\vec{H}$ be a digraph with $v(\vec{H}) \le e(\vec{H})+1$. When looking for ODCs by amalgamated $\vec{H}$-families, it is again profitable to concentrate on a particular class of ODCs. Let V be the vertex set of $\vec{D}_n$ and consider an orthogonal directed cover of $\vec{D}_n$ by some digraph $\vec{G}$. Every page $\vec{G}_i$, $i \in V$, in this ODC is a copy of $\vec{G}$ under an isomorphism ϕ_i. We speak of a *rooted ODC* if $\vec{G}$ contains a vertex x such that $\phi_i(x) = i$ holds for every $i \in V$. The vertex x under inspection is called the *root* of this ODC.

It is natural to ask for rooted ODCs by amalgamated $\vec{H}$-families such that the root of the ODC coincides with the root of the $\vec{H}$-family. The set of all integers n admitting such an ODC of $\vec{D}_n$ is PBD-closed again. Clearly every n in this set is congruent 1 mod $e(\vec{H})$. Surprisingly however, this condition is not eventually sufficient. In fact there are suitable digraphs $\vec{H}$ allowing infinitely many exceptions. As an example consider the dipath $\vec{P}_3$. Whenever we take an even number of copies and glue them together in their non-pendant vertex, the resultant digraph admits an ODC. Conversely, this observation does not hold for any odd number of copies.

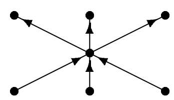

Figure 11. An amalgamated $\vec{P}_3$-family not admitting an ODC

THEOREM 4.7 (Hartmann [39]). *Let x be non-pendant vertex of the digraph $\vec{P}_3$. An ODC of $\vec{D}_n$ by an amalgamated $\vec{P}_3$-family with root x exists if and only if* $n \equiv 1 \bmod 4$.

Fortunately this example is not the usual case. For the large majority of suitable digraphs the necessary conditions detailed above happen to be eventually sufficient.

THEOREM 4.8 (Hartmann [39]). *Let $\vec{H}$ be a digraph with $v(\vec{H}) \le e(\vec{H})+1$. Fix a vertex x in $\vec{H}$ such that, when* $e(\vec{H}) \equiv 2 \bmod 4$, *neither its indegree nor its outdegree equal $\frac{1}{2}e(\vec{H})$. Then there exists a rooted*

ODC of $\vec{D}_n$ by an amalgamated $\vec{H}$-family with root x for almost every n satisfying the necessary condition $n \equiv 1 \bmod e(\vec{H})$.

The notion of rooted ODCs by an amalgamated digraph families is easily transmitted to the undirected case. Suppose we are given a suitable graph H and a fixed vertex x. In order to apply the preceding theorem we simply assign every edge of H an orientation. Attention is called for only in the case $e(H) \equiv 2 \bmod 4$, where it is easy to guarantee that x obtains neither indegree nor outdegree $\frac{1}{2}e(H)$.

COROLLARY 4.9 (Hartmann [39]). *Let H be a graph with $v(H) \leq e(H) + 1$, and let x be some vertex in H. Then there exists a rooted ODC of K_n by an amalgamated H-family with root x for almost every $n \equiv 1 \bmod e(H)$.*

The preceding result is of particular interest for trees. If H is a tree then every amalgamated H-family is a tree again. Whenever we take sufficiently many copies of an arbitrary tree and glue them together in some root x we come up with a new tree which admits an ODC.

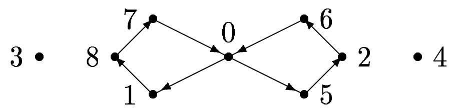

Figure 12. An ODC-generating digraph with respect $\mathbb{Z}_9$

As an example for Theorem 4.8, consider the dicycle $\vec{C}_k$. Choose some vertex x on the dicycle and consider the amalgamated $\vec{H}$-family with root x. We call the resultant digraph a *directed k-flower* and denote it by $\vec{F}_{k,m}$. Figure 12 particularly shows the directed 4-flower $\vec{F}_{4,2}$ obtained by gluing together two copies of $\vec{C}_4$ in the vertex $x = 0$. Of course, the dicycle $\vec{C}_k$ satisfies the presumption of Theorem 4.8, since every potential root has indegree 1 and outdegree 1, whereas $e(\vec{C}_k) = k \geq 3$.

COROLLARY 4.10. *Given $k \geq 3$, there exists an ODC of $\vec{D}_n$ by a k-flower for almost every n satisfying the necessary condition $n \equiv 1 \bmod k$.*

We continue with some results for small values of k, for details see [39]. Let $\mathbb{N}_{1(k)}$ denote the set of all positive integers congruent 1 mod k. Group-generated ODCs by directed 3-flowers turned out to exist for all prime powers $q \equiv 1 \bmod 3$, $q \neq 7$. The PBD-closure of $\{4, 19, 31, 43\}$ equals $\mathbb{N}_{1(3)} \backslash \{7, 10, 22, 34, 46, 55\}$. For $n = 7$ it was not too difficult to check the non-existence of an ODC by the directed 3-flower $\vec{F}_{3,2}$. There does not even exist an orthogonal double cover in the undirected case. For $n = 10$ an exhaustive computer search shows that there is no ODC by the directed 3-flower $\vec{F}_{3,3}$. For the remaining values of n, the existence problem is still open.

THEOREM 4.11. *An ODC of $\vec{D}_n$ by a directed 3-flower exists for all $n \equiv 1 \bmod 3$ apart from $n = 7, 10$, and with the possible exceptions $n = 22, 34, 46$ and 55.*

At this point a short remark is called for. Though there is no ODC of $\vec{D}_{10}$ by $\vec{F}_{3,3}$, we found an orthogonal double cover of K_{10} by the undirected 3-flower $F_{3,3}$. Since the PBD-closure of $\{4, 10, 19\}$ equals $\mathbb{N}_{1(3)} \backslash \{7, 22\}$, this yields $n = 22$ as the only open value in the undirected case.

For $k = 4$, however, we are able to determine the complete spectrum. With respect to the additive group of the finite field $\mathrm{GF}(q)$, a group-generated ODC by a directed 4-flower exists for every prime power $q \equiv 1 \bmod 4$, $q \neq 9$. Moreover, for $n = 9$ and 33 we found ODCs generated by the cyclic group. A solution for $n = 9$ is shown in Figure 12. Since the PBD-closure of $\{5, 9, 13, 17, 29, 33\}$ equals $\mathbb{N}_{1(4)}$ we conclude as follows:

THEOREM 4.12. *An ODC of $\vec{D}_n$ by a directed 4-flower exists if and only if $n \equiv 1 \bmod 4$.*

4.3. Embedding results for ODCs

During our study of ODCs we came across a considerable number of graphs and digraphs which do not admit an ODC. For $n \leq 10$ details were given in Section 3.1. It is natural to ask whether these graphs and digraphs may be characterized by forbidden substructures. However, Theorem 4.3 indicates that such a characterization does not exist.

THEOREM 4.13. *Let $\vec{H}$ be a given digraph. For almost every n there exists a suitable digraph $\vec{G}^{(n)}$ which admits an idempotent ODC of $\vec{D}_n$ by the vertex-disjoint union $\vec{H} \cup \vec{G}^{(n)}$.*

Proof. The idea is to find two digraphs $\vec{H}_1'$ and $\vec{H}_2'$ such that the set $\mathcal{H}$ containing the vertex-disjoint unions $\vec{H} \cup \vec{H}_1'$ and $\vec{H} \cup \vec{H}_2'$ satisfies the assumption of Theorem 4.3. That is, we require

$$e(\vec{H}_j') - v(\vec{H}_j') \geq v(\vec{H}) - e(\vec{H})$$

for $j = 1, 2$. If the arc numbers $e(\vec{H} \cup \vec{H}'_1)$ and $e(\vec{H} \cup \vec{H}'_2)$ are relatively prime, Theorem 4.3 guarantees an ODC of $\vec{D}_n$ by an $\mathcal{H}$-family for every sufficiently large n. Due to our choice of $\mathcal{H}$ every non-trivial $\mathcal{H}$-family contains a copy of $\vec{H}$ and, in fact, may be regarded as the vertex-disjoint union of a copy of $\vec{H}$ and some digraph $\vec{G}^{(n)}$.

Given a positive integer e' let $\vec{H}'$ be a digon-free digraph with $e(\vec{H}') = e'$ arcs and as few vertices as possible. Clearly, we have

$$\binom{v(\vec{H}')}{2} \geq e' > \binom{v(\vec{H}') - 1}{2},$$

and thus $2\sqrt{e'} + 2 > v(\vec{H}')$. For our purposes here we choose two different integers e'_1, e'_2 such that $e(\vec{H}) + e'_j$ is a prime and $e'_j - 2\sqrt{e'_j} - 2 \geq v(\vec{H}) - e(\vec{H})$ holds for $j = 1, 2$. It is straightforward to check that the corresponding digraphs $\vec{H}'_1$ and $\vec{H}'_2$ have all the desired properties. This enables us to apply Theorem 4.3 as suggested. $\square$

Suppose we are given a digraph $\vec{H}$ with $v(\vec{H}) = e(\vec{H}) + 1 = n$. Even if there is no ODC of $\vec{D}_n$ by $\vec{H}$ it is usually possible to find some copies of $\vec{H}$ in $\vec{D}_n$ which are pairwise orthogonal. Generally, every packing of $\vec{D}_n$ by mutually orthogonal spanning subdigraphs is called a *partial ODC*. Clearly, there are partial ODCs of $\vec{D}_n$ which cannot be extended to ODCs of $\vec{D}_n$. This raises the question whether every partial ODC may at least be embedded into an ODC of some complete digraph of larger size. Traditionally much effort has been expended in discrete mathematics towards producing embedding theorems for partial algebraic or combinatorial structures. Let W and $V \subset W$ be the vertex sets of the complete digraphs $\vec{D}_m$ and $\vec{D}_n$. Suppose we are given a partial ODC $\mathcal{P} = \{\vec{H}_i : i \in U \subseteq V\}$ of $\vec{D}_n$ and an ODC $\mathcal{O} = \{\vec{G}_i : i \in W\}$ of $\vec{D}_m$. For every $i \in U$ we consider the subdigraph $\vec{G}'_i$ of $\vec{G}_i$ induced by the vertices from V. We say that $\mathcal{P}$ is *embedded* into $\mathcal{O}$ if $\vec{G}'_i$ is isomorphic to $\vec{H}_i$ for every $i \in U$. The following result verifies that every partial ODC actually appears as a part of some ODC of a sufficiently large complete digraph.

THEOREM 4.14. *Let $\mathcal{P}$ be a partial ODC of $\vec{D}_n$. Then there exists a cyclic ODC $\mathcal{O}$ of $\vec{D}_m$ for some $m > n$ such that $\mathcal{P}$ is embedded into $\mathcal{O}$.*

Proof. Put $m = 4 \cdot 2^{n^2} + 1$. Suppose $W = \mathbb{Z}_m$ and $V = \{2^i : i = 0, \ldots, n-1\} \subset W$ are the vertex sets of the complete digraphs $\vec{D}_m$ and $\vec{D}_n$, and let $\mathcal{P}$ consist of digraphs $\vec{H}_{2^i}$ with $2^i \in U \subseteq V$. Our aim is to embed $\mathcal{P}$ into a cyclic ODC $\mathcal{O}$. Thus $\mathcal{O}$ will be the orbit $\vec{G} + \mathbb{Z}_m$ of a digraph $\vec{G}$ to be constructed. Note that within this proof all calculation is performed with respect to $\mathbb{Z}_m$.

For every $2^i \in U$, let $\vec{H}'_{2^i}$ be the copy $\vec{H}_{2^i} + 2^{in}$ of the digraph $\vec{H}_{2^i}$ in $\mathcal{P}$. It is noteworthy that the new digraphs $\vec{H}'_{2^i}$ are mutually vertex-disjoint. By $\vec{A}$ we denote the union of the digraphs $\vec{H}_{2^i}$, $2^i \in U$, and by $V(\vec{A})$ its vertex set. The idea is to derive the desired digraph $\vec{G}$ as the union of $\vec{A}$ and a second digraph $\vec{B}$ with vertex set $W \backslash V(\vec{A})$.

As mentioned we want $\vec{G}$ to be the generator of a cyclic ODC, that is, $\vec{G}$ has to satisfy the conditions assembled in Section 2.1 with respect to $\mathbb{Z}_m$. Consider an arc $(2^a, 2^b)$ in one of the digraphs $\vec{H}_{2^i}$. Its image in $\vec{A}$ is the arc $(2^{in} + 2^a, 2^{in} + 2^b)$ whose length is $2^b - 2^a$. Since $\mathcal{P}$ is a packing of $\vec{D}_n$, the arcs in $\vec{A}$ have mutually distinct lengths. It easy to see that any two arcs in $\vec{A}$ having inverse lengths arise from a pair of reverse arcs covered by $\mathcal{P}$, say $(2^a, 2^b)$ in $\vec{H}_{2^i}$ and $(2^b, 2^a)$ in $\vec{H}_{2^j}$. Their images in $\vec{A}$ have the inverse lengths $2^b - 2^a$ and $2^a - 2^b$, and the distance between them is $\pm(2^{jn} - 2^{in})$.

In summary, the arcs in $\vec{A}$ comprise all lengths in $L_A = \{2^b - 2^a : (2^a, 2^b) \text{ is covered by } \mathcal{P}\}$ and all distances in $D_A = \{\pm(2^{jn} - 2^{in}) : \vec{H}_{2^i} \text{ and } \vec{H}_{2^j} \text{ are orthogonal}\}$. It suffices to find a digraph $\vec{B}$ such that its arcs settle all the remaining lengths and distances. Clearly, the permitted lengths are just the values in $L_B = \mathbb{Z}_m \backslash (\{0\} \cup L_A)$, and the permitted distances are the values in $D_B = \mathbb{Z}_m \backslash (\{0\} \cup D_A)$.

Now we are going to generate the arc set of $\vec{B}$. We start with the empty digraph on the vertex set $W \backslash V(\vec{A})$ and insert its arcs incrementally. Suppose the size of D_B exceeds the size of L_B. Then there must be some $\ell \in L_B$ whose inverse $-\ell$ lies not in L_B. Hence $\vec{A}$ contains no arc of length ℓ, but an arc e of length $\ell(e) = -\ell$, say $e = (2^{in} + 2^a, 2^{in} + 2^b)$. This happens whenever $\mathcal{P}$ covers $(2^a, 2^b)$ but not $(2^b, 2^a)$. We take a distance which is still permitted, that is, some $\delta \in D_B$, and insert the new arc $(\delta + 2^{in} + 2^b, \delta + 2^{in} + 2)$ into $\vec{B}$. Of course, δ has to be chosen such that both vertices of the new arc belong to $W \backslash V(\vec{A})$. This is possible since $|D_B| > |L_B| \geq (m-1) - (n-1)n > 2n^2 \geq 2n|\mathcal{P}| = 2|V(\vec{A})|$ holds. Moreover, e and the new arc have inverse lengths, and the distance between them is $\pm\delta$. Thus we delete ℓ from the set L_B of permitted lengths, and $\pm\delta$ from the set D_B of permitted distances. Then we continue with the next value $\ell \in L_B$ satisfying $-\ell \notin L_B$.

Afterwards we have $-\ell \in L_B$ for every $\ell \in L_B$, and the sets L_B and D_B are of equal size. We select some $\ell \in L_B$ as well as some $\delta \in D_B$, and insert the new arcs $(x, x + \ell)$ and $(x + \ell + \delta, x + \delta)$ into $\vec{B}$. Herein x is chosen from W such that all four vertices x, $x + \ell$, $x + \ell + \delta$ and $x + \delta$ belong to $W \backslash V(\vec{A})$. This is possible since $|W| = m > 4n^2 = 4n|\mathcal{P}| =$

$4|V(\vec{A})|$ holds. Clearly, the new arcs have length $\pm\ell$ and distance $\pm\delta$. Thus we delete $\pm\ell$ from the set L_B of permitted lengths, and $\pm\delta$ from the set D_B of permitted distances. Then we choose the next values from L_B and D_B, and repeat the whole procedure, until both L_B and D_B are empty.

Due to our construction, the resultant digraph $\vec{G} = \vec{A} \cup \vec{B}$ is an ODC-generator with respect to the cyclic group $\mathbb{Z}_m$. The orbit $\vec{G} + \mathbb{Z}_m$ is the claimed ODC. In every page $\vec{G} - 2^{in}$ with $2^i \in U$, the subset V of $W = \mathbb{Z}_m$ induces just the digraph $\vec{H}'_{2^i} - 2^{in} = \vec{H}_{2^i}$. Hence the partial ODC $\mathcal{P}$ is embedded in the ODC $\mathcal{O}$ as claimed. □

5. Generalizations

The notion of an ODC has been generalized in various directions, like suborthogonal double covers, ODCs of general graphs or generalized orthogonal covers.

5.1. Suborthogonal double covers

A *λ-fold decomposition* of a complete graph K_n is a collection $\mathcal{O}$ of subgraphs of K_n such that every edge of K_n occurs in exactly λ of these subgraphs. As usual, the subgraphs in $\mathcal{O}$ shall be called the *pages* of $\mathcal{O}$. If, in particular, all pages are isomorphic copies of some graph G, we speak of a λ-fold decomposition of K_n by G.

Evidently, ODCs are 2-fold decompositions of K_n with the additional property that any two pages have exactly one edge in common. In [32], Gronau, Mullin and Rosa conjectured the existence of an ODC of K_n by every tree on n vertices with one genuine exception, namely the path P_4 on 4 vertices.

The suspected exceptional case P_4 inspired the following question: Does there exist a 2-fold decomposition of the complete graph K_n where any two pages have *at most* one edge in common?

A 2-fold decomposition $\mathcal{S}$ satisfying this relaxed condition will be called a *suborthogonal double cover (SODC)* of K_n. Again, we speak of an SODC by G if all pages are isomorphic to some graph G. Note, that this time the number of pages in $\mathcal{S}$ does not necessarily coincide with the vertex number of K_n. However, if $\mathcal{S}$ consists of exactly n pages then it is just an ODC.

Suborthogonal double covers have first been studied by Hartmann and Schumacher [45]. An SODC of the complete graph K_n by the path P_4 exists if and only if $n \geq 6$ and $n \equiv 0, 1 \bmod 3$. Hence, the non-existence of an ODC by P_4, i.e. of an SODC of the complete graph K_4

by P_4 is indeed an exception when dealing with suborthogonal double covers.

Given a graph G, let $e(G)$ denote its edge number and $d(G)$ the greatest common divisor of its vertex degrees. The existence of an SODC of K_n by G immediately implies the following two conditions:

$$n(n-1) \equiv 0 \bmod e(G), \tag{1}$$

$$2(n-1) \equiv 0 \bmod d(G). \tag{2}$$

Note that these two conditions are well-known to be necessary for every 2-fold decomposition of K_n.

THEOREM 5.1 ([45]). *Let G be a graph on at most 4 vertices. There exists an SODC of K_n by G if and only if $n \geq e(G)+1$ and the necessary conditions (1), (2) hold, with the well-known exception $G = P_4$ and $n = 4$.*

Let S_k be the star with k vertices and $k-1$ edges. For stars the existence problem of SODCs has been completely settled by Schumacher.

THEOREM 5.2 ([63]). *There is an SODC of K_n by the star S_k if and only if the necessary conditions (1), (2) hold, and n exceeds $(k-1)t-\binom{t}{2}$, where $t := \lceil \frac{n-1}{k-1} \rceil$.*

Again, we may study the directed analogue of the question under inspection. Formally, a *suborthogonal directed cover (SODC)* is decomposition of the complete (symmetric) digraph $\vec{D}_n$ into subdigraphs (called pages) such that the union of any two different pages contains exactly one pair of oppositely directed arcs. If all pages are isomorphic copies of some digraph $\vec{G}$, we speak of an SODC by $\vec{G}$.

In the directed case, necessary conditions for the existence of decompositions are slightly more involved. Consider a digraph $\vec{G}$. For each vertex w of $\vec{G}$, let $\deg^-(w)$ and $\deg^+(w)$ denote the indegree and the outdegree of w, respectively. Further, let $d(\vec{G})$ be the smallest positive integer in the set

$$\{y \in \mathbb{Z} \;:\; \forall w \in \vec{G}\; \exists x_w \in \mathbb{Z} \text{ s.t. } \sum_{w \in \vec{G}} x_w \deg^-(w) = \sum_{w \in \vec{G}} x_w \deg^+(w) = y\}.$$

Whenever $\vec{D}_n$ admits an decomposition into copies of $\vec{G}$, we have

$$n(n-1) \equiv 0 \bmod e(\vec{G}), \tag{3}$$

$$n-1 \equiv 0 \bmod d(\vec{G}). \tag{4}$$

A well-known result by Wilson [67] shows that these two conditions are eventually sufficient for the existence of a decomposition of D_n into copies of $\vec{G}$. In general, the obtained decompositions are not SODCs. However, Hartmann [36] proved the same two conditions to be sufficient even for the existence of a suborthogonal directed cover of $\vec{D}_n$ by $\vec{G}$ for almost all n. On applying this result, a similar observation has been verified for suborthogonal double covers of K_n by a given (undirected) graph G.

THEOREM 5.3 ([36, 45]).

(i) There exists an SODC of K_n by a given graph G for almost all n satisfying the necessary conditions (1) and (2).

(ii) There exists an SODC of $\vec{D}_n$ by a given digraph $\vec{G}$ for almost all n satisfying the necessary conditions (3) and (4).

A (n, k, λ) *block design* with index λ is a pair $(V, \mathcal{B})$, where V is an n-element set and $\mathcal{B}$ is a collection of k-element subsets of V (called *blocks*) such that every 2-element subset of V occurs in exactly λ blocks. Well-known necessary conditions for the existence of a (n, k, λ) block design are:

$$\lambda n(n-1) \equiv 0 \bmod k(k-1), \tag{5}$$
$$\lambda(n-1) \equiv 0 \bmod k-1. \tag{6}$$

A block design is said to be *simple* if it does not contain repeated blocks. Motivated by the repeated block issue in design theory, Gronau and Mullin [31] asked for *supersimple* block designs, i.e. block designs where any two different blocks share at most two vertices. It is easy to check, that every supersimple $(n, k, 2)$ design corresponds to a suborthogonal double cover of K_n by the complete graph K_k. The blocks of the design are just the pages of the SODC.

THEOREM 5.4 ([31], see also [38, 52]). *There is a supersimple $(n, 4, 2)$-block design if and only if $n \geq 7$ and $n \equiv 0, 1 \bmod 3$.*

Evidently, for $k = 3$, every simple block design is also supersimple. The existence problem for simple triple systems has been completely settled by Dehon [16] who proved a simple $(n, 3, \lambda)$ block design to exist exactly when $n \geq \lambda + 2$ and the necessary conditions (5) and (6) hold.

Since then, a number of further results on supersimple designs with block size 4 and index larger than 2 have been established by Khodkar [52], Chen [12] as well as Adams, Bryant and Khodkar [1].

THEOREM 5.5 ([1, 12, 52]). *There is a supersimple $(n, 4, \lambda)$-block design with $\lambda = 3, 4$ if and only if $n \geq 2\lambda + 2$ and the necessary conditions (5) and (6) hold.*

An asymptotic existence result for supersimple block designs was presented by Hartmann [40]. This paper also contains similar results on graph and digraph designs where every two blocks are almost disjoint.

THEOREM 5.6 ([40]). *There is a supersimple (n, k, λ) block design for almost all positive integers n satisfying the necessary conditions (5) and (6).*

In [40], the reader will also find a discussion of possible generalizations of the concept of suborthogonality to λ-fold decompositions of complete graphs and digraphs where λ is larger than two.

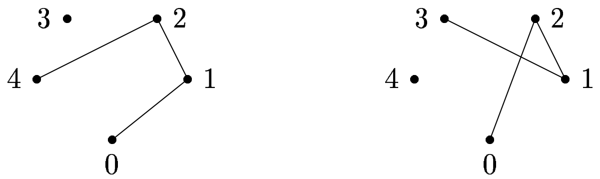

Figure 13. Two copies of P_4 generating a suborthogonal 3-cover of K_5

As an example consider the complete graph K_5 and the cyclic group $\mathbb{Z}_5$ acting on its vertex set. Figure 13 shows two copies P' and P'' of the path P_4. The union of the orbits $P' + \mathbb{Z}_5$ and $P'' + \mathbb{Z}_5$ gives us a 3-fold decomposition of K_5 where any two pages share at most one edge, i.e. a suborthogonal 3-cover of K_5 by P_4.

5.2. ODCs of general graphs

In Section 2.4 we briefly touched double covers of $K_{n,m}$ with a certain orthogonality property. This motivates some interest in a generalization of ODCs to arbitrary graphs H as the underlying graphs instead of K_n. An orthogonal double cover (ODC) of H is a collection $\mathcal{O}$ of spanning subgraphs G_i, $i \in V(H)$, of the graph H such that every edge of H occurs in exactly two subgraphs, and any two subgraphs G_i,G_j share an edge if and only if i, j are adjacent in H. Again, if all the pages are isomorphic to some graph G we speak of an ODC by G.

Note that this definition is consistent with the one in the introduction for $H = K_n$.

A necessary condition for the existence of an ODC by G has been established by Hartmann and Schumacher:

PROPOSITION 5.7 ([44]). *There is an ODC of a given graph H by some graph G if and only if H is regular.*

Obviously, the graph $K_{n,m}$ is regular if and only if $n = m$ and, therefore, does not admit an ODC by some graph G for $n \neq m$. For $n = m$ several results were obtained which are put in the following section.

Hartmann and Schumacher observed that there does not exist a restriction as in Proposition 5.7 of the graphs that are able to play the role of G in an ODC of H by G.

PROPOSITION 5.8 ([44]). *For every graph G there is some graph H admitting an ODC of H by G.*

Among the graph classes for H considered in [44] are r-dimensional cube graphs covered by forests containing r edges. We remark that in this paper the equivalent to Conjecture 3.10 has been proved to be true for cube graphs. The authors also described all almost all graphs H which admit an ODC of H by P_4. Moreover, they studied ODCs whose pages are isomorphic sets of independent edges and obtained the following result.

THEOREM 5.9 ([44]). *Every r-regular graph H with vertex number $\nu(H) \geq \frac{16(r-1)^2+2}{r}$ admits an ODC by rK_2.*

Furthermore, clique graphs G have been investigated. Hartmann and Schumacher were able to characterize all graphs H admitting an ODC by K_3 or by K_4.

If we do not demand all pages to be isomorphic, then also non-regular graphs H are of interest as for example $K_{n,m}$. The thin edges in Figure 6 represent an ODC of $K_{4,5}$ whose pages are isomorphic to either $S_4 \cup E_4$ or $S_4 \cup K_2 \cup E_2$. In general, not much is known about ODCs of $K_{n,m}$ for $n \neq m$.

5.3. ODCs of complete bipartite graphs

In this section, ODCs of $K_{n,n}$ by G will be studied. In the sequel, we will label the vertices of $K_{n,n}$ by $\Gamma \times \{0,1\}$ such that $\Gamma \times \{0\}$ and $\Gamma \times \{1\}$ are independent sets of vertices. As an immediate consequence of the double cover property, G contains exactly n edges. Moreover, the orthogonality property forces the graphs $G_{a,0}$ for all $a \in \Gamma$ and the graphs $G_{a,1}$ for all $a \in \Gamma$ to form two orthogonal edge-decompositions of $K_{n,n}$. See Figure 14 for an example.

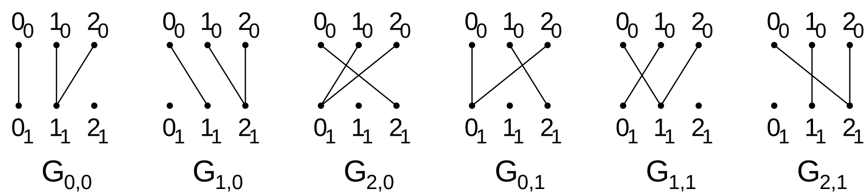

Figure 14. An ODC of $K_{3,3}$ by $G = S_2 \cup K_2 \cup E_1$

To each of the two edge decompositions we may associate bijectively an $n \times n$-square with entries belonging to Γ by $L_i = L_i(k,l)$ $i = 0,1; k,l \in \Gamma$ with $L_i(k,l) = m$ if and only if $\{k_0, l_1\} \in E(G_{m,i})$. For the squares, the orthogonality condition reads as $|\{(L_0(k,l), L_1(k,l)) : k,l \in \Gamma\}| = n^2$. Obviously, a 1-factorization ($G = nK_2$) corresponds bijectively to a Latin square. As a consequence of the existence of mutually orthogonal Latin squares, see [15], we have the following fact.

THEOREM 5.10 ([23, 44]). *Let $n \neq 2, 6$ be a positive integer and $G = nK_2$. Then there exists an ODC of $K_{n,n}$ by G.*

Actually, we can take any pair of $n \times n$-squares with every entry of Γ occurring exactly n times, as long as we can ensure that the graphs described are mutually isomorphic. For example, define $L_0(k,l) = k$ and $L_1(k,l) = l$. The graphs associated with these squares are stars and we obtain an ODC of $K_{n,n}$ by S_n for all positive integers n.

With a certain kind of blowing-up construction (a variation of McNeish's recursive construction [60] for orthogonal Latin squares) we may obtain larger ODCs from small ingredient ODCs. Starting with an ODC $\mathcal{O}$ of $K_{m,m}$ by mK_2 we replace every point by n new points and every edge by an ODC of $K_{n,n}$. The verification that this construction works as claimed will be given in the forthcoming paper of El-Shanawany and Gronau [22]. We just state the theorem.

THEOREM 5.11. *Let $m \neq 2, 6$ be a positive integer, and assume that there exist ODCs $\mathcal{O}_i$ of $K_{n,n}$ by G_i for $i = 0, 1, \ldots, m-1$. Then there exists an ODC of $K_{mn,mn}$ by $G_0 \cup G_1 \cup \ldots \cup G_{m-1}$.*

To make the blowing-up construction work, one needs ingredient ODCs. For small graphs, ODCs of $K_{n,n}$ have been studied by El-Shanawany, Gronau and Grüttmüller who established the following result.

THEOREM 5.12 ([23]). *There exists an ODC of $K_{n,n}$ by all spanning subgraphs G of $K_{n,n}$ with n edges for $1 \leq n \leq 9$, except when $G = 2K_2$ or $G = 6K_2$.*

The main construction tool for that result has been called ODC-generating graph, in analogy to Section 2.1 (symmetric starter in [23]). By this, we mean a graph G such that the translates $G_{i,0}$ ($i \in \Gamma$) with $E(G_{i,0}) = \{\{(k+i)_0, (l+i)_1\} : \{k_0, l_1\} \in E(G)\}$ form an edge decomposition of $K_{n,n}$ which is orthogonal to its symmetric decomposition (G is symmetric to G' if $\{k_0, l_1\}$ is an edge in G if and only if $\{l_0, k_1\}$ is an edge in G'). Necessary and sufficient conditions for a graph G to be an ODC-generating graph are $\{l-k : \{k_0, l_1\} \in E(G)\} = \Gamma$ and $\{l-k' : \{k_0, l_1\}, \{k'_0, l'_1\} \in E(G), l_1 - k_0 = -(l'_1 - k'_0)\} = \Gamma$.

The authors of [23] also proved the existence of an ODC $K_{n,n}$ by G for certain infinite classes of graphs G . These classes are mentioned in the following.

PROPOSITION 5.13. *Let $n \geq 3$ be an odd integer and $G = S_2 \cup S_{n-2}$ or $G = S_2 \cup (n-2)K_2$, or let $n \geq 3$ be an integer and $G = K_2 \cup S_{n-1}$. Then there exists an ODC of $K_{n,n}$ by G.*

Given an ODC of $K_{n,n}$ generated by G we can construct a group-generated ODC of $\vec{D}_n$ by $\vec{G}'$: For every edge $\{k_0, l_1\}$ in G with $k \neq l$ put an arc (k, l) in $\vec{G}'$. Since G is an ODC-generating graph, it contains exactly one edge $\{k_0, k_1\}$ and, therefore, we obtain exactly $n-1$ arcs in $\vec{G}'$. The construction also works in the opposite direction with an additional edge $\{k_0, k_1\}$ (for some fixed $k \in \Gamma$) added to the edges obtained from $\vec{G}'$. This allows us to use results from Section 2.6.

THEOREM 5.14. *Let $n = 2^e t + 1$ be a prime power, where $t \geq t_0(e)$ is an odd integer. Then there exists an ODC of $K_{n,n}$ by P_n.*

Proof. Start with an almost-hamiltonian cycle from Theorem 2.28 and give opposite directions to consecutive edges, in order to obtain a group-generated ODC of $\vec{D}_n$. The anti-directed cycle has the property that edges with inverse lengths occur consecutively. Take one such pair of edges and flip their directions. The resulting directed graph still is an ODC-generator which can be used to construct, as described above, a subgraph of $K_{n,n}$ consisting of two disjoint paths. Adding an edge $\{k_0, k_1\}$ (k_0, k_1 are endpoints of the two paths) provides an ODC generating P_n. □

5.4. Generalized orthogonal covers

As pointed out earlier, biplanes give rise to ODCs by complete graphs (augmented by a suitable number of isolated vertices). Biplanes are also known as symmetric block designs of index $\lambda = 2$. In this subsection we shall investigate a common generalization of symmetric block designs of arbitrary index and orthogonal double covers. This generalization is due to Gronau, Mullin, Rosa and Schellenberg [33].

A λ-*fold factorization* of a complete graph K_n is a collection $\mathcal{O}$ of spanning subgraphs of K_n such that every edge of K_n occurs in exactly λ of these subgraphs. As usual, the subgraphs in $\mathcal{O}$ shall be called the *pages* of $\mathcal{O}$. If, in particular, all pages are isomorphic copies of some graph G, we speak of a λ-fold factorization of K_n by G.

Let G and F be given graphs on n vertices. A λ-fold factorization $\mathcal{O}$ of K_n by G is said to be a *generalized orthogonal cover* $(n, G, \lambda; F)$-GOC if $\mathcal{O}$ has exactly n pages and the intersection of any two of the pages is isomorphic to F. Generalized orthogonal covers are better known as *symmetric graph designs* due to a proposal of Alex Rosa. Later, however, he suggested to rename them and brought in the notion of a generalized orthogonal cover.

Clearly, every $(n, K_k \cup E_{n-k}, \lambda; F)$-GOC corresponds to a symmetric (n, k, λ) block design.

An easy calculation shows that in an GOC of K_n the graph G has exactly $\lambda(n-1)/2$ edges, whereas the graph F has $\lambda(\lambda-1)/2$ edges.

For $\lambda = 1$, F has to be the empty graph on n vertices. Hence an $(n, G, 1; E_n)$-GOC is just a factorization of K_n by some graph G with $(n-1)/2$ edges. This problem has been widely studied in literature. For example, Ringel's conjecture can be restated as the question whether there exists an $(n, T \cup E_{(n-1)/2}, 1; E_n)$-GOC for any tree T with $(n-1)/2$ edges. The interested reader may consult e.g. Bosák's book [8] for an overview of existing results on graph factorizations. Moreover, cyclic GOCs with $\lambda = 1$ lead to ρ-labelings of graphs. Here we refer to the dynamic survey of Gallian [26].

For $\lambda = 2$, F consists of n vertices and exactly one edge. Evidently, GOC with $\lambda = 2$ correspond to ODCs. In this sense, GOCs may be regarded as an extension of the concept of orthogonal double covers.

The investigation of proper GOCs starts with $\lambda = 3$. Here we have various possibilities to choose the graph F, i.e. the intersection of the pages of the GOC. However, the complete graph K_λ augmented by $n - \lambda$ isolated vertices seems to be the most natural choice. Examples of GOCs may be found almost immediately as we shall see below.

Recall that an amalgamated H-family is a graph consisting of m copies of H glued together at some vertex x. The well-chosen vertex x is also said to be the *root* of the amalgamated H-family under discussion. Now, let H be the complete graph K_k augmented by $k(k-2)/2$ isolated vertices. For this graph, the augmented H-family is also known as the *friendship graph* $F_{k,m}$.

A simple observation provides GOCs by friendship graphs: Take an $(n, k, 1)$ block design $(V, \mathcal{B})$ and replace every block by an complete graph on k vertices. Let G_v be the graph with vertex set V which consists of the edges of all the blocks in $\mathcal{B}$ containing the element v. Then the collection $\mathcal{O}$ of the arising graphs G_v, $v \in V$, form an $(n, F_{k,m}, k; K_k \cup E_{n-k})$-GOC where m equals $(n-1)/(k-1)$. Note that, conversely, every GOC of this kind yields an $(n, k, 1)$ block design.

Due to this correspondence, for $\lambda = k = 3$, Steiner triple systems yield $(n, F_{3,m}, 3; K_3 \cup E_{n-3})$-GOCs and vice versa. Hence, these GOCs exist precisely for all $n \equiv 1$ or 3 mod 6.

As mentioned earlier, $K_3 \cup E_{n-3}$ is not the only possibility for F. In fact, we may select any graph with 3 edges. Let V be the vertex set of the complete graph K_n. Suppose, there exists an $(n, F_{3,m}, 3; F)$-GOC $\mathcal{O}$ with pages G_v, $v \in V$. Without loss of generality, we may assume that the vertex v is the root of the page G_v. Then any two pages G_v and G_w share the edge $\{v, w\}$. Since G_v is a copy of the friendship graph $F_{3,m}$, it contains some triangle $\{v, w, x_v\}$. Analogously, G_w contains some triangle $\{v, w, x_w\}$. We treat two cases: If x_v equals x_w, the two pages under consideration intersect in the triangle $\{v, w, x_v\}$. Otherwise, if x_v differs from x_w, the two pages intersect in a 4-path with vertices x_v, v, w and x_w.

Thus, rather surprisingly, it turns out that an $(n, F_{3,m}, 3; F)$-GOC exists only if F is either $K_3 \cup E_{n-3}$ or $P_4 \cup E_{n-4}$, as pointed out by Fronček and Rosa [25]. In the same paper, they determine the complete spectrum for GOCs by friendship graphs with $\lambda = 3$:

THEOREM 5.15 ([25]). *There exists an $(n, F_{3,m}, 3; P_4 \cup E_{n-4})$-GOC if and only if n is odd and $n \geq 5$.*

Further results on GOCs have been obtained by Gronau, Mullin, Rosa and Schellenberg [33] who provide a triplication construction for GOCs with $\lambda = 3$, and by Cameron [11] who gives a classification of GOCs with a 2-transitive automorphism group.

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Address for Offprints: Universität Rostock
Fachbereich Mathematik
18051 Rostock
Germany