Point estimates for Green's matrix to boundary value problems for second order elliptic systems in a polyhedral cone

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Abstract: In this paper we are concerned with boundary value problems for general second order elliptic equations and systems in a polyhedral cone. We obtain point estimates of Green's matrix in different areas of the cone. The proof of these estimates is essentially based on weighted L_2 estimates for weak solutions and their derivatives. As examples, we consider the Neumann problem to the Laplace equation and the Lamé system.

1. Introduction

We deal with the Dirichlet, Neumann and mixed problems for elliptic systems of second order equations in a polyhedral cone $\mathcal{K}$. Our main goal is to obtain point estimates for Green's matrix. In a forthcoming work we will prove, by means of such estimates, solvability theorems and regularity assertions in weighted L_p Sobolev and Hölder spaces.

As is well-known, the nonsmoothness of the boundary causes singularities of the solutions at the edges even if the right-hand side of the differential equation and the boundary data are smooth. Therefore, Green's matrix $G(x,\xi)$ is singular not only at the diagonal but also for x or ξ near the vertex or an edge. For a cone without edges these singularities were described by Maz′ya and Plamenevskiĭ [12] in terms of eigenvalues, eigenfunctions and generalized eigenfunctions of a certain operator pencil. The presence of edges on the boundary makes the investigation of Green's functions more difficult. In [10] Maz′ya and Plamenevskiĭ obtained estimates for Green's functions of boundary value problems in a dihedral angle. The results in [10] are applicable, e.g., to the Dirichlet problem for elliptic equations but not to the Neumann problem. Green's functions for the Dirichlet problem in polyhedral domains were studied in papers by Maz′ya and Plamenevskiĭ [13] (Lamé and Stokes systems), Maz′ya and Roßmann [15] (strongly elliptic $2m$ order equations). Concerning the Neumann problem for the Laplace equation in domains with edges, we refer to the preprints of Solonnikov [22], Grachev and Maz′ya [5].

We outline the main results of our paper. Let $\mathcal{K} = \{x \in \mathbb{R}^3 : \omega = x/|x| \in \Omega\}$ be a polyhedral cone with faces $\Gamma_j = \{x : x/|x| \in \gamma_j\}$ and edges M_j, $j = 1, \dots, n$. Here Ω is curvilinear polygon on the unit sphere bounded by the sides $\gamma_1, \dots, \gamma_n$. Suppose that $\mathcal{K}$ coincides with a dihedral angle $\mathcal{D}_j$ in a neighborhood of an arbitrary edge point $x \in M_j$. By $\mathcal{S}$ we denote the set $M_1 \cup \dots \cup M_n \cup \{0\}$. We consider the boundary value problem

$$L(\partial_x)\, u = -\sum_{i,j=1}^{3} A_{i,j}\, \partial_{x_i}\partial_{x_j} u = f \quad \text{in } \mathcal{K}, \tag{1.1}$$

$$u = g_j \ \text{ on } \Gamma_j \ \text{ for } j \in J_0, \tag{1.2}$$

$$B(\partial_x)\, u = \sum_{i,j=1}^{3} A_{i,j}\, n_j\, \partial_{x_i} u = g_k \quad \text{on } \Gamma_k \text{ for } k \in J_1. \tag{1.3}$$

where $A_{i,j}$ are constant $\ell \times \ell$ matrices such that $A_{i,j} = A^*_{j,i}$, $J_0 \cup J_1 = \{1, \dots, n\}$, $J_0 \cap J_1 = \emptyset$, u, f, g are vector-valued functions, and (n_1, n_2, n_3) denotes the exterior normal to $\partial\mathcal{K}\backslash\mathcal{S}$.

Weak solutions of problem (1.1)–(1.3) can be defined by means of the sesquilinear form

$$b_{\mathcal{K}}(u,v) = \int_{\mathcal{K}} \sum_{i,j=1}^{3} A_{i,j} \partial_{x_i} u \cdot \partial_{x_j} \overline{v}\, dx, \tag{1.4}$$

where $u \cdot \overline{v}$ is the scalar product in $\mathbb{C}^\ell$ of the vectors u and v. We denote by $\mathcal{H}$ the closure of the set $\{u \in C_0^\infty(\overline{\mathcal{K}})^\ell : \, u = 0 \text{ on } \Gamma_j \text{ for } j \in J_0\}$ with respect to the norm

$$\|u\|_{\mathcal{H}} = \Big(\int_{\mathcal{K}} \sum_{j=1}^{3} |\partial_{x_j} u|^2_{\mathbb{C}^\ell}\, dx \Big)^{1/2}. \tag{1.5}$$

Here $C_0^\infty(\overline{\mathcal{K}})$ is the set of all infinitely differentiable functions on $\overline{\mathcal{K}}$ with compact supports.

From the above assumptions on the coefficients $A_{i,j}$ it follows that $b_{\mathcal{K}}(u,v) = \overline{b_{\mathcal{K}}(v,u)}$ for $u, v \in \mathcal{H}$. Throughout this paper, it will be assumed that the form $b_{\mathcal{K}}$ is $\mathcal{H}$-coercive, i.e.,

$$b_{\mathcal{K}}(u,u) \geq c\, \|u\|^2_{\mathcal{H}} \quad \text{for all } u \in \mathcal{H}. \tag{1.6}$$

By Lax-Milgram's lemma, this implies that the variational problem

$$b_{\mathcal{K}}(u,v) = (F, v)_{\mathcal{K}} \quad \text{for all } v \in \mathcal{H} \tag{1.7}$$

is uniquely solvable in $\mathcal{H}$ for arbitrary $F \in \mathcal{H}^*$. Here $(\cdot,\cdot)_{\mathcal{K}}$ denotes the scalar product in $L_2(\mathcal{K})^\ell$ or its extension to $\mathcal{H}^* \times \mathcal{H}$.

In Section 2 we consider the boundary value problem in a dihedron $\mathcal{D} = K \times \mathbb{R}$, where K is an infinite angle in the x_1, x_2-plane with opening θ. The main goal of this section is the estimation of Green's matrix. We give here the estimates in the case of the Neumann problem to the Laplace equation, which was also considered in [22]. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ be arbitrary multi-indices. Then

$$\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \leq c\, |x-\xi|^{-1-|\alpha|-|\gamma|} \, \Big(\frac{|x'|}{|x-\xi|} \Big)^{\min(0,\pi/\theta-\alpha_1-\alpha_2-\varepsilon)} \, \Big(\frac{|\xi'|}{|x-\xi|} \Big)^{\min(0,\pi/\theta-\gamma_1-\gamma_2-\varepsilon)}$$

for $|x-\xi| \geq \min(|x'|, |\xi'|)$, where $x' = (x_1, x_2)$, $\xi' = (\xi_1, \xi_2)$, and ε is an arbitrarily small positive number. For $|x-\xi| < \min(|x'|, |\xi'|)$ there is the estimate $\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \leq c\, |x-\xi|^{-1-|\alpha|-|\gamma|}$. The same inequalities hold for Green's matrix of the Neumann problem to the Lamé system if $\theta < \pi$, while in the case $\theta > \pi$ the number π/θ in the exponent has to be replaced by $\xi_+(\theta)/\theta$, where $\xi_+(\theta)$ is the smallest positive root of the equation

$$\frac{\sin \xi}{\xi} + \frac{\sin \theta}{\theta} = 0. \tag{1.8}$$

For the proof of these inequalities, we use weighted L_2 estimates for weak solutions and their derivatives.

Section 3 concerns the parameter-dependent boundary value problems

$$\mathcal{L}(\lambda)\, u = f \;\text{ in } \Omega, \quad u = g_j \;\text{ on } \gamma_j,\; j \in J_0, \quad \mathcal{B}(\lambda)\, u = g_k \;\text{ on } \gamma_k,\; k \in J_1 \tag{1.9}$$

generated by problem (1.1)–(1.3) on the intersection Ω of the cone $\mathcal{K}$ with the unit sphere S^2. Here

$$\mathcal{L}(\lambda) u = \rho^{2-\lambda}\, L(\partial_x)\, \big(\rho^\lambda u(\omega)\big), \quad \mathcal{B}(\lambda) u = \rho^{1-\lambda}\, B(\partial_x)\, \big(\rho^\lambda u(\omega)\big), \tag{1.10}$$

$\rho = |x|$, and $\omega = x/|x|$. Let $\mathfrak{A}(\lambda)$ be the operator of problem (1.9). We prove that problem (1.9) is uniquely solvable (in a certain class of weighted Sobolev spaces) for all λ, except finitely many, in a double angle of the complex plane containing the imaginary axis. Furthermore, we obtain an a priori estimate of the solution.

In Section 4, by means of these results, solvability theorems for the boundary value problem (1.1)–(1.3) in weighted Sobolev spaces are obtained. In particular, we prove the existence of weak solutions $u \in V^1_\beta(\mathcal{K})^\ell$, where $V^1_\beta(\mathcal{K})$ is the weighted Sobolev space with the norm

$$\|u\|_{V^1_\beta(\mathcal{K})} = \Big(\int_{\mathcal{K}} |x|^{2\beta} \big(|\nabla u|^2 + |x|^{-2} |u|^2 \big)\, dx \Big)^{1/2}. \tag{1.11}$$

Here, for example, by a weak solution of the Neumann problem we mean a vector function $u \in V^1_\beta(\mathcal{K})^\ell$ satisfying

$$b_{\mathcal{K}}(u,v) = (F,v)_{\mathcal{K}} \quad \text{for all } v \in V^1_{-\beta}(\mathcal{K})^\ell,$$

where F is a given continuous functional on $V^1_{-\beta}(\mathcal{K})^\ell$. We prove that the absence of eigenvalues of the pencil $\mathfrak{A}$ on the line $\operatorname{Re}\lambda = -\beta - 1/2$ ensures the unique existence of a weak solution $u \in V^1_\beta(\mathcal{K})^\ell$. Furthermore, we prove regularity assertions for the solution. For example, we conclude from our results that the second derivatives of the solution $u \in \mathcal{H}$ of the Dirichlet and Neumann problems for the Laplace equation (and other second order differential equations, including the Lamé system) are square summable if the angles at the edges are less than π and there are no eigenvalues of the pencil $\mathfrak{A}$ with positive real part $\le 1/2$. In particular, the W^2 regularity holds for the Dirichlet problem to the Laplace equation and to the Lamé system if $\mathcal{K}$ is convex. This follows from the monotonicity of real eigenvalues of the pencil $\mathfrak{A}$ in the interval $[0,1]$ (see, e.g., the monograph by Kozlov, Maz′ya and Roßmann [8, Ch.2,3]). For the Neumann problem to the Laplace equation the W^2 regularity was proved by Dauge [3, 4].

The absence of eigenvalues of the pencil $\mathfrak{A}$ on the line $\operatorname{Re}\lambda = -\beta - 1/2$ guarantees also the existence of a unique solution $G(x,\xi)$ of the problem

$$L(\partial_x)\, G(x,\xi) = \delta(x-\xi)\, I_\ell, \quad x,\xi \in \mathcal{K}, \tag{1.12}$$

$$G(x,\xi) = 0, \quad x \in \Gamma_j,\ \xi \in \mathcal{K},\ j \in J_0, \tag{1.13}$$

$$B(\partial_x)\, G(x,\xi) = 0, \quad x \in \Gamma_j,\ \xi \in \mathcal{K},\ j \in J_1 \tag{1.14}$$

(I_ℓ denotes the $\ell \times \ell$ identity matrix) such that the function $x \to \zeta\Big(\frac{|x-\xi|}{r(\xi)}\Big)\, G(x,\xi)$ belongs to the space $V^1_\beta(\mathcal{K})^{\ell\times\ell}$ for every fixed $\xi \in \mathcal{K}$ and for every smooth function ζ on $(0,\infty)$ equal to one in $(1,\infty)$ and to zero in $(0,\frac{1}{2})$. We obtain point estimates for the derivatives of $G(x,\xi)$ in different areas of $\mathcal{K}\times\mathcal{K}$. For example, Green's function of the Neumann problem to the Laplace equation satisfies the following estimate for $|x| < |\xi|/2$:

$$\begin{aligned} \big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \ &\le\ c\,|x|^{\Lambda_- - |\alpha| + \varepsilon} \prod_j \Big(\frac{r_j(x)}{|x|}\Big)^{\min(0,\pi/\theta_j - |\alpha| - \varepsilon)} \\ &\qquad \times |\xi|^{-1-\Lambda_- - |\gamma| - \varepsilon} \prod_j \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{\min(0,\pi/\theta_j - |\gamma| - \varepsilon)}. \end{aligned}$$

Here $\Lambda_- < \operatorname{Re}\lambda < \Lambda_+$ is the widest strip in the complex plane containing the line $\operatorname{Re}\lambda = -\beta - 1/2$ which is free of eigenvalues of the pencil $\mathfrak{A}$, θ_j is the angle at the edge M_j, r_j is the distance to M_j, and ε is an arbitrarily small positive number. The same estimate holds for the Lamé system if $\theta_j < \pi$ for $j = 1,\dots,n$. If $\theta_j > \pi$, then the number π in the exponent has to be replaced by $\xi_+(\theta_j)$. In the case $\beta = 0$, when $\Lambda_+ = 0$ and $\Lambda_- = -1$, these estimates can be improved.

2. The boundary value problem in a dihedron

Let $\mathcal{D}$ be the dihedron $\{x = (x', x_3) : x' = (x_1, x_2) \in K,\ x_3 \in \mathbb{R}\}$, where K is the angle $\{x' = (x_1, x_2) : 0 < r < \infty,\ 0 < \varphi < \theta\}$. Here r, φ are the polar coordinates in the (x_1, x_2)-plane. Furthermore, let $\Gamma^- = \{x : \varphi = 0\}$ and $\Gamma^+ = \{x : \varphi = \theta\}$ be the sides of $\mathcal{D}$, $M = \overline{\Gamma^+} \cap \overline{\Gamma^-}$ the edge, and $d^\pm \in \{0, 1\}$. We consider the boundary value problem

$$L(\partial_x)u = f \quad \text{in } \mathcal{D}, \qquad d^\pm\, u + (1 - d^\pm)\, B(\partial_x)u = g^\pm \quad \text{on } \Gamma^\pm. \tag{2.1}$$

This means, for $d^+ = d^- = 1$ we are concerned with the Dirichlet problem, for $d^+ = d^- = 0$ with the Neumann problem, and for $d^+ \neq d^-$ with the mixed problem.

We denote by $\mathcal{H}_\mathcal{D}$ the closure of the set $\{u \in C_0^\infty(\overline{\mathcal{D}})^\ell : d^\pm u = 0 \text{ on } \Gamma^\pm\}$ with respect to the norm (1.5), where $\mathcal{K}$ is replaced by $\mathcal{D}$, and by $b_\mathcal{D}$ the sesquilinear form

$$b_\mathcal{D}(u, v) = \int_\mathcal{D} \sum_{i,j=1}^{3} A_{i,j} \partial_{x_i} u \cdot \partial_{x_j} \overline{v}\, dx. \tag{2.2}$$

Suppose again that

$$b_\mathcal{D}(u, u) \geq c \int_\mathcal{D} |\nabla u|^2_{\mathbb{C}^\ell}\, dx \quad \text{for all } u \in \mathcal{H}_\mathcal{D}. \tag{2.3}$$

Then the variational problem

$$b_\mathcal{D}(u, v) = (F, v)_\mathcal{D} \quad \text{for all } v \in \mathcal{H}_\mathcal{D} \tag{2.4}$$

has a unique solution $u \in \mathcal{H}_\mathcal{D}$ for arbitrary $F \in \mathcal{H}^*_\mathcal{D}$.

A large part of this section deals with the regularity of weak solutions. For the Dirichlet and mixed problems, which are handled at the end of the section, we give only the formulation of a theorem which follows from results of Maz′ya and Plamenevskiĭ [9], Nazarov and Plamenevskiĭ [18]. The more complicated case of the Neumann problem is studied in Sections 2.2–2.5. The results here were partially obtained by Zajaczkowski and Solonnikov [23], Nazarov [16, 17], Roßmann [20], Nazarov and Plamenevskiĭ [18].

The proof of point estimates for Green's matrix in this section is essentially based on weighted L_2 estimates for weak solutions and their derivatives. As examples, we consider the Neumann problem for the Laplace equation and the Lamé system.

2.1. Weighted Sobolev spaces in a dihedron and in an angle

Let $\delta > -1$. Then $L^k_\delta(\mathcal{D})$ denotes the closure of $C_0^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{L^k_\delta(\mathcal{D})} = \int_\mathcal{D} \sum_{|\alpha|=k} r^{2\delta}\, |\partial_x^\alpha u|^2\, dx\Big)^{1/2},$$

where $r = |x'| = (x_1^2 + x_2^2)^{1/2}$. Furthermore, we set

$$W^k_\delta(\mathcal{D}) = \bigcap_{j=0}^{k} L^j_\delta(\mathcal{D}).$$

For arbitrary real δ let $V^k_\delta(\mathcal{D})$ be the closure of $C_0^\infty(\overline{\mathcal{D}} \backslash M)$ with respect to the norm

$$\|u\|_{V^k_\delta(\mathcal{D})} = \int_\mathcal{D} \sum_{|\alpha|\leq k} r^{2(\delta-k+|\alpha|)}\, |\partial_x^\alpha u|^2\, dx\Big)^{1/2}. \tag{2.5}$$

Analogously, we define the spaces $L^k_\delta(K)$, $V^k_\delta(K)$ and $W^k_\delta(K)$ for a plane angle K with vertex in the origin (then in the above norms $\mathcal{D}$ has to be replaced by K).

By Hardy's inequality, every function $u \in C_0^\infty(\overline{\mathcal{D}})$ satisfies the inequality

$$\int_{\mathcal{D}} r^{2(\delta-1)} |u|^2 \, dx \le c \int_{\mathcal{D}} r^{2\delta} \, |\nabla u|^2 \, dx \tag{2.6}$$

for $\delta > 0$ with a constant c depending only on δ . Consequently, the space $L^k_\delta(\mathcal{D})$ is continuously imbedded into $L^{k-1}_{\delta-1}(\mathcal{D})$ if $\delta > 0$. If $\delta > k-1$, then $L^k_\delta(\mathcal{D}) = V^k_\delta(\mathcal{D})$. Furthermore, from Hardy's inequality it follows that

$$\int_{\mathcal{D}} |x - x_0|^{-2} \, |u(x)|^2 \, dx \le c \, \|\nabla u\|^2_{L_2(\mathcal{D})^\ell} \tag{2.7}$$

for every $u \in \mathcal{H}_{\mathcal{D}}$ and for an arbitrary point $x_0 \in M$. This means that any vector function $u \in \mathcal{H}_{\mathcal{D}}$ is square integrable on every bounded subset of $\mathcal{D}$. From (2.6) an (2.7) we conclude that

$$\int_{\mathcal{D}} r^{2\delta} \, |\phi u|^2 \, dx \le c \, \|\nabla u\|^2_{L_2(\mathcal{D})^\ell}. \tag{2.8}$$

for $\delta > -1$ if $u \in \mathcal{H}_{\mathcal{D}}$ and ϕ is a function in $C^1(\overline{\mathcal{D}})$ with compact support.

The spaces of the traces of functions from $L^k_\delta(\mathcal{D})$, $V^k_\delta(\mathcal{D})$ and $W^k_\delta(\mathcal{D})$, $k \ge 1$, on the sides $\Gamma^\pm = \gamma^\pm \times \mathbb{R}$ of $\mathcal{D}$ are denoted by $L^{k-1/2}_\delta(\Gamma^\pm)$, $V^{k-1/2}_\delta(\Gamma^\pm)$ and $W^{k-1/2}_\delta(\Gamma^\pm)$, respectively. The norm in $L^{k-1/2}_\delta(\Gamma^\pm)$ is defined as

$$\|u\|_{L^{k-1/2}_\delta(\Gamma^\pm)} = \inf \big\{ \|v\|_{L^k_\delta(\mathcal{D})} \ : \ v \in L^k_\delta(\mathcal{D}), \ v|_{\Gamma^\pm} = u \big\}.$$

Analogously, the norms in $V^{k-1/2}_\delta(\Gamma^\pm)$ and $W^{k-1/2}_\delta(\Gamma^\pm)$ are defined. An equivalent norm in $V^{k-1/2}_\delta(\Gamma^\pm)$ is given by (see [9, Le.1.4])

$$\begin{aligned} \|u\| \ &= \ \Big(\int_{\gamma^\pm} \int_{\mathbb{R}} \int_{\mathbb{R}} r^{2\delta} \, \big| \partial^{k-1}_{x_3} u(r,x_3) - \partial^{k-1}_{y_3}(r,y_3) \big|^2 \, \frac{dx_3\, dy_3}{|x_3 - y_3|^2} \, dr \\ &\quad + \int_{\mathbb{R}} \int_{\gamma^\pm} \int_{\gamma^\pm} \big| r_1^\delta (\partial_r^{k-1} u)(r_1, x_3) - (\partial_r^{k-1} u)(r_2, x_3) \big|^2 \, \frac{dr_1\, dr_2}{|r_1 - r_2|^2} \, dx_3 \\ &\quad + \int_{\Gamma^\pm} \sum_{j=0}^{k-1} r^{2(\delta-k+j)+1} \, |\partial_r^j u(r,x_3)|^p \, dr \, dx_3 \Big)^{1/2}. \end{aligned} \tag{2.9}$$

For $\delta > k-1$ this is also an equivalent norm in $L^{k-1/2}_\delta(\Gamma^\pm)$.

2.2. The operator pencil corresponding to the boundary value problem

Let $\mathcal{H}_{(0,\theta)} = \{ u \in W^1(0,\theta)^\ell \ : \ d^- u(0) = d^+ u(\theta) = 0 \}$, where W^1 denotes the usual Sobolev space and $d^\pm$ are the numbers introduced in the beginning of this section. Furthermore, let

$$a_K(u,v;\lambda) = \frac{1}{\log 2} \int\limits_{\substack{K \\ 1<|x'|<2}} \sum_{i,j=1}^2 A_{i,j} \partial_{x_i} U \cdot \partial_{x_j} \overline{V} \, dx',$$

where $U = r^\lambda u(\varphi)$, $V = r^{-\overline{\lambda}} v(\varphi)$, $u, v \in \mathcal{H}_{(0,\theta)}$, $\lambda \in \mathbb{C}$. The form $a_K(\cdot,\cdot;\lambda)$ generates a continuous operator $A(\lambda) : \mathcal{H}_{(0,\theta)} \to \mathcal{H}^*_{(0,\theta)}$ by

$$\big(A(\lambda) u, v \big) = a_K(u,v;\lambda), \quad u, v \in \mathcal{H}_{(0,\theta)}.$$

Here $(\cdot,\cdot)$ denotes the scalar product in $L_2((0,\theta))^\ell$. As is known, the spectrum of the pencil A consists of isolated points, the eigenvalues. The line $\mathrm{Re}\,\lambda = 0$ contains no eigenvalues if $d^+ \neq d-$ or $d^+ = d^- = 1$. In the case $d^+ = d^- = 0$ (the case of the Neumann problem), the line $\mathrm{Re}\,\lambda = 0$ contains the single eigenvalue $\lambda = 0$. The eigenvectors corresponding to this eigenvalue are constant vectors. Every of these eigenvectors has exactly one generalized eigenvector (see [8, Ch.12]). We set

$$L(\partial_{x'},0) = -\sum_{i,j=1}^{2} A_{i,j}\,\partial_{x_i}\,\partial_{x_j}\,, \quad B(\partial_{x'},0) = \sum_{i,j=1}^{2} A_{i,j}\,n_j\,\partial_{x_i}$$

and denote by $\gamma^\pm$ be sides of K.

Remark 2.1 The vector function $u = r^{\lambda_0} \sum_{k=0}^{s} \frac{1}{k!}\,(\log r)^k\, v_{s-k}(\varphi)$ is a solution of the problem

$$L(\partial_{x'},0)\,u = 0 \text{ in } K, \quad p^\pm u + (1-p^\pm)\,B(\partial_{x'},0)\,u = 0 \text{ on } \gamma^\pm$$

if and only if λ_0 is an eigenvalue of the pencil $A(\lambda)$ and $v_0, v_1, \dots, v_s$ is a Jordan chain corresponding to this eigenvalue (see [8, Le.12.1.1]).

We denote by λ_1 the eigenvalue of the pencil $A(\lambda)$ with smallest positive real part and by μ_1 its real part.

2.3. Regularity results for the solution of the Neumann problem

Let $d^+ = d^- = 0$. We assume that F is a functional on $\mathcal{H}_\mathcal{D}$ which has the form

$$(F,v)_\mathcal{D} = \int_\mathcal{D} f\cdot \overline{v}\,dx + \sum_\pm \int_{\Gamma^\pm} g^\pm \cdot \overline{v}\,d\sigma_\pm\,, \quad v \in \mathcal{H}_\mathcal{D}\,, \tag{2.10}$$

where $f \in L^0_\delta(\mathcal{D})^\ell$, $g^\pm \in L^{1/2}_\delta(\Gamma^\pm)^\ell$, $0 < \delta < 1$. Then the solution of (2.4) belongs to the Sobolev space $W^2_{loc}(\mathcal{D})$ and satisfies the equations

$$L(\partial_x)u = f \ \text{ in } \mathcal{D}, \quad B(\partial_x)u = g^\pm \ \text{ on } \Gamma^\pm. \tag{2.11}$$

Note that the right-hand side of (2.10) always defines a functional on $\mathcal{H}_\mathcal{D}$ if $f \in L^0_\delta(\mathcal{D})^\ell$, $g^\pm \in L^{1/2}_\delta(\Gamma^\pm)^\ell$, and the supports of f and $g^\pm$ are compact. For the first term on the right of (2.10), this can be easily proved by means of (2.8). Furthermore, we have $L^{1/2}_\delta(\Gamma^\pm) = V^{1/2}_\delta(\Gamma^\pm)$ for $\delta > 0$ and, due to the equivalence of the norm in $V^{k-1/2}_\delta(\Gamma^\pm)$ to (2.9),

$$\int_{\Gamma^\pm} r^{2\delta-1}|g^\pm|^2\,d\sigma_\pm \le c\,\|g^\pm\|^2_{V^{1/2}_\delta(\Gamma^\pm)}.$$

This implies

$$\begin{aligned} \Big|\int_{\Gamma^\pm} g^\pm\cdot\overline{v}\,d\sigma_\pm\Big|^2 &\le c\int_{\Gamma^\pm} r^{2\delta-1}|g^\pm|^2\,d\sigma_\pm \cdot \int_{\Gamma^\pm} r^{1-2\delta}|\phi v|^2\,d\sigma_\pm \\ &\le c\,\|g^\pm\|^2_{V^{1/2}_\delta(\Gamma^\pm)^\ell}\ \|\phi v\|^2_{V^{1/2}_{1-\delta}(\Gamma^\pm)^\ell} \le c\,\|g^\pm\|^2_{L^{1/2}_\delta(\Gamma^\pm)^\ell}\ \|\phi v\|^2_{\mathcal{H}_\mathcal{D}}\,. \end{aligned}$$

The following lemma can be found in [9, Le.3.1].

Lemma 2.1 *Let* $g^\pm \in V_\delta^{l+d^\pm-3/2}(\Gamma^\pm)^\ell$, *where* $l \ge 1$ *if* $d^+ = d^- = 1$, $l \ge 2$ *else. Then there exists a vector function* $u \in V_\delta^l(\mathcal{D})^\ell$ *such that* $d^\pm u + (1-d^\pm)\, Bu = g^\pm$ *on* $\Gamma^\pm$ *and*

$$\|u\|_{V_\delta^l(\mathcal{D})^\ell} \le c \sum_\pm \|g^\pm\|_{V_\delta^{l+d^\pm-3/2}(\Gamma^\pm)^\ell}$$

with a constant c *independent of* g^+ *and* g^-.

Since $V_\delta^{1/2}(\Gamma^\pm) = L_\delta^{1/2}(\Gamma^\pm)$ for $\delta > 0$ and $V_\delta^2(\mathcal{D}) \subset L_\delta^2(\mathcal{D})$, we conclude that for all $g^\pm \in L_\delta^{1/2}(\Gamma^\pm)^\ell$ there exists a vector function $v \in L_\delta^2(\mathcal{D})^\ell$ such that $B(\partial_x)v = g^\pm$ on $\Gamma^\pm$.

For the proof of the following lemma we refer to [23] and [20] (for general elliptic problems see also [17, 18]).

Lemma 2.2 *Let* ϕ, ψ *be infinitely differentiable functions on* $\overline{\mathcal{D}}$ *with compact supports such that* $\psi = 1$ *in a neighborhood of* $\operatorname{supp}\phi$. *If* $u \in \mathcal{H}_\mathcal{D}$ *is a solution of* (2.4) *and* F *is a functional of the form* (2.10), *where* $\psi f \in L_\delta^0(\mathcal{D})^\ell$ *and* $\psi g^\pm \in L_\delta^{1/2}(\Gamma^\pm)^\ell$, $\max(1-\mu_1, 0) < \delta < 1$, *then* $\phi u \in L_\delta^2(\mathcal{D})^\ell$ *and*

$$\|\phi u\|_{L_\delta^2(\mathcal{D})^\ell} \le c\Big(\|\psi f\|_{L_\delta^0(\mathcal{D})^\ell} + \sum_\pm \|\psi g\|_{L_\delta^{1/2}(\Gamma^\pm)^\ell} + \|\psi u\|_{\mathcal{H}_\mathcal{D}}\Big) \tag{2.12}$$

Corollary 2.1 *Let* $\max(1-\mu_1, 0) < \delta < 1$. *Then for every* $u \in L_\delta^2(\mathcal{D})^\ell$ *the estimate*

$$\|u\|_{L_\delta^2(\mathcal{D})^\ell} \le c\Big(\|L(\partial_x)u\|_{L_\delta^0(\mathcal{D})^\ell} + \sum_\pm \|B(\partial_x)u\|_{L_\delta^{1/2}(\Gamma^\pm)^\ell}\Big)$$

is valid. Here the constant c *is independent of* u.

Proof: Due to Lemma 2.1, we may assume, without loss of generality, that $B(\partial_x)u = 0$. If the support of u is contained in the ball $|x| \le 1$, then by Lemma 2.2, we have

$$\|u\|_{L_\delta^2(\mathcal{D})^\ell} \le c\Big(\|L(\partial_x)u\|_{L_\delta^0(\mathcal{D})^\ell} + \|u\|_{L_0^1(\mathcal{D})^\ell}\Big). \tag{2.13}$$

Let $\operatorname{supp} u$ be contained in the ball $|x| < N$. Then the support of the function $v(x) = u(Nx)$ is contained in the unit ball $|x| \le 1$. Furthermore, $B(\partial_x)v = 0$ on $\Gamma^\pm$. Therefore, v satisfies (2.13). From this inequality, by means of the coordinate change $x = y/N$, one obtains

$$\|u\|_{L_\delta^2(\mathcal{D})^\ell} \le c\Big(\|L(\partial_x)u\|_{L_\delta^0(\mathcal{D})^\ell} + N^{\delta-1}\,\|u\|_{L_0^1(\mathcal{D})^\ell}\Big)$$

with the same constant c as in (2.13). The result follows. ■

The following theorem generalizes Lemma 2.2.

Theorem 2.1 *Let* ϕ, ψ *be the same functions as in Lemma* 2.2. *If* $u \in \mathcal{H}_\mathcal{D}$ *is a solution of* (2.4) *and the functional* F *has the form* (2.10), *where* $\psi\partial_{x_3}^j f \in L_\delta^0(\mathcal{D})^\ell$ *and* $\psi\partial_{x_3}^j g^\pm \in L_\delta^{1/2}(\Gamma^\pm)^\ell$ *for* $j = 0, \dots, k$, $\max(1-\mu_1, 0) < \delta < 1$, *then* $\phi\partial_{x_3}^j u \in L_\delta^2(\mathcal{D})^\ell$ *for* $j = 0, \dots, k$ *and*

$$\sum_{j=0}^k \|\phi\partial_{x_3}^j u\|_{L_\delta^2(\mathcal{D})^\ell} \le c\Big(\sum_{j=0}^k \|\psi\partial_{x_3}^j f\|_{L_\delta^0(\mathcal{D})^\ell} + \sum_{j=0}^k \sum_\pm \|\psi\partial_{x_3}^j g^\pm\|_{L_\delta^{1/2}(\Gamma^\pm)^\ell} + \|\psi u\|_{L_\delta^1(\mathcal{D})^\ell}\Big) \tag{2.14}$$

with a constant c *independent of* u.

Proof: We prove the theorem by induction in k. For $k=0$ the assertion follows from Lemma 2.2 and from the unique solvability of problem (2.4) in $\mathcal{H}_{\mathcal{D}}$. Suppose the theorem is proved for $k-1$. Then, under our assumptions on F, we have $\chi\partial_{x_3}^j u \in L^2_\delta(\mathcal{D})^\ell$ for $j=0,\dots,k-1$. Let $v=\partial_{x_3}^{k-1}u$. Then $\phi v \in L^2_\delta(\mathcal{D})^\ell$. We consider the vector function

$$v_h(x) = h^{-1}\,\big(v(x',x_3+h)-v(x',x_3)\big),$$

where h is a sufficiently small real number. Obviously, v_h is a solution of the problem $Lv_h=\Phi_h$ in $\mathcal{D}$, $Bv_h=\Psi_h^\pm$ on $\Gamma^\pm$, where $\Phi=\partial_{x_3}^{k-1}f$, $\Psi^\pm=\partial_{x_3}^{k-1}g^\pm$. Consequently,

$$\|\phi v_h\|_{L^2_\delta(\mathcal{D})^\ell} \le c\,\Big(\|\chi\Phi_h\|_{L^0_\delta(\mathcal{D})^\ell} + \sum_\pm \|\chi\Psi_h^\pm\|_{L^{1/2}_\delta(\Gamma^\pm)^\ell} + \|\chi v_h\|_{L^1_\delta(\mathcal{D})}\Big) \tag{2.15}$$

with a constant c independent of h. Here $\chi\Phi_h=(\chi\Phi)_h-\chi_h\Phi$ and, for sufficiently small $|h|$,

$$\begin{aligned}
\|(\chi\Phi)_h\|^2_{L^0_\delta(\mathcal{D})^\ell} &= \int_{\mathcal{D}} r^{2\delta}\,h^{-2}\,\big|(\chi\Phi)(x',x_3+h)-(\chi\Phi)(x',x_3)\big|^2\,dx\\
&= \int_{\mathcal{D}} r^{2\delta}\Big|\int_0^1 \frac{\partial(\chi\Phi)}{\partial x_3}(x',x_3+th)\,dt\Big|^2\,dx \le \int_{\mathcal{D}} r^{2\delta}\,\big|\partial_{x_3}(\chi(x)\Phi(x)\big|^2\,dx\\
&\le c\,\Big(\|\psi\partial_{x_3}^{k-1}f\|^2_{L^0_\delta(\mathcal{D})^\ell} + \|\psi\partial_{x_3}^{k}f\|^2_{L^0_\delta(\mathcal{D})^\ell}\Big),\\
\|\chi_h\Phi\|^2_{L^0_\delta(\mathcal{D})^\ell} &\le c\,\|\psi\partial_{x_3}^{k-1}f\|^2_{L^0_\delta(\mathcal{D})^\ell}
\end{aligned}$$

Analogously,

$$\|\chi\Psi_h^\pm\|_{L^{1/2}_\delta(\Gamma^\pm)^\ell} \le c\,\Big(\|\psi\partial_{x_3}^{k-1}g^\pm\|^2_{L^{1/2}_\delta(\Gamma^\pm)^\ell} + \|\psi\partial_{x_3}^{k}g^\pm\|^2_{L^{1/2}_\delta(\Gamma^\pm)^\ell}\Big).$$

For the proof of the last inequality one can use the equivalence of the norm in $L^{1/2}_\delta(\Gamma^\pm)$ with the norm (2.9). Furthermore,

$$\|\chi v_h\|_{L^1_\delta(\mathcal{D})} \le c\,\Big(\|\eta\partial_{x_3}^{k-1}u\|_{L^1_\delta(\mathcal{D})^\ell} + \|\eta\partial_{x_3}^{k}u\|_{L^1_\delta(\mathcal{D})^\ell}\Big), \tag{2.16}$$

where η is a smooth function such that $\eta=1$ in a neighborhood of $\operatorname{supp}\chi$ and $\psi=1$ in a neighborhood of $\operatorname{supp}\eta$. Since the theorem was assumed to be true for $k-1$, the right-hand side of (2.16) is majorized by the right-hand side of (2.14). Consequently, the limit (as $h\to 0$) of the left-hand side of (2.15) is majorized by the right-hand side of (2.14). This proves the theorem. ■

Lemma 2.3 *Let u be a solution of problem* (2.11) *such that $\psi u \in W^l_\delta(\mathcal{D})^\ell$, $\psi f \in W^{l+k-2}_{\delta+k}(\mathcal{D})^\ell$ and $\psi g^\pm \in W^{l+k-3/2}_{\delta+k}(\Gamma^\pm)^\ell$, $l\ge 1$, $\delta>-1$. Here ϕ, ψ are the same functions as in Lemma* 2.2. *Then $\phi u \in W^{l+k}_{\delta+k}(\mathcal{D})^\ell$ and*

$$\|\phi u\|_{W^{l+k}_{\delta+k}(\mathcal{D})^\ell} \le c\,\Big(\|\psi f\|_{W^{l+k-2}_{\delta+k}(\mathcal{D})^\ell} + \sum_\pm \|\psi g^\pm\|_{W^{l+k-3/2}_{\delta+k}(\Gamma^\pm)^\ell} + \|\psi u\|_{W^l_\delta(\mathcal{D})^\ell}\Big). \tag{2.17}$$

Proof: By [21, Cor.2,Rem.2], the vector function $\phi u \in W^l_\delta(\mathcal{D})^\ell$ admits the representation $\phi u = v+w$, where $v\in V^l_\delta(\mathcal{D})^\ell$ and $w\in W^{l+k}_{\delta+k}(\mathcal{D})^\ell$. Let first $k=1$. Then $Lv=\phi f+[L,\phi]u-Lw \in W^{l-1}_{\delta+1}(\mathcal{D})^\ell \cap V^{l-2}_\delta(\mathcal{D})^\ell \subset V^{l-1}_{\delta+1}(\mathcal{D})^\ell$ (here $[L,\phi]=L\phi-\phi L$ denotes the commutator of L and ϕ) and, analogously, $Bv=\phi g^\pm+[B,\phi]u-Bw \in V^{l-1/2}_{\delta+1}(\Gamma^\pm)^\ell$. Using [9, Th.10.2], we obtain $v\in V^{l+1}_{\delta+1}(\mathcal{D})^\ell$ and, therefore, $\phi u\in W^{l+1}_{\delta+1}(\mathcal{D})^\ell$. This proves the lemma for $k=1$. Repeating this argument, we obtain the assertion for $k\ge 2$. ■

2.4. Higher regularity of the solution to the Neumann problem

We improve the results of the previous subsection for the case $\mu_1 > 1$. Let us consider first the Neumann problem in the plane angle K.

Lemma 2.4 *Let the integer $k > 0$ be not an eigenvalue of the pencil $A(\lambda)$. Then for arbitrary homogeneous polynomials p_{k-2}, $q^\pm_{k-1}$ of degrees $k-2$ and $k-1$, respectively ($p_{k-2} = 0$ if $k = 1$) there exists a homogeneous polynomial p_k of degree k such that*

$$L(\partial_{x'}, 0)\, p_k = p_{k-2} \ \text{in } K, \quad B(\partial_{x'}, 0)\, p_k = q^\pm_{k-1} \ \text{on } \gamma^\pm. \tag{2.18}$$

Proof: Let $p_{k-2} = \sum_{j=0}^{k-2} b_j\, x_1^j\, x_2^{k-2-j}$ and $q^\pm_{k-1}\big|_{\gamma^\pm} = c^\pm\, r^{k-1}$ with $b_j, c^\pm \in \mathbb{C}^\ell$ be given. Inserting

$$p_k = \sum_{j=0}^{k} a_j\, x_1^j\, x_2^{k-j} \tag{2.19}$$

into (2.18) and comparing the coefficients of $x_1^j x_2^{k-2-j}$ and r^{k-1}, respectively, we get a linear system of $k+1$ equations with $k+1$ unknowns $a_0, a_1, \ldots, a_k$. Since k is not an eigenvalue of the pencil $A(\lambda)$, the corresponding homogeneous system has only the trivial solution (see Remark 2.1). Therefore, there exists a unique polynomial (2.19) satisfying (2.18). ■

Lemma 2.5 *Let $u \in W^{l-1}_\delta(K)^\ell$ be a solution of the problem*

$$L(\partial_{x'}, 0)\, u = f \ \text{in } K, \quad B(\partial_{x'}, 0)\, u = g^\pm \ \text{on } \gamma^\pm \tag{2.20}$$

with $f \in W^{l-2}_\delta(K)^\ell$, $g^\pm \in W^{l-3/2}_\delta(\gamma^\pm)^\ell$, $l \ge 2$, $0 < \delta < l-1$, δ not integer. Suppose that the strip $l-2-\delta \le \operatorname{Re}\lambda \le l-1-\delta$ does not contain eigenvalues of the pencil $A(\lambda)$. Then $u \in W^l_\delta(K)^\ell$ and

$$\|u\|_{W^l_\delta(K)^\ell} \le c\left(\|u\|_{W^{l-1}_\delta(K)^\ell} + \|f\|_{W^l_\delta(K)^\ell} + \sum_\pm \|g^\pm\|_{W^{l-3/2}_\delta(\gamma^\pm)^\ell}\right)$$

with a constant c independent of u.

Proof: Let $k = \langle l-1-\delta\rangle$ be the greatest integer less than $l-1-\delta$. The vector function u has continuous derivatives up to order $k-1$ at the point $x = 0$ (see [7, Le.7.1.3]). We denote by p_{k-1} the Taylor polynomial of degree $k-1$ of u and by ζ a smooth cut-off function equal to one near the origin and to zero outside the unit ball. Then $v = u - \zeta p_{k-1}$ belongs to $V^{l-1}_\delta(K)^\ell$ (see [7, Th.7.1.1]). Consequently,

$$\begin{aligned} L(\partial_{x'}, 0)\, v &= f - L(\partial_{x'}, 0)\,(\zeta p_{k-1}) \in W^{l-2}_\delta(K)^\ell \cap V^{l-3}_\delta(K)^\ell, \\ B(\partial_{x'}, 0)\, v\big|_{\gamma^\pm} &= g^\pm - B(\partial_{x'}, 0)\,(\zeta p_{k-1})\big|_{\gamma^\pm} \in W^{l-3/2}_\delta(K)^\ell \cap V^{l-5/2}_\delta(K)^\ell. \end{aligned}$$

By [7, Th.7.1.1], there are the representations

$$L(\partial_{x'}, 0)\, v = \zeta p^\circ_{k-2} + F \ \text{ in } K, \quad B(\partial_{x'}, 0)\, v = \zeta q^\pm_{k-1} + G^\pm \ \text{ on } \gamma^\pm,$$

where p°_{k-2}, $q^\pm_{k-1}$ are homogeneous polynomials of degrees $k-2$ and $k-1$, respectively, $F \in V^{l-2}_\delta(K)^\ell$, $G^\pm \in V^{l-3/2}_\delta(\gamma^\pm)^\ell$. By Lemma 2.4, there exists a homogeneous polynomial p°_k of degree k such that $L(\partial_{x'}, 0)p^\circ_k = p^\circ_{k-2}$ in K and $B(\partial_{x'}, 0)p^\circ_k = q^\pm_{k-1}$ on $\gamma^\pm$. Then $v - \zeta p^\circ_k \in V^{l-1}_\delta(K)^\ell$, $L(\partial_{x'}, 0)\,(v - \zeta p^\circ_k) \in V^{l-2}_\delta(K)^\ell$, $B(\partial_{x'}, 0)\,(v - \zeta p^\circ_k)\big|_{\gamma^\pm} \in V^{l-3/2}_\delta(\gamma^\pm)^\ell$. Applying [11] (in the case $p = 2$ see also [7, Th.6.1.4]), we obtain $v - \zeta p^\circ_k \in V^l_\delta(K)^\ell$ and, therefore, $u \in W^l_\delta(K)^\ell$. Furthermore, the desired estimate holds. ■

We prove an analogous result for the problem in the dihedron $\mathcal{D}$.

Lemma 2.6 *Let u be a solution of problem (2.11), and let ϕ, ψ be smooth functions on $\overline{\mathcal{D}}$ with compact supports such that $\phi\psi = \phi$. Suppose that $\psi u \in W_\delta^{l-1}(\mathcal{D})^\ell$, $\psi\partial_{x_3} u \in W_\delta^{l-1}(\mathcal{D})^\ell$, $\psi f \in W_\delta^{l-2}(\mathcal{D})^\ell$, $\psi g^\pm \in W_\delta^{l-3/2}(\Gamma^\pm)^\ell$, $0 < \delta < l-1$, δ is not integer, and the strip $l-2-\delta \le \operatorname{Re}\lambda \le l-1-\delta$ does not contain eigenvalues of the pencil $A(\lambda)$. Then $\phi u \in W_\delta^l(\mathcal{D})^\ell$ and*

$$\|\phi u\|_{W_\delta^l(\mathcal{D})^\ell} \le c\Big(\sum_{j=0}^{1} \|\psi\partial_{x_3}^j u\|_{W_\delta^{l-1}(\mathcal{D})^\ell} + \|\psi f\|_{W_\delta^{l-2}(\mathcal{D})^\ell} + \sum_\pm \|\psi g^\pm\|_{W_\delta^{l-3/2}(\Gamma^\pm)^\ell}\Big). \qquad (2.21)$$

Here the constant c depends only on the C^l norm of ζ.

Proof: From the equation $L(\partial_{x'}, \partial_{x_3})u = f$ it follows that

$$L(\partial_{x'}, 0)\,(\phi u) = F, \quad \text{where } F = \phi f + \phi L_1 \partial_{x_3} u + [L(\partial_{x'}, 0), \phi]\, u.$$

Here $[L(\partial_{x'}, 0), \phi] = L(\partial_{x'}, 0)\,\phi - \phi L(\partial_{x'}, 0)$ is the commutator of $L(\partial_{x'}, 0)$ and ϕ, and L_1 is a first order differential operator with constant coefficients, $L_1\partial_{x_3} u = \big(L(\partial_{x'}, 0) - L(\partial_{x'}, \partial_{x_3})\big)u$. An analogous representation holds for $G^\pm = B(\partial_{x'}, 0)\,(\phi u)\big|_{\Gamma^\pm}$. For almost all x_3 we have $\phi(\cdot, x_3)\, u(\cdot, x_3) \in W_\delta^{k+1}(K)^\ell$. Furthermore, by the conditions of the lemma, $F(\cdot, x_3) \in W_\delta^{l-2}(K)^\ell$ and $G^\pm(\cdot, x_3) \in W_\delta^{l-3/2}(\gamma^\pm)^3$. Consequently, by Lemma 2.5, we obtain $\phi(\cdot, x_3)\, u(\cdot, x_3) \in W_\delta^l(K)^\ell$ and

$$\int_{\mathbb{R}} \|\phi(\cdot, x_3)\, u(\cdot, x_3)\|^2_{W_\delta^l(K)^\ell}\, dx_3 \;\le\; c \int_{\mathbb{R}} \Big(\|\phi(\cdot, x_3)\, u(\cdot, x_3)\|^2_{W_\delta^{l-1}(K)^\ell} + \|F(\cdot, x_3)\|^2_{W_\delta^{l-2}(K)^\ell} + \sum_\pm \|G^\pm(\cdot, x_3\|^2_{W_\delta^{l-3/2}(\gamma^\pm)^\ell}\Big)\, dx_3.$$

Here the right-hand side of the last inequality can be estimated by the right-hand side of (2.21). This together with the assumption that $\psi\partial_{x_3} u \in W_\delta^{l-1}(\mathcal{D})^\ell$ implies the assertion of the lemma. ■

Theorem 2.2 *Let $u \in \mathcal{H}_{\mathcal{D}}$ be a solution of problem (2.11), and let ϕ, ψ be smooth functions on $\overline{\mathcal{D}}$ with compact supports such that $\psi = 1$ in a neighborhood of* $\operatorname{supp}\phi$. *We suppose that $\psi f \in W_\delta^{l-2}(\mathcal{D})^\ell$, $\psi g^\pm \in W_\delta^{l-3/2}(\Gamma^\pm)$, $l \ge 2$, δ is not integer, and* $\max(l-1-\mu_1, 0) < \delta < l-1$. *Then $\phi u \in W_\delta^l(\mathcal{D})^\ell$.*

Proof: We prove the theorem by induction in $\langle l-1-\delta\rangle$. Here $\langle s\rangle$ denotes the greatest integer less than s.

1) If $\langle l-1-\delta\rangle = 0$, then $\max(1-\mu_1, 0) < \delta-l+2 < 1$, $\psi f \in W^0_{\delta-l+2}(\mathcal{D})^\ell$, $\psi g^\pm \in W^{1/2}_{\delta-l+2}(\Gamma^\pm)^\ell$. Consequently, according to Theorem 2.1, we have $\chi u \in W^2_{\delta-l+2}(\mathcal{D})^\ell$, where χ is a smooth function equal to one near $\operatorname{supp}\phi$ such that $\psi = 1$ near $\operatorname{supp}\chi$. Applying Lemma 2.3, we obtain $\phi u \in W_\delta^l(\mathcal{D})^\ell$.

2) Let $\langle l-1-\delta\rangle = 1$. Then $\max(2-\mu_1, 0) < \delta-l+3 < 1 < 1$ and, by means of Theorem 2.1, we obtain $\chi\partial_{x_3}^j u \in W^2_{\delta-l+3}(\mathcal{D})^\ell$ for $j = 0, 1$. Consequently, it follows from Lemmas 2.6 and 2.3 that $\phi u \in W_\delta^l(\mathcal{D})^\ell$.

3) Let $k < l-\delta-1 < k+1$, where k is an integer, $k \ge 2$. We assume that the theorem is proved for $l-\delta-1 < k$. Then, by the induction hypothesis, $\chi u \in W_\delta^{l-1}(\mathcal{D})^\ell$, $l \ge 4$, and $\chi\partial_{x_3} u \in W_\delta^{l-2}(\mathcal{D})^\ell \subset \mathcal{H}_{\mathcal{D}}$. Since $\psi\partial_{x_3} f \in W_\delta^{l-3}(\mathcal{D})$ and $\psi\partial_{x_3} g^\pm \in W_\delta^{l-5/2}(\Gamma^\pm)^\ell$, we obtain, by the induction hypothesis, that $\chi\partial_{x_3} u \in W_\delta^{l-1}(\mathcal{D})^\ell$. By the assumptions of the lemma, there are no eigenvalues of the pencil $A(\lambda)$ in the strip $0 < \operatorname{Re}\lambda \le l-\delta-1$. Thus, Lemma 2.6 implies $\phi u \in W_\delta^l(\mathcal{D})^\ell$. The proof is complete. ■

Corollary 2.2 *Let $u \in \mathcal{H}_\mathcal{D}$ be a solution of problem (2.11), where $\psi\partial_{x_3}^j f \in W_\delta^{l-2}(\mathcal{D})^\ell$ and $\psi\partial_{x_3}^j g^\pm \in W_\delta^{l-3/2}(\Gamma^\pm)^\ell$ for $j = 0, \ldots, k$, δ is not integer,* $\max(l-1-\mu_1, 0) < \delta < l-1$. *Then $\phi\partial_{x_3}^j u \in W_\delta^l(\mathcal{D})^\ell$ and*

$$\sum_{j=0}^k \|\phi\partial_{x_3}^j u\|_{W_\delta^l(\mathcal{D})^\ell} \le c\Big(\sum_{j=0}^k \|\psi\partial_{x_3}^j f\|_{W_\delta^{l-2}(\mathcal{D})^\ell} + \sum_{j=0}^k\sum_\pm \|\psi\partial_{x_3}^j g^\pm\|_{W_\delta^{l-3/2}(\Gamma^\pm)^\ell} + \|\psi u\|_{L_\delta^1(\mathcal{D})^\ell}\Big).$$

Proof: Let first $l-1-\delta < 1$. Then $\max(1-\mu_1, 0) < \delta - l + 2 < 1$, $W_\delta^{l-2}(\mathcal{K}) \subset W_{\delta-l+2}^0(\mathcal{K})$ and $W_\delta^{l-3/2}(\Gamma^\pm) \subset W_{\delta-l+2}^{1/2}(\Gamma^\pm)$. Consequently, by Theorem 2.1, we have $\chi\partial_{x_3}^j u \in W_{\delta-l+2}^2(\mathcal{K})^\ell$, where χ is a smooth function such that $\chi = 1$ in a neighborhood of $\operatorname{supp}\phi$ and $\psi = 1$ in a neighborhood of $\operatorname{supp}\chi$. Applying Lemma 2.3, we get $\phi\partial_{x_3}^j u \in W_\delta^l(\mathcal{K})^\ell$ for $j = 0, \ldots, k$.

Now let $l-1-\delta > 1$. Then $l \ge 3$ and, by Theorem 2.2, we obtain $\chi u \in W_\delta^l(\mathcal{K})^\ell$, $\chi\partial_{x_3} u \in W_\delta^{l-1}(\mathcal{K})^\ell \subset \mathcal{H}_\mathcal{D}$. Since $\psi\partial_{x_3} f \in W_\delta^{l-2}(\mathcal{D})^\ell$ and $\psi\partial_{x_3} g^\pm \in W_\delta^{l-3/2}(\Gamma^\pm)^\ell$, we conclude again from Theorem 2.2 that $\phi\partial_{x_3} u \in W_\delta^l(\mathcal{K})^\ell$. Repeating this argument, we get $\phi\partial_{x_3}^j u \in W_\delta^l(\mathcal{K})^\ell$ for j=2,…,k. ■

Example. We consider the Neumann problem

$$-\Delta u = f \ \text{ in } \mathcal{D}, \quad \frac{\partial u}{\partial \nu} = g^\pm \ \text{ on } \Gamma^\pm. \tag{2.22}$$

Here the eigenvalues of the corresponding operator pencil $A(\lambda)$ are the numbers $\lambda_j = j\pi/\theta$, $j = 0, \pm1, \pm2, \ldots$. Consequently, the assertion of Theorem 2.2 with $\mu_1 = \pi/\theta$ holds.

2.5. The Neumann problem to the Lamé system

We consider a special case, where $\lambda = 1$ is an eigenvalue of the pencil A and the eigenfunctions corresponding to this eigenvalue are restrictions of linear functions to the unit circle. A necessary and sufficient condition for this case is given in the following lemma.

Lemma 2.7 *Let $\omega \ne \pi$, $\omega \ne 2\pi$. Then the homogeneous boundary value problem*

$$L(\partial_{x'}, 0)\, u = 0 \ \text{ in } K, \quad B(\partial_{x'}, 0)\, u = 0 \ \text{ on } \gamma^\pm$$

has a solution of the form $u = c\, x_1 + d\, x_2$, $c, d \in \mathbb{C}^\ell$, if and only if the $2\ell \times 2\ell$ matrix

$$A' = \begin{pmatrix} A_{1,1} & A_{2,1} \\ A_{1,2} & A_{2,2} \end{pmatrix}$$

is not invertible.

Proof: The linear function $u = c\, x_1 + d\, x_2$ satisfies the homogeneous boundary conditions $B(\partial_{x'}, 0)\, u = 0$ on $\gamma^\pm$ if and only if

$$\begin{pmatrix} n_1^+ & n_2^+ \\ n_1^- & n_2^- \end{pmatrix} \begin{pmatrix} A_{1,1} & A_{2,1} \\ A_{1,2} & A_{2,2} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 0.$$

Here the first matrix is invertible for $\omega \ne \pi$, $\omega \ne 2\pi$. This proves the lemma. ■

Let r' denote the rank of the matrix A'. From the proof of the last lemma it follows that there are $2\ell - r'$ linearly independent eigenvectors of the form $c\cos\varphi + d\sin\varphi$ corresponding to the eigenvalue $\lambda = 1$. Furthermore, the inhomogeneous boundary conditions $B(\partial_{x'}, 0)u = g^\pm$ on $\gamma^\pm$ can be satisfied for a vector function $u \in W_\delta^3(K)^\ell$ only if g^+ and g^- satisfy $2\ell - r'$ compatibility conditions at $x = 0$.

Such compatibility conditions must be also satisfied, in general, for the boundary data of the Neumann problem in the dihedron $\mathcal{D}$. If $u \in W^3_\delta(\mathcal{D})^\ell$, $0<\delta<1$, then the restriction of $B(\partial_x)u$ to the edge M belongs to the space $W_2^{1-\delta}(M)^\ell$ (see, e.g., [14], [21]), and we obtain

$$(A_{1,1}n_1^\pm + A_{1,2}n_2^\pm)\,\partial_{x_1}u|_M + (A_{2,1}n_1^\pm + A_{2,2}n_2^\pm)\,\partial_{x_2}u|_M + (A_{3,1}n_1^\pm + A_{3,2}n_2^\pm)\,\partial_{x_3}u|_M = g^\pm|_M.$$

The last system can be written in the form

$$\begin{pmatrix} n_1^+ & n_2^+ \\ n_1^- & n_2^- \end{pmatrix} \begin{pmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{pmatrix} \begin{pmatrix} \partial_{x_1}u|_M \\ \partial_{x_2}u|_M \\ \partial_{x_3}u|_M \end{pmatrix} = \begin{pmatrix} g^+|_M \\ g^-|_M \end{pmatrix}.$$

From this it follows that $2\ell - r''$ compatibility conditions must be satisfied for g^+ and g^- on the edge M, where r'' is the rank of the matrix

$$A'' = \begin{pmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{pmatrix}.$$

This means, there exist $2\ell - r''$ constant vectors $c^{(k)}$ such that

$$c^{(k)} \cdot \big(g^+|_M, g^-|_M\big) = 0 \quad \text{for } k = 1,\dots,2\ell - r''. \tag{2.23}$$

We suppose that $r' = r''$. Then there are the same compatibility conditions for the Neumann problem (2.11) in the dihedron and the corresponding Neumann problem (2.20) in the angle K. This condition is satisfied, e.g., for the Neumann problem in isotropic and anisotropic elasticity. Furthermore, we assume that the geometric and algebraic multiplicity of the eigenvalue $\lambda = 1$ is equal to $2\ell - r'$. This means that all eigenvectors corresponding to this eigenvalue have the form $c\cos\varphi + d\sin\varphi$ and that there are no generalized eigenvectors corresponding to this eigenvalue.

Lemma 2.8 *Suppose that there are no eigenvalues of the pencil $A(\lambda)$ in the strip $0 < \operatorname{Re}\lambda < 1$ and the line $\operatorname{Re}\lambda = 1$ contains the single eigenvalue $\lambda = 1$ having geometric and algebraic multiplicity $2\ell - r' = 2\ell - r''$. Denote by λ_2 the eigenvalue with smallest real part greater than 1 and by μ_2 its real part. Furthermore, let ϕ, ψ be the same functions as in Theorem 2.1 and let $u \in \mathcal{H}_\mathcal{D}$ be a solution of problem (2.11), where $\psi f \in W^1_\delta(\mathcal{D})^\ell$, $\psi g^\pm \in W^{3/2}_\delta(\Gamma^\pm)^\ell$, $\max(2-\mu_2, 0) < \delta < 1$, and g^+ and g^- satisfy the compatibility condition (2.23). Then $\phi u \in W^3_\delta(\mathcal{D})^\ell$ and*

$$\|\phi u\|_{W^3_\delta(\mathcal{D})^\ell} \le c\Big(\|\psi f\|_{W^1_\delta(\mathcal{D})^\ell} + \sum_\pm \|\psi g\|_{W^{3/2}_\delta(\Gamma^\pm)^\ell} + \|\psi u\|_{L^1_\delta(\mathcal{D})^\ell}\Big) \tag{2.24}$$

with a constant c independent of u.

Proof: Let χ be a smooth function on $\overline{\mathcal{D}}$ such that $\chi\phi = \phi$ and $\chi\psi = \chi$. From Theorem 2.1 it follows that $\chi u \in W^2_\delta(\mathcal{D})^\ell$ and $\chi\partial_{x_3}u \in W^2_\delta(\mathcal{D})^\ell$. Consequently, for almost all x_3 we have

$$\begin{aligned} L(\partial_{x'},0)\,u(\cdot,x_3) &= f(\cdot,x_3) - \big(L(\partial_{x'},\partial_{x_3}) - L(\partial_{x'},0)\big)\,u(\cdot,x_3) = F(\cdot,x_3), \\ B(\partial_{x'},0)\,u(\cdot,x_3) &= g^\pm(\cdot,x_3) - \sum_{j=1}^2 A_{3,j}n_j^\pm\,\partial_{x_3}u(\cdot,x_3) = G^\pm(\cdot,x_3), \end{aligned}$$

where $\chi(\cdot,x_3)F(\cdot,x_3) \in W^1_\delta(K)^\ell$, $\chi(\cdot,x_3)G^\pm(\cdot,x_3) \in W^{3/2}_\delta(\gamma^\pm)^\ell$. Since $r' = r''$ and g^+, g^- satisfy the compatibility condition (2.23), there exist vectors $c(x_3), d(x_3) \in \mathbb{C}^\ell$ such that

$$\begin{pmatrix} n_1^+ & n_2^+ \\ n_1^- & n_2^- \end{pmatrix} \begin{pmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{pmatrix} \begin{pmatrix} c(x_3) \\ d(x_3) \\ (\partial_{x_3}u)(0,x_3) \end{pmatrix} = \begin{pmatrix} g^+(0,x_3) \\ g^-(0,x_3) \end{pmatrix}$$

for all x_3. From this it follows that $p(x) = c(x_3)\, x_1 + d(x_3)\, x_2$ satisfies

$$B(\partial'_x, 0) w(\cdot, x_3) = G^{\pm}(0, x_3) \text{ on } \gamma^{\pm}$$

for all x_3. Therefore, for $v = u - p$ we obtain

$$L(\partial_{x'}, 0)\, v(\cdot, x_3) = F(\cdot, x_3) \text{ in K}, \quad B(\partial_{x'}, 0)\, v(\cdot, x_3) = G^{\pm}(\cdot, x_3) - G^{\pm}(0, x_3) \text{ on } \gamma^{\pm}.$$

Here, according to [7, Th.7.1.1], $\chi(\cdot, x_3)\, \big(G^{\pm}(\cdot, x_3) - G^{\pm}(0, x_3)\big) \in V_\delta^{3/2}(\gamma^{\pm})$. By the assumptions of the theorem, $\lambda = 1$ is the only eigenvalues of the pencil A in the strip $0 < \operatorname{Re}\lambda \le 2 - \delta$, all eigenfunctions are restrictions of linear functions to the unit circle, and generalized eigenfunctions corresponding to the eigenvalue $\lambda = 1$ do not exist. Thus, by [6, Th.1.2] (see also [7, Th.6.1.4]), ϕv admits the representation

$$\phi(x) v(x) = c^{(0)}(x_3) + c^{(1)}(x_3)\, x_1 + c^{(2)}(x_3)\, x_2 + w(x),$$

where $w(\cdot, x_3) \in V_\delta^3(K)^\ell$ and

$$\begin{aligned} \|w(\cdot, x_3)\|^2_{V_\delta^3(K)^\ell} &\le c\Big(\|\phi(\cdot, x_3)\, F(\cdot, x_3)\|^2_{V_\delta^1(K)^\ell} + \sum_{\pm} \|\phi(\cdot, x_3)\, G^{\pm}(\cdot, x_3)\|^2_{V_\delta^{3/2}(\gamma^{\pm})^\ell} \\ &\qquad + \|\chi(\cdot, x_3)\, u(\cdot, x_3)\|^2_{W_\delta^2(K)^\ell}\Big) \\ &\le c\Big(\|\chi(\cdot, x_3)\, f(\cdot, x_3)\|^2_{V_\delta^1(K)^\ell} + \sum_{\pm} \|\chi(\cdot, x_3)\, g^{\pm}(\cdot, x_3)\|^2_{V_\delta^{3/2}(\gamma^{\pm})^\ell} \\ &\qquad + \|\chi(\cdot, x_3)\, u(\cdot, x_3)\|^2_{W_\delta^2(K)^\ell} + \|\chi(\cdot, x_3)\, \partial_{x_3} u(\cdot, x_3)\|^2_{W_\delta^2(K)^\ell}\Big) \end{aligned}$$

with a constant c independent of x_3. Since $\partial^\alpha_{x'}(\phi u) = \partial^\alpha_{x'} w + \partial^\alpha_{x'}(\phi p)$ for $|\alpha| = 3$, the last estimate implies

$$\begin{aligned} \|\phi(\cdot, x_3) u(\cdot, x_3)\|^2_{L_\delta^3(K)^\ell} &\le c\Big(\|\chi(\cdot, x_3)\, f(\cdot, x_3)\|^2_{V_\delta^1(K)^\ell} + \sum_{\pm} \|\chi(\cdot, x_3)\, g^{\pm}(\cdot, x_3)\|^2_{V_\delta^{3/2}(\gamma^{\pm})^\ell} \\ &\qquad + \|\chi(\cdot, x_3)\, u(\cdot, x_3)\|^2_{W_\delta^2(K)^\ell} + \|\chi(\cdot, x_3)\, \partial_{x_3} u(\cdot, x_3)\|^2_{W_\delta^2(K)^\ell}\Big). \end{aligned}$$

Integrating this inequality with respect to x_3 and using (2.14), we obtain (2.24). The lemma is proved. ∎

Now, analogously to Theorem 2.2, the following statement holds.

Theorem 2.3 *Suppose that there are no eigenvalues of the pencil $A(\lambda)$ in the strip $0 < \operatorname{Re}\lambda < 1$ and the line $\operatorname{Re}\lambda = 1$ contains the single eigenvalue $\lambda = 1$ having geometric and algebraic multiplicity $2\ell - r' = 2\ell - r''$. Furthermore, we assume that $u \in \mathcal{H}_{\mathcal{D}}$ is a solution of problem* (2.11), *where $\psi f \in W_\delta^{l-2}(\mathcal{D})^\ell$, $\psi g^{\pm} \in W_\delta^{l-3/2}(\Gamma^{\pm})^\ell$, $l \ge 2$, $\max(l - 1 - \mu_2, 0) < \delta < l - 1$, and g^+, g^- satisfy the compatibility condition* (2.23). *Then $\phi u \in W_\delta^l(\mathcal{D})^\ell$ and*

$$\|\phi u\|_{W_\delta^l(\mathcal{D})^\ell} \le c\Big(\|\psi f\|_{W_\delta^{l-2}(\mathcal{D})^\ell} + \sum_{\pm} \|\psi g\|_{L_\delta^{l-3/2}(\Gamma^{\pm})^\ell} + \|\psi u\|_{L_\delta^1(\mathcal{D})^\ell}\Big) \qquad (2.25)$$

with a constant c independent of u.

Proof: If $0 < l - \delta - 1 < 1$, then the results holds in the same way as in the first step of the proof of Theorem 2.2.

Suppose that $1 < l - \delta - 1 < 2$. Then $\max(2 - \mu_2, 0) < \delta - l + 3 < 1$, and Lemma 2.8 implies $\chi u \in W^3_{\delta - l + 3}(\mathcal{D})^\ell$, where χ is a smooth function such that $\chi = 1$ in a neighborhood of $\operatorname{supp}\phi$ and $\psi = 1$ in a neighborhood of $\operatorname{supp}\chi$. Applying Lemma 2.3, we obtain $\phi u \in W_\delta^l(\mathcal{D})^\ell$.

The proof for the case $k < l-\delta-1 < k+1$, where k is an integer, $k \geq 2$, proceeds analogously to the third step in the proof of Theorem 2.2. ■

Moreover, the assertion of Corollary 2.2 with μ_2 instead of μ_1 is valid.

Example. We consider the Neumann problem for the Lamé system

$$\Delta u + \frac{1}{1-2\nu}\nabla\nabla\cdot u = f \ \text{ in } \mathcal{D}, \quad \sigma(u)\, n = g^\pm \ \text{ on } \Gamma^\pm. \tag{2.26}$$

Here $\sigma(u) = \{\sigma_{i,j}(u)\}$ is the stress tensor connected with the strain tensor

$$\Big\{\varepsilon_{i,j}(u)\Big\} = \Big\{\frac{1}{2}\,(\partial_{x_j} u_i + \partial_{x_i} u_j)\Big\}$$

by the Hooke law

$$\sigma_{i,j}(u) = 2\mu\,\Big(\frac{\nu}{1-2\nu}(\varepsilon_{1,1}+\varepsilon_{2,2}+\varepsilon_{3,3})\,\delta_{i,j} + \varepsilon_{i,j}\Big)$$

(μ is the shear modulus, ν is the Poisson ratio, $\nu < 1/2$, and $\delta_{i,j}$ denotes the Kronecker symbol).
The corresponding problem (2.20) in the angle K is:

$$\Delta_{x'}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{1}{1-2\nu}\nabla_{x'}\nabla_{x'}\cdot\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \Delta_{x'} u_3 = f_3 \quad \text{in } K,$$
$$\sigma(u_1,u_2)\, n = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad \frac{\partial u_3}{\partial n} = g_3 \quad \text{on } \gamma^\pm.$$

If the opening θ of the angle K is greater than π, then the eigenvalue with smallest positive real part of the pencil $A(\lambda)$ is $\xi_+(\theta)/\theta$, where $\xi_+(\theta)$ is the smallest positive root of the equation (1.8). This is shown, e.g., in [8, Sect.4.2]. Note that $\xi_+(\theta) < \pi$ for $\pi < \theta < 2\pi$. If $\theta < \pi$, then the eigenvalues with smallest positive real parts are $\lambda_1 = 1$ and $\lambda_2 = \pi/\theta$. The eigenvalue λ_1 is simple, the corresponding eigenvector is $(\sin\varphi, -\cos\varphi)$.
Let $n^\pm$ be the exterior normal to $\Gamma^\pm$. If $u \in W^3_\delta(\mathcal{D})^3$, $\delta < 1$, then it follows from the Neumann boundary conditions that

$$\sigma(u)\, n^\pm\big|_M = g^\pm\big|_M$$

and consequently, $n^- \cdot \sigma\, n^+\big|_M = n^- \cdot g^+\big|_M$ and $n^+ \cdot \sigma\, n^-\big|_M = n^+ \cdot g^-\big|_M$. Here $a \cdot b$ denotes the scalar product in $\mathbb{R}^3$. Since σ is symmetric, we have $n^- \cdot \sigma\, n^+ = n^+ \cdot \sigma\, n^-$. Consequently, g^+ and g^- must satisfy the compatibility condition

$$n^- \cdot g^+ = n^+ \cdot g^- \ \text{ on } M.$$

Applying Theorem 2.3, we get the following result:

1) *Let* $u \in \mathcal{H}_{\mathcal{D}}$ *be a solution of problem* (2.26), *where* $\psi f \in W^0_\delta(\mathcal{D})^3$, $\psi g^\pm \in W^{1/2}_\delta(\Gamma^\pm)^3$, $0 < \delta < 1$ *for* $\theta < \pi$, $1 - \xi_+(\theta)/\theta < \delta < 1$ *for* $\theta > \pi$. *Then* $\phi u \in W^2_\delta(\mathcal{D})^3$.
2) *Let* $\theta < \pi$ *and let* $u \in \mathcal{H}_{\mathcal{D}}$ *be a solution of problem* (2.26), *where* $\psi f \in W^{l-2}_\delta(\mathcal{D})^3$, $l \geq 3$, $\psi g^\pm \in W^{l-3/2}_\delta(\Gamma^\pm)^3$, $n^-\cdot g^+|_M = n^+\cdot g^-|_M$, $\max(l-1-\pi/\theta, 0) < \delta < l-1$. *Then* $\phi u \in W^l_\delta(\mathcal{D})^3$.

In particular, ϕu belongs to the Sobolev space $W^2(\mathcal{D})^3$ if $\theta < \pi$, $f \in W^1_\delta(\mathcal{D})^3$, $g^\pm \in W^{3/2}_\delta(\Gamma^\pm)^3$, $\delta < 1$, $n^- \cdot g^+|_M = n^+ \cdot g^-|_M$.

2.6. Estimates for Green's matrix to the Neumann problem

From the unique solvability of the Neumann problem in $\mathcal{H}_\mathcal{D}$ and from classical results on fundamental solutions of elliptic boundary value problems in a half-space we obtain the following assertions (for the Laplace equation see [22]).

Theorem 2.4 1) *There exists a unique solution $G(x,\xi)$ of the boundary value problem*

$$L(\partial_x)\, G(x,\xi) = \delta(x-\xi)\, I_\ell \quad \text{for } x,\xi \in \mathcal{D}, \tag{2.27}$$

$$B(\partial_x)\, G(x,\xi) = 0 \quad \text{for } x \in \partial\mathcal{D}\backslash M,\ \xi \in \mathcal{D} \tag{2.28}$$

such the function $x \to \zeta(|\xi'|^{-1}|x'|)\, G(x,\xi)$ belongs to $\mathcal{H}_\mathcal{D}^\ell$ for arbitrary fixed $\xi = (\xi',\xi_3) \in \mathcal{D}$. Here I_ℓ is the $\ell\times\ell$ identity matrix and ζ is a smooth function on $(0,\infty)$ equal to zero in the interval $(3/4,3/2)$ and to one outside the interval $(1/2,2)$.

2) *The function $G(x,\xi)$ is infinitely differentiable with respect to $x,\xi \in \overline{\mathcal{D}}\backslash M$, $x \neq \xi$. For $|x-\xi| < \min(|x'|,|\xi'|)$ there is the estimate*

$$|\partial_x^\alpha \partial_\xi^\beta G(x,\xi)| \le c\, |x-\xi|^{-1-|\alpha|-|\beta|},$$

where c is independent of x and ξ.

3) *The function $G(x,\xi)$ is also the unique solution of the problem*

$$L(\partial_\xi)\overline{G(x,\xi)} = \delta(x-\xi)\, I_\ell \quad \text{for } x,\xi \in \mathcal{D},$$

$$B(\partial_\xi)\, \overline{G(x,\xi)} = 0 \quad \text{for } x \in \mathcal{D},\ \xi \in \partial\mathcal{D}\backslash M$$

such that the function $\xi \to \zeta(|\xi'|^{-1}|x'|)\, G(x,\xi)$ belongs to $\mathcal{H}_\mathcal{D}^\ell$ for arbitrary fixed $x \in \mathcal{D}$.

We establish now an estimate for the derivatives of Green's function $G(x,\xi)$ in the case $|x-\xi| \ge \min(|x'|,|\xi'|)$. For this we need the following lemma analogous to Lemma 2.2 in [10].

Lemma 2.9 *Let $\mathcal{B}$ be a ball with radius 1 and center x_0 such that $\mathrm{dist}(x_0,M) \le 4$. Furthermore, let ϕ, ψ be infinitely differentiable functions with supports in $\mathcal{B}$ such that $\psi = 1$ on $\mathrm{supp}\,\phi$. If $\psi u \in \mathcal{H}_\mathcal{D}$, $Lu = 0$ in $\mathcal{D}\cap\mathcal{B}$ and $Bu = 0$ on $(\partial\mathcal{D}\backslash M)\cap\mathcal{B}$, then*

$$\sup_{x\in\mathcal{D}} |x'|^{\max(|\alpha|-\mu_1+\varepsilon,0)} \left|\phi(x)\, \partial_{x'}^\alpha \partial_{x_3}^j u(x)\right| \le c\, \|\psi u\|_{\mathcal{H}_\mathcal{D}}\,, \tag{2.29}$$

where ε is an arbitrarily small positive number. The constant c in (2.29) is independent of u and x_0.

Proof: Let ε be such that $\mu_1 - \varepsilon \in (k,k+1)$. Then $\delta = k+1-\mu_1+\varepsilon \in (0,1)$. Furthermore, let χ be a function from $C_0^\infty(\mathcal{B})$ such that $\phi\chi = \phi$ and $\chi\psi = \chi$. From Theorems 2.1 and 2.2 it follows that $\partial_{x_3}^j(\chi u) \in W_\delta^{k+2}(\mathcal{D})^\ell$ for $j = 0,1,\ldots$ and

$$\|\chi\partial_{x_3}^j u\|_{W_\delta^{k+2}(\mathcal{D})^\ell} \le c\, \|\psi u\|_{\mathcal{H}_\mathcal{D}}\,.$$

Hence we have $\partial_{x'}^\alpha \partial_{x_3}^j(\chi u) \in W_\delta^2(\mathcal{D})^\ell$ for $|\alpha| \le k$. Since $W_\delta^2(K)$ is continuously imbedded into $C(\overline{K})$, we have

$$\sup_{x'\in K,\, x_3\in\mathbb{R}} \left|\partial_{x'}^\alpha \partial_{x_3}^j(\chi u)\right| \le c \sup_{x_3\in\mathbb{R}} \|\partial_{x'}^\alpha \partial_{x_3}^j(\chi u)(\cdot,x_3)\|_{W_\delta^2(K)^\ell}\,.$$

Using the continuity of the imbedding $W_2^1(M) \subset C(M)$, we obtain

$$\begin{aligned}\sup_{x_3\in\mathbb{R}} \|\partial_{x'}^\alpha \partial_{x_3}^j(\chi u)(\cdot,x_3)\|_{W_\delta^2(K)^\ell} &\le c\left(\|\partial_{x'}^\alpha \partial_{x_3}^j(\chi u)\|_{W_\delta^2(\mathcal{D})^\ell} + \|\partial_{x'}^\alpha \partial_{x_3}^{j+1}(\chi u)\|_{W_\delta^2(\mathcal{D})^\ell}\right)\\ &\le c\, \|\psi u\|_{\mathcal{H}_\mathcal{D}}\,.\end{aligned}$$

This proves (2.29) for $|\alpha| \le k$. Now let $|\alpha| \ge k+1$. By Theorems 2.1 and 2.2, we have $\partial_{x_3}^j(\chi u) \in W_{\delta-k+|\alpha|}^{|\alpha|+2}(\mathcal{D})^\ell$ and, therefore, $\partial_{x'}^\alpha \partial_{x_3}^j(\chi u) \in W_{\delta-k+|\alpha|}^2(\mathcal{D})^\ell \subset V_{\delta-k+|\alpha|}^2(\mathcal{D})^\ell$. Using Sobolev's lemma, it can be easily shown that

$$\sup_{x'\in K} |x'|^{\beta-k+1}\, |v(x')| \le c\, \|v\|_{V_\beta^k(K)} \quad \text{for arbitrary } v \in V_\beta^k(K),\ k \ge 2 \tag{2.30}$$

with a constant c independent of v and x'. Applying this inequality to $\partial_{x'}^\alpha \partial_{x_3}^j(\chi u)$, we obtain

$$\sup_{x'\in K,\, x_3\in\mathbb{R}} |x'|^{\delta-k+|\alpha|-1} \left|\partial_{x'}^\alpha \partial_{x_3}^j(\chi u)\right| \le c \sup_{x_3\in\mathbb{R}} \|\partial_{x'}^\alpha \partial_{x_3}^j(\chi u)(\cdot, x_3)\|_{W_{\delta-k+|\alpha|}^2(K)^\ell}.$$

Using again the continuity of the imbedding $W_2^1(M) \subset C(M)$, we arrive at (2.29). ■

Theorem 2.5 *For* $|x-\xi| \ge \min(|x'|, |\xi'|)$ *there is the estimate*

$$\begin{aligned} &\left|\partial_{x'}^\alpha \partial_{x_3}^j \partial_{\xi'}^\beta \partial_{\xi_3}^k G(x,\xi)\right| \\ &\le c\, |x-\xi|^{-1-|\alpha|-|\beta|-j-k} \Big(\frac{|x'|}{|x-\xi|}\Big)^{\min(0,\mu_1-|\alpha|-\varepsilon)} \Big(\frac{|\xi'|}{|x-\xi|}\Big)^{\min(0,\mu_1-|\beta|-\varepsilon)}, \end{aligned} \tag{2.31}$$

where ε *is an arbitrarily small positive number.*

Proof: Since $G(Tx, T\xi) = T^{-1}\, G(x,\xi)$, we may assume, without loss of generality, that $|x-\xi| = 2$. Then $\max(|x'|, |\xi'|) \le 4$. Let $\mathcal{B}_x$ and $\mathcal{B}_\xi$ be balls with centers x and ξ, respectively, and radius 1. Furthermore, let η and ψ be infinitely differentiable functions with supports in $\mathcal{B}_x$ and $\mathcal{B}_\xi$, respectively.

Applying Lemma 2.9 to the function $\partial_{x'}^\alpha \partial_{x_3}^j \overline{G(x,\cdot)}$, we obtain

$$|\xi'|^{\max(|\beta|-\mu_1+\varepsilon,0)} \left|\partial_{x'}^\alpha \partial_{x_3}^j \partial_{\xi'}^\beta \partial_{\xi_3}^k G(x,\xi)\right| \le c\, \|\psi(\cdot) \partial_{x'}^\alpha \partial_{x_3}^j G(x,\cdot)\|_{\mathcal{H}_\mathcal{D}}\,. \tag{2.32}$$

We consider the solution

$$u(x) = \big(\psi(\cdot)\, F(\cdot)\,,\, \overline{G(x,\cdot)}\big)_\mathcal{D}$$

of problem (2.4), where $F \in \mathcal{H}_\mathcal{D}^*$. Since ψF vanishes in the ball $\mathcal{B}_x$, we conclude from Lemma 2.9 that

$$|x'|^{\max(|\alpha|-\mu_1+\varepsilon,0)}\, |\partial_{x'}^\alpha \partial_{x_3}^j u(x)| \le c\, \|\eta u\|_{\mathcal{H}_\mathcal{D}}\,.$$

Consequently, the mapping

$$\mathcal{H}_\mathcal{D}^* \ni F \to |x'|^{\max(|\alpha|-\mu_1+\varepsilon,0)}\, \partial_{x'}^\alpha\, \partial_{x_3}^j u(x) = |x'|^{\max|\alpha|-\mu_1+\varepsilon,0)} \big(F(\cdot)\,,\, \overline{\psi(\cdot)\partial_{x'}^\alpha\, \partial_{x_3}^j G(x,\cdot)}\big)_\mathcal{D} \in \mathbb{C}$$

represents a linear and continuous functional on $\mathcal{H}_\mathcal{D}^*$ for arbitrary $x \in \mathcal{D}$. The norm of this functional is independent of x. This implies

$$|x'|^{\max(|\alpha|-\mu_1+\varepsilon,0)}\, \|\psi(\cdot)\, \partial_{x'}^\alpha\, \partial_{x_3}^j G(x,\cdot)\|_{\mathcal{H}_\mathcal{D}} \le c$$

what together with (2.32) yields the desired estimate. ■

Using Theorem 2.3 instead of Theorem 2.2 in the proof of Lemma 2.9, we obtain the following result.

Theorem 2.6 *Suppose that there are no eigenvalues of the pencil* $A(\lambda)$ *in the strip* $0 < \operatorname{Re}\lambda < 1$ *and the line* $\operatorname{Re}\lambda = 1$ *contains the single eigenvalue* $\lambda = 1$ *having geometric and algebraic multiplicity* $2\ell - r' = 2\ell - r''$ *(*r' *and* r'' *were defined in Section 2.5). Then* $G(x,\xi)$ *satisfies (2.31) with* μ_2 *instead of* $\mu_1 = 1$.

Examples. 1) Green's matrix of the Neumann problem (2.22) for the Laplace equation satisfies (2.31) with $\mu_1 = \pi/\theta$.

2) For $\theta > \pi$ Green matrix of the Neumann problem (2.26) for the Lamé system satisfies (2.31) with $\mu_1 = \xi_+(\theta)/\theta$. In the case $\theta < \pi$, the number μ_1 has to be replaced by π/θ.

2.7. Estimates for Green's matrices to the Dirichlet and mixed problems

We consider problem (2.1) for the case when the Dirichlet condition is given on at least one of the sides Γ^+, Γ^-, i.e., not both numbers d^+, d^- equal zero. Then $\mathcal{H}_\mathcal{D} \subset V_0^1(\mathcal{D})^\ell$. From Lax-Milgram's lemma and Lemma 2.1 it follows that the problem

$$b_\mathcal{D}(u,v) = (F,v)_\mathcal{D} \ \text{ for all } v \in \mathcal{H}_\mathcal{D}, \quad u = g^\pm \ \text{ on } \Gamma^\pm \ \text{ for } d^\pm = 1 \tag{2.33}$$

has a unique solution $u \in V_0^1(\mathcal{D})^\ell$ for arbitrary $F \in \mathcal{H}_\mathcal{D}^*$, $g^\pm \in V_0^{1/2}(\Gamma^\pm)^\ell$.

For the following theorem we refer to [9, Th.4.1,7.2] and [18, Ch.11,Prop.1.4].

Theorem 2.7 *Let $u \in V_0^1(\mathcal{D})^\ell$ be a solution of problem (2.33), where the functional F has the form*

$$(F,v)_\mathcal{D} = (f,v)_\mathcal{D} + \sum_\pm (1-d^\pm)\, \big(g^\pm, v\big)_{\Gamma^\pm}$$

with $\psi\partial_{x_3}^j f \in V_\delta^{l-2}(\mathcal{D})^\ell$ and $\psi\partial_{x_3}^j g^\pm \in V_\delta^{l+d^\pm-3/2}(\Gamma^\pm)^\ell$ for $j = 0,1,\dots,k$, $l-1-\mu_1 < \delta < l-1$. Here ϕ and ψ are the same cut-off functions as in Theorem 2.2. Then $\phi\partial_{x_3}^j u \in V_\delta^l(\mathcal{D})^\ell$ and

$$\sum_{j=0}^k \|\phi\partial_{x_3}^j u\|_{V_\delta^l(\mathcal{D})^\ell} \le c\Big(\sum_{j=0}^k \|\psi\partial_{x_3}^j f\|_{V_\delta^{l-2}(\mathcal{D})^\ell} + \sum_{j=0}^k \sum_\pm \|\psi\partial_{x_3}^j g^\pm\|_{V_\delta^{l+\delta^\pm-3/2}(\Gamma^\pm)^\ell} + \|\psi u\|_{V_0^1(\mathcal{D})^\ell}\Big).$$

Analogously to Theorem 2.4, there exists a unique solution $G(x,\xi)$ of the problem

$$\begin{aligned} &L(\partial_x)\, G(x,\xi) = \delta(x-\xi)\, I_\ell \text{ for } x,\xi \in \mathcal{D}, \\ &d^\pm G(x,\xi) + (1-d^\pm)\, B(\partial_x)\, G(x,\xi) = 0 \text{ for } x \in \Gamma^\pm,\ \xi \in \mathcal{D} \end{aligned}$$

such that the function $x \to \zeta\big(|\xi'|^{-1}|x'|\big)G(x,\xi)$ belongs to $\mathcal{H}_\mathcal{D}^\ell$ for arbitrary $\xi \in \mathcal{D}$ and for an arbitrary smooth function ζ on $(0,\infty)$ equal to zero in the interval $(3/4,3/2)$ and to one outside the interval $(1/2,2)$. We call the matrix-valued function $G(x,\xi)$ Green's matrix of problem (2.11). Using Theorem 2.7, one can prove the following estimates.

Theorem 2.8 *The matrix $G(x,\xi)$ satisfies the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta G(x,\xi)| \le c\, |x-\xi|^{-1-|\alpha|-|\beta|}$$

for $|x-\xi| < \min(|x'|,|\xi'|)$ and

$$\big|\partial_{x'}^\alpha \partial_{x_3}^j \partial_{\xi'}^\beta \partial_{\xi_3}^k G(x,\xi)\big| \le c\, |x-\xi|^{-1-|\alpha|-|\beta|-j-k} \Big(\frac{|x'|}{|x-\xi|}\Big)^{\mu_1-|\alpha|-\varepsilon} \Big(\frac{|\xi'|}{|x-\xi|}\Big)^{\mu_1-|\beta|-\varepsilon}$$

for $|x-\xi| > \min(|x'|,|\xi'|)$, where ε is an arbitrarily small positive number.

3. The parameter-dependent problem on a domain of the sphere

In this section we study the parameter-dependent boundary value problem (1.9). We prove that this problem is uniquely solvable in a certain class of weighted Sobolev spaces for all λ, except finitely many, in a double angle of the complex plane containing the imaginary axis. This result is essentially known. For a smooth domain Ω on the sphere (and Sobolev spaces without weight) it was proved by Agranovich, M. S. and Vishik, M. I. [1].

3.1. The parameter dependent Neumann problem in an angle

Let again $\mathcal{D}$ be the dihedron $K \times \mathbb{R}$, where K is the angle $\{x' = (x_1, x_2) : \varphi \in (0, \theta)\}$, and let $b_{\mathcal{D}}$ be the sesquilinear form (2.2). We denote by $\tilde{u}$ and $\tilde{v}$ the Fourier transforms with respect to x_3 of the vector-functions u and v. Then, by Parseval's equality, we have

$$b_{\mathcal{D}}(u, v) = \int_{\mathbb{R}} b_K\big(\tilde{u}(\cdot, \eta), \tilde{v}(\cdot, \eta); \eta\big)\, d\eta,$$

where

$$b_K(u, v; \eta) = \int_K \Big(\sum_{i,j=1}^{2} A_{i,j} \partial_{x_i} u \cdot \overline{\partial_{x_j} v} + i\eta \sum_{i=1}^{2} \big(A_{3,i} u \cdot \overline{\partial_{x_i} v} - A_{i,3} \partial_{x_i} u \cdot \overline{v} \big) + \eta^2 A_{3,3} u \cdot \overline{v} \Big)\, dx'.$$

We consider the variational problem

$$b_K(u, v; \eta) = (F, v)_K \quad \text{for all } v \in W_2^1(K)^\ell, \tag{3.1}$$

which corresponds to the parameter-depending Neumann problem

$$L(\partial_{x'}, i\eta)\, u = - \sum_{i,j=1}^{2} A_{i,j} \partial_{x_i} \partial_{x_j} u - i\eta \sum_{i=1}^{2} \big(A_{i,3} + A_{3,i} \big) \partial_{x_i} u + \eta^2 A_{3,3} u = f \quad \text{in } K, \tag{3.2}$$

$$B(\partial_{x'}, i\eta)\, u = \sum_{i,j=1}^{2} A_{i,j} \partial_{x_i} u\, n_j + i\eta \sum_{j=1}^{2} A_{3,j} u\, n_j = g^\pm \quad \text{on } \gamma^\pm, \tag{3.3}$$

where $\gamma^\pm$ are the sides of K.

Theorem 3.1 *The boundary value problem* (3.2), (3.3) *is uniquely solvable in* $W_\delta^2(K)^\ell$ *for arbitrary* $f \in W_\delta^0(K)^\ell$, $g^\pm \in W_\delta^{1/2}(\gamma^\pm)^\ell$, $\max(1 - \mu_1, 0) < \delta < 1$, $\eta \in \mathbb{R}$, $\eta \neq 0$. *The solution satisfies the inequality*

$$\sum_{j=0}^{2} |\eta|^{2-j} \|u\|_{L_\delta^j(K)^\ell} \le c \Big(\|f\|_{L_\delta^0(K)^\ell} + \sum_\pm \|g^\pm\|_{L_\delta^{1/2}(\gamma^\pm)^\ell} + |\eta|^{1/2} \sum_\pm \|r^\delta g^\pm\|_{L_2(\gamma^\pm)^\ell} \Big) \tag{3.4}$$

with a constant c *independent of* f, $g^\pm$ *and* η.

Proof: Let the functional F be given by

$$(F, v)_K = \int_K f \cdot \overline{v}\, dx' + \sum_\pm \int_{\gamma^\pm} g^\pm \cdot \overline{v}\, dr, \quad v \in W_0^1(K)^\ell,$$

where $f \in L_\delta^0(K)$, $g^\pm \in W_\delta^{1/2}(\gamma^\pm)$, $0 < \delta < 1$. It can be easily seen that this functional belongs to $W_0^1(K)^*$. We set $u(x) = N^{-1/2} e^{i\eta x_3}\, \phi(x_3/N)\, v(x')$, where $v \in W_2^1(K)^\ell$, and $\phi \in C_0^\infty(\mathbb{R})$ is a real-valued function such that $\int_{-\infty}^{+\infty} \phi(t)\, dt = 1$. Then

$$\int_{\mathcal{D}} \sum_{j=1}^{3} |\partial_{x_j} u|^2\, dx \ge \int_K \Big(|\nabla_{x'} v|^2 + \frac{|\eta|^2}{2} |v|^2 \Big)\, dx' - c\, N^{-2} \int_K |v|^2\, dx',$$

where c is independent of v and N. Analogously,

$$b_{\mathcal{D}}(u, u) \le b_K(v, v; \eta) + c\, N^{-2} \int_K |v|^2\, dx'.$$

Consequently, (2.3) yields

$$b_K(v,v;\eta) \ge c \int_K \left(|\nabla_{x'} v|^2 + \eta^2 \, |v|^2\right) dx'.$$

Thus, by Lax-Milgram's lemma, for all real $\eta \neq 0$ there exists a unique solution $u \in W_0^1(K)^\ell$ of problem (3.1) which is also a solution of problem (3.2), (3.3).

We show that $u \in L^2_\delta(K)^\ell$. Let χ be an arbitrary smooth cut-off function with compact support equal to one near the vertex of K. Then $\chi u \in V^1_\varepsilon(K)^\ell$ with an arbitrary positive ε and, therefore, also $\chi u \in V^2_{1+\varepsilon}(K)^\ell$ (see, e.g. [7, Le.6.3.1]). Furthermore, $L(\partial_{x'}, i\eta)(\chi u) \in V^0_\delta(K)^\ell$, $B(\partial_{x'}, i\eta)(\chi u) \in V^{1/2}_\delta(\gamma^\pm)^\ell$. Hence, according to [7, Th.6.4.1] and [8, Th.12.3.3], the vector-function χu has the asymptotics

$$\chi u = c + d \log r + w, \quad \text{where } c, d \in \mathbb{C}^\ell, \ w \in V^2_\delta(K)^\ell.$$

Since $u \in W^1_2(K)^\ell$, the vector d is equal to zero. This implies $\chi u \in L^2_\delta(K)^\ell$. We consider the vector-function $(1-\chi)u$. Obviously, $(1-\chi)u \in W^0_{\delta-2}(K)^\ell$, $L(\partial_{x'}, i\eta)\big((1-\chi)u\big) \in W^0_\delta(K)^\ell$, while $B(\partial_{x'}, i\eta)\big((1-\chi)u\big) \in V^{1/2}_\delta(\gamma^\pm)^\ell \cap W^{1/2}_\delta(\gamma^\pm)^\ell$. Consequently, by [9, Th.4.1'], we obtain $(1-\chi)u \in V^2_\delta(K)^\ell \cap W^2_\delta(K)^\ell$. Thus, we have shown that $u \in L^2_\delta(K)$.

Estimate (3.4) holds by applying the inequality of Corollary 2.1 to the vector function $v(x) = N^{-1/2} e^{i\eta x_3} \phi(N^{-1}x_3)\, u(x')$, where $\phi \in C_0^\infty(\mathbb{R})$ and N is a large number. ■

An analogous result holds for the parameter-dependent Dirichlet and mixed problems in the angle K. Here the spaces L^j_δ can be replaced by V^j_δ.

3.2. Solvability of problem (1.9)

Let $\mathcal{H}_\Omega = \{u \in W^1(\Omega)^\ell : u = 0 \text{ on } \gamma_j \text{ for } j \in J_0\}$. We introduce the parameter-dependent sesquilinear form

$$a(u,v;\lambda) = \frac{1}{\log 2} \int\limits_{\substack{\mathcal{K} \\ 1<|x|<2}} \sum_{i,j=1}^{3} A_{i,j} \partial_{x_i} U \cdot \partial_{x_j} \overline{V} \, dx,$$

where $U(x) = \rho^\lambda(\omega)$, $V(x) = \rho^{-1-\overline{\lambda}} v(\omega)$, and define the operator $\mathfrak{A}(\lambda) : \mathcal{H}_\Omega \to \mathcal{H}^*_\Omega$ by

$$\big(\mathfrak{A}(\lambda)u, v\big)_\Omega = a(u,v;\lambda), \quad u, v \in \mathcal{H}_\Omega \,.$$

The pencil $\mathfrak{A}$ has following properties (see [8, Ch.10,12]).

(i) The spectrum of the pencil $\mathfrak{A}$ consists of isolated points, the eigenvalues of this pencil. All eigenvalues have finite algebraic multiplicity.

(ii) If λ is an eigenvalue of the pencil $\mathfrak{A}$, then $-1-\overline{\lambda}$ is also an eigenvalue with the same geometric and algebraic multiplicity.

(iii) The vector function $u = r^{\lambda_0} \sum\limits_{k=0}^{s} \frac{1}{k!} (\log r)^k \, u_{s-k}(\omega)$ satisfies the equality $b_{\mathcal{K}}(u,v) = 0$ for all $v \in \mathcal{H}$ equal to zero in a neighborhood of the origin and infinity if and only if λ_0 is an eigenvalue of the pencil $\mathfrak{A}$ and $u^{(0)}, \dots, u^{(s)}$ is a Jordan chain corresponding to this eigenvalue.

We denote by $\tilde{J}$ the set all $j \in \{1, 2, \dots, n\}$ such that the Dirichlet condition in problem (1.1)–(1.3) is given on at least one side adjacent to the edge M_j, i.e. $M_j \subset \overline{\Gamma}_k$ for at least one

$k \in J_0$. Let $\vec{\delta} = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$, $\delta_j > -1$ for $j \notin \tilde{J}$. Then we define the norm in the weighted Sobolev space $\mathcal{W}^l_{\vec{\delta}}(\Omega; \tilde{J})$ by

$$\|u\|_{\mathcal{W}^l_{\vec{\delta}}(\Omega;\tilde{J})} = \Big(\int\limits_{\substack{\mathcal{K} \\ 1<|x|<2}} \sum_{|\alpha|\le l} \prod_{j\in\tilde{J}} r_j^{2(\delta_j - l + |\alpha|)} \prod_{j\notin\tilde{J}}^{n} r_j^{2\delta_j} \, |\partial_x^\alpha u(x)|^2 \, dx \Big)^{1/2},$$

where u is extended by the equality $u(x) = u(x/|x|)$ to the cone $\mathcal{K}$ and $r_j(x) = \text{dist}\,(x, M_j)$. Furthermore, we set $V^l_{\vec{\delta}}(\Omega) = \mathcal{W}^l_{\vec{\delta}}(\Omega; \{1, \dots, n\})$ and $W^l_{\vec{\delta}}(\Omega) = \mathcal{W}^l_{\vec{\delta}}(\Omega; \emptyset)$. From Hardy's inequality it follows that $\mathcal{W}^l_{\vec{\delta}}(\Omega; \tilde{J}) = V^l_{\vec{\delta}}(\Omega)$ if $\delta_j > l-1$ for $j \notin \tilde{J}$. Furthermore, $\mathcal{H}_\Omega \subset \mathcal{W}^1_{\vec{0}}(\Omega; \tilde{J})^\ell$. The trace spaces for $V^l_{\vec{\delta}}(\Omega)$, $W^l_{\vec{\delta}}(\Omega)$ and $\mathcal{W}^l_{\vec{\delta}}(\Omega; \tilde{J})$, $l \ge 1$, on the arc γ_j are denoted by $V^{l-1/2}_{\vec{\delta}}(\gamma_j)$, $W^{l-1/2}_{\vec{\delta}}(\gamma_j)$ and $\mathcal{W}^{l-1/2}_{\vec{\delta}}(\gamma_j; \tilde{J})$, respectively. In particular, $\mathcal{W}^{l-1/2}_{\vec{\delta}}(\gamma_j; \tilde{J}) = V^{l-1/2}_{\vec{\delta}}(\gamma_j)$ for $j \in J_0$.

Let $\mathcal{D}_j$ be the dihedron which coincides with $\mathcal{K}$ near the point $M_j \cap S^2$. The boundary value problem for the system (1.1) in $\mathcal{D}_j$ is connected with a pencil $A_j(\lambda)$ on an interval $(0, \theta_j)$, where θ_j is the interior angle at the edge M_j (see the definition of the pencil $A(\lambda)$ in Section 2). We denote by $\lambda_1^{(j)}$ the eigenvalue with smallest positive real part and set $\mu_j = \text{Re}\,\lambda_1^{(j)}$. Furthermore, let the operator $\mathfrak{A}_{\vec{\delta}}(\lambda)$ be defined as

$$\begin{aligned} \mathcal{W}^2_{\vec{\delta}}(\Omega; \tilde{J})^\ell \ni u \to \big(\mathcal{L}(\lambda)u\,,\ \{u|_{\gamma_j}\}_{j\in J_0}\,,\ \{\mathcal{B}(\lambda)u|_{\gamma_j}\}_{j\in J_1} \big) \\ \in W^0_{\vec{\delta}}(\Omega)^\ell \times \prod_{j\in J_0} \mathcal{W}^{3/2}_{\vec{\delta}}(\gamma_j; \tilde{J})^\ell \times \prod_{j\in J_1} \mathcal{W}^{1/2}_{\vec{\delta}}(\gamma_j; \tilde{J})^\ell, \end{aligned}$$

where $\mathcal{L}$ and $\mathcal{B}$ are given by (1.10).

Theorem 3.2 *Let* $1 - \mu_j < \delta_j < 1$ *for* $j \in \tilde{J}$ *and* $\max(1 - \mu_j, 0) < \delta_j < 1$ *for* $j \notin \tilde{J}$.

1) *Then the spectra of the pencils* $\mathfrak{A}$ *and* $\mathfrak{A}_{\vec{\delta}}$ *coincide.*

2) *There exist positive real constants* N *and* ϵ *such that for all* λ *in the set*

$$\big\{ \lambda \in \mathbb{C} \,:\ |\lambda| > N,\ |\text{Re}\,\lambda| < \epsilon\,|\text{Im}\,\lambda| \big\} \tag{3.5}$$

the operator $\mathfrak{A}_{\vec{\delta}}(\lambda)$ *is an isomorphism. Furthermore, every solution* $u \in W^2_{\vec{\delta}}(\Omega)^\ell$ *of the problem* (1.9) *with* λ *in the set* (3.5) *satisfies the inequality*

$$\begin{aligned} \sum_{j=0}^{2} |\lambda|^{2-j}\, \|u\|_{\mathcal{W}^j_{\vec{\delta}}(\Omega;\tilde{J})^\ell} \le c \Big(\|f\|_{V^0_{\vec{\delta}}(\Omega)^\ell} + \sum_{j\in J_0} \big(\|g\|_{V^{3/2}_{\vec{\delta}}(\gamma_j)^\ell} + |\lambda|^{3/2}\, \|g\|_{V^0_{\vec{\delta}}(\gamma_j)^\ell} \big) \\ + \sum_{j\in J_1} \big(\|g\|_{V^{1/2}_{\vec{\delta}}(\gamma_j)^\ell} + |\lambda|^{1/2}\, \|g\|_{V^0_{\vec{\delta}}(\gamma_j)^\ell} \big) \Big), \end{aligned} \tag{3.6}$$

where c *is independent of* u *and* λ.

Proof: 1) We consider the differential operators $\mathcal{L}(\lambda)$ and $\mathcal{B}(\lambda)$ in a neighborhood of $M_1 \cap S^2$. Without loss of generality, we may assume that M_1 coincides with the x_3-axis and $\mathcal{D}_1$ is the dihedron $K \times \mathbb{R}$. By $\omega_1 = x_1/\rho$, $\omega_2 = x_2/\rho$ we denote local coordinates on the unit sphere near the north pole $N = M_1 \cap S^2$. Since

$$\begin{aligned} &\partial_{x_1} = \omega_1 \partial_\rho + \frac{1-\omega_1^2}{\rho} \partial_{\omega_1} - \frac{\omega_1\omega_2}{\rho}\, \partial_{\omega_2}, \quad \partial_{x_2} = \omega_2 \partial_\rho - \frac{\omega_1\omega_2}{\rho} \partial_{\omega_1} + \frac{1-\omega_2^2}{\rho}\, \partial_{\omega_2}, \\ &\partial_{x_3} = (1 - \omega_1^2 - \omega_2^2)^{1/2}\, \big(\partial_\rho - \frac{\omega_1}{\rho}\, \partial_{\omega_1} - \frac{\omega_2}{\rho}\, \partial_{\omega_2} \big), \end{aligned}$$

the operator $\mathcal{L}(\lambda)$ has the form

$$\begin{aligned}\mathcal{L}(\lambda) &= -\sum_{i,j=1}^{2} A_{i,j}\partial_{\omega_i}\partial_{\omega_j} - (\lambda-1)\sum_{i=1}^{2}\big(A_{i,3}+A_{3,i}\big)\,\partial_{\omega_i} - \lambda\,(A_{1,1}+A_{2,2}) - \lambda(\lambda-1)\,A_{3,3} \\ &\quad +\lambda^2\mathcal{P}_0(\omega) + \lambda\,\mathcal{P}_1(\omega,\partial_\omega) + \mathcal{P}_2(\omega,\partial_\omega),\end{aligned}$$

where $\mathcal{P}_j$ are differential operators of order j with coefficients vanishing at the point $(\omega_1,\omega_2)=0$. Analogously,

$$\mathcal{B}(\lambda) = \sum_{i,j=1}^{2} A_{i,j}\, n_j\, \partial_{\omega_i} + \lambda \sum_{j=1}^{2} A_{3,j} n_j + \lambda\,\mathcal{Q}_0(\omega) + \mathcal{Q}_1(\omega,\partial_\omega)$$

near N, where $\mathcal{Q}_j$ are differential operators of order j with coefficients vanishing at $(\omega_1,\omega_2)=0$. Furthermore Ω coincides with the wedge K in the coordinate system ω_1,ω_2 near $M_1\cap S^2$. Hence we conclude, analogously to the proof of Theorem 3.1, that every weak solution $u\in W_2^1(\Omega)^\ell$ of problem (1.9) with support near N belongs to the space $\mathcal{W}^2_{\vec\delta}(\Omega;\tilde J)^\ell$ if $f\in V^0_{\vec\delta}(\Omega)^\ell$, $g_j\in V^{3/2}_{\vec\delta}(\gamma_j)^\ell$ for $j\in J_0$ and $g_j\in V^{1/2}_{\vec\delta}(\gamma_j)^\ell$ for $j\in J_1$. By means of a partition of unity on Ω, we obtain this result for arbitrary weak solutions. This implies, in particular, that every eigenfunction of the pencil $\mathfrak{A}$ is an eigenfunction of $\mathfrak{A}_{\vec\delta}$ corresponding to the same eigenvalue. The same is true for generalized eigenfunctions.

2) We prove the second assertion first for purely imaginary $\lambda=i\eta$. Let $\zeta_0,\zeta_1,\dots,\zeta_n$ be a partition of unity on Ω such that $\zeta_j=1$ near $M_j\cap S^2$ and $\operatorname{supp}\zeta_j$ is sufficiently small for $j=1,\dots,n$. We consider the vector-function $\zeta_1 u$ and assume, as above, that the edge M_1 coincides with the x_3-axis. The difference of the operator $\mathcal{L}(\lambda)$ (in the coordinates ω_1, ω_2 introduced above) and the operator (3.2) is small for large $|\lambda|$ and small $\omega_1^2+\omega_2^2$. This means, there is the inequality

$$\big\|\big(\mathcal{L}(\lambda) - L(\partial_{\omega_1},\partial_{\omega_2},\lambda)\big)\,(\zeta_1 u)\big\|_{L^0_{\delta_1}(K)^\ell} \le \varepsilon \sum_{j=0}^{2} |\lambda|^{2-j}\|\zeta_1 u\|_{L^j_{\delta_1}(K)^\ell}\,,$$

where ε is small if $\operatorname{supp}\zeta_1$ is small and $|\lambda|$ is large. The same is true for the difference of the operators $\mathcal{B}(\lambda)$ and (3.3). Hence in the case of the Neumann problem it follows from Theorem 3.1 that

$$\sum_{j=0}^{2-j} |\lambda|^{2-j}\|\zeta_1 u\|_{L^j_{\vec\delta}(\Omega)^\ell} \le c\,\Big(\|\mathcal{L}(\lambda)u\|_{W^0_{\vec\delta}(\Omega)^\ell} + \sum_{j=1}^{n}\big(\|\mathcal{B}(\lambda)u\|_{W^{1/2}_{\vec\delta}(\gamma_j)^\ell} + |\lambda|^{1/2}\,\|\mathcal{B}(\lambda)u\|_{W^0_{\vec\delta}(\gamma_j)^\ell}\big)\Big)$$

for sufficiently large $|\lambda|$. The same inequality is true for the vector-functions $\zeta_j u$, $j=1,\dots,n$. The validity of this inequality for $\zeta_0 u$ follows from a result of Agranovich and Vishik [1] (see also [7, Th.3.6.1]). An analogous estimate holds for the Dirichlet and mixed problems. This implies (3.6) for purely imaginary λ, $|\lambda|>N$. For λ in the set (3.5) this estimate can be proved in the same way as in [1, 7]. ■

4. The boundary value problem in a polyhedral cone

In the last section we consider problem (1.1)–(1.3) in the cone $\mathcal{K}$. We prove the existence of strong and weak solutions, obtain regularity assertions for the solutions and point estimates for Green's matrices. As in Section 2 we concentrate on the case of the Neumann problem. Analogous assertions for the Dirichlet and mixed problem are formulated at the end of the section and can by obtained by obvious modifications in the proofs. For the Dirichlet problem we refer also to the papers by Maz′ya and Plamenevskiĭ [13] (Lamé and Stokes systems), Maz′ya and Roßmann [15] (scalar $2m$ order elliptic equations) which include solvability theorems in weighted Sobolev and Hölder spaces and estimates for Green's functions. The solvability of the Neumann problem for diagonalizable second order equations in Sobolev spaces without weight was studied by Dauge [3, 4].

4.1. Weighted Sobolev space in $\mathcal{K}$

For an arbitrary point $x \in \mathcal{K}$ let $\rho(x) = |x|$ be the distance to the vertex of the cone and $r_j(x)$ the distance to the edge M_j. Furthermore, we denote by $r(x)$ the regularized distance to $\mathcal{S}$, i.e., an infinitely differentiable function in $\mathcal{K}$ which coincides with $\text{dist}(x, \mathcal{S})$ in a neighborhood of $\mathcal{S}$.

Let l be a nonnegative integer, $\tilde{J}$ the same subset of $\{1, 2, \dots, n\}$ as in Section 3, $\beta \in \mathbb{R}$, $\vec{\delta} = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$, $\delta_j > -1$ for $j \notin \tilde{J}$. By $\mathcal{W}^l_{\beta,\vec{\delta}}(\mathcal{K}; \tilde{J})$ we denote the weighted Sobolev space with the norm

$$\|u\|_{\mathcal{W}^l_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})} = \Big(\int_{\mathcal{K}} \sum_{|\alpha|\le l} \rho^{2(\beta-l+|\alpha|)} \prod_{j\in\tilde{J}} \big(\frac{r_j}{\rho}\big)^{2(\delta_j-l+|\alpha|)} \prod_{j\notin\tilde{J}} \big(\frac{r_j}{\rho}\big)^{2\delta_j} \, |\partial_x^\alpha u|^2 \, dx \Big)^{1/2}.$$

Furthermore, we define $V^l_{\beta,\vec{\delta}}(\mathcal{K}) = \mathcal{W}^l_{\beta,\vec{\delta}}(\mathcal{K}; \{1, \dots, n\})$ and $W^l_{\beta,\vec{\delta}}(\mathcal{K}) = \mathcal{W}^l_{\beta,\vec{\delta}}(\mathcal{K}; \emptyset)$. Passing to spherical coordinates ρ, ω, one obtains the following equivalent norm in $\mathcal{W}^l_{\beta,\vec{\delta}}(\mathcal{K}; \tilde{J})$:

$$\|u\| = \Big(\int_0^\infty \rho^{2(\beta-l+1)} \sum_{k=0}^{l} \|(\rho\partial_\rho)^k u(\rho,\cdot)\|^2_{\mathcal{W}^{l-k}_{\vec{\delta}}(\Omega;\tilde{J})} \, d\rho \Big)^{1/2}.$$

Lemma 4.1 *Let $\vec{\delta} = (\delta_1, \dots, \delta_n)$, $\vec{\delta}' = (\delta'_1, \dots, \delta'_n)$ be such that $\delta'_j - \delta_j \le 1$ for $j = 1, \dots, n$ and $\delta_j > -1$, $\delta'_j > -1$ for $j \notin \tilde{J}$. Then $\mathcal{W}^{l+1}_{\beta+1,\vec{\delta}'}(\mathcal{K}; \tilde{J})$ is continuously imbedded into $\mathcal{W}^l_{\beta,\vec{\delta}}(\mathcal{K}; \tilde{J})$.*

Proof: It suffices to note that, by Hardy's inequality, the space $\mathcal{W}^{l+1-k}_{\vec{\delta}'}(\Omega; \tilde{J})$ is continuously imbedded into $\mathcal{W}^{l-k}_{\vec{\delta}}(\Omega; \tilde{J})$, $k = 0, \dots, l$. ■

Obviously, $V^l_{\beta,\vec{\delta}}(\mathcal{K}) \subset \mathcal{W}^l_{\beta,\vec{\delta}}(\mathcal{K}; \tilde{J})$. If $\delta_j > l - 1$ for all $j \notin \tilde{J}$, then, according to Lemma 4.1, $V^l_{\beta,\vec{\delta}}(\mathcal{K}) = \mathcal{W}^l_{\beta,\vec{\delta}}(\mathcal{K}; \tilde{J})$.

We denote the trace spaces for $V^l_{\beta,\vec{\delta}}(\mathcal{K})$, $W^l_{\beta,\vec{\delta}}(\mathcal{K})$ and $\mathcal{W}^l_{\beta,\vec{\delta}}(\mathcal{K}; \tilde{J})$, $l \ge 1$, on Γ_j by $V^{l-1/2}_{\beta,\vec{\delta}}(\Gamma_j)$, $W^{l-1/2}_{\beta,\vec{\delta}}(\Gamma_j)$ and $\mathcal{W}^{l-1/2}_{\beta,\vec{\delta}}(\Gamma_j; \tilde{J})$, respectively. Using Lemma 2.1, we obtain the following assertion.

Lemma 4.2 *Let $g_j \in V^{l-1/2}_{\beta,\vec{\delta}}(\Gamma_j)^\ell$ for $j \in J_0$ and $g_j \in V^{l-3/2}_{\beta,\vec{\delta}}(\Gamma_j)^\ell$ for $j \in J_1$. Here $l \ge 2$ if $J_1 \ne \emptyset$ and $l \ge 1$ else. Then there exists a vector function $u \in V^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ such that $u = g_j$ on Γ_j for $j \in J_0$, $Bu = g_j$ on Γ_j for $j \in J_1$, and*

$$\|u\|_{V^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \le c \Big(\sum_{j\in J_0} \|g_j\|_{V^{l-1/2}_{\beta,\vec{\delta}}(\Gamma_j)^\ell} + \sum_{j\in J_1} \|g_j\|_{V^{l-3/2}_{\beta,\vec{\delta}}(\Gamma_j)^\ell} \Big) \tag{4.1}$$

with a constant c independent of g_j, $j = 1, \dots, n$.

Proof: Let ζ_k be smooth functions depending only on $\rho = |x|$ such that

$$\text{supp}\, \zeta_k \subset (2^{k-1}, 2^{k+1}), \qquad \sum_{k=-\infty}^{+\infty} \zeta_k = 1, \quad |(\rho\partial_\rho)^j \zeta_k(\rho)| \le c_j \tag{4.2}$$

with constants c_j independent of k and ρ. We set $h_{k,j}(x) = \zeta_k(2^k x)\, g(2^k x)$ for $j \in J_0$, $h_{k,j}(x) = 2^k\, \zeta_k(2^k x)\, g(2^k x)$ for $j \in J_1$. These functions vanish for $|x| < \frac{1}{2}$ and $|x| > 2$. Consequently, by Lemma 2.1, there exist vector functions $v_k \in V^l_{\beta,\vec\delta}(\mathcal{K})^\ell$ such that $v_k = h_{k,j}$ on Γ_j for $j \in J_0$, $Bv_k = h_{k,j}$ on Γ_j for $j \in J_1$,

$$\|v_k\|_{V^l_{\beta,\vec\delta}(\mathcal{K})^\ell} \le c \Big(\sum_{j\in J_0} \|h_{k,j}\|_{V^{l-1/2}_{\beta,\vec\delta}(\Gamma_j)^\ell} + \sum_{j\in J_1} \|h_{k,j}\|_{V^{l-3/2}_{\beta,\vec\delta}(\Gamma_j)^\ell} \Big), \tag{4.3}$$

and $v_k(x) = 0$ for $|x| < \frac{1}{4}$ and $|x| > 4$. Hence for the functions $u_k(x) = v_k(2^{-k}x)$ we obtain $u_k = \zeta_k g_j$ on Γ_j for $j \in J_0$, $Bu_k = \zeta_k g_j$ on Γ_j for $j \in J_1$, $u_k(x) = 0$ for $|x| < 2^{k-2}$ and $|x| > 2^{k+2}$. Furthermore, u_k satisfies (4.3) with $\zeta_k g_j$ instead of $h_{k,j}$ and a constant c independent of k and g_j. Consequently, for $u = \sum u_k$ we have $u = g_j$ on Γ_j for $j \in J_0$ and $Bu = g_j$ on Γ_j for $j \in J_1$. Inequality (4.1) follows from the equivalence of the norms in $V^l_{\beta,\vec\delta}(\mathcal{K})$ and $V^{l-1/2}_{\beta,\vec\delta}(\Gamma_j)$ with the norms

$$\|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{V^l_{\beta,\vec\delta}(\mathcal{K})} \Big)^{1/2} \quad \text{and} \quad \|g_j\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k g_j\|^2_{V^{l-1/2}_{\beta,\vec\delta}(\Gamma_j)} \Big)^{1/2}, \tag{4.4}$$

respectively (cf. [7, Sect.6.1]). ■

4.2. Solvability of the boundary value problem

The following results can be proved in a standard way (cf. [6], [7, Th.6.1.1,6.1.4]) by means of Theorem 3.2.

Theorem 4.1 *Suppose that there are no eigenvalues of the pencil $\mathfrak{A}$ on the line $\operatorname{Re}\lambda = -\beta+1/2$ and that the components of $\vec\delta$ satisfy the inequalities $1-\mu_j < \delta_j < 1$ for $j \in \tilde J$ and $\max(1-\mu_j, 0) < \delta_j < 1$ for $j \notin \tilde J$. Then the boundary value problem (1.1)–(1.3) is uniquely solvable in $\mathcal{W}^2_{\beta,\vec\delta}(\mathcal{K}; \tilde J)^\ell$ for arbitrary $f \in V^0_{\beta,\vec\delta}(\mathcal{K})^\ell$, $g_j \in V^{3/2}_{\beta,\vec\delta}(\Gamma_j)^\ell$, $j \in J_0$, $g_k \in V^{1/2}_{\beta,\vec\delta}(\Gamma_k)$, $k \in J_1$.*

Theorem 4.2 *Let $u \in \mathcal{W}^2_{\beta,\vec\delta}(\mathcal{K}; \tilde J)^\ell$ be a solution of the boundary value problem (1.1)–(1.3), where $f \in V^0_{\beta',\vec\delta'}(\mathcal{K})^\ell$, $g_j \in V^{3/2}_{\beta',\vec\delta'}(\Gamma_j)^\ell$ for $j \in J_0$, $g_k \in V^{1/2}_{\beta',\vec\delta'}(\Gamma_k)^\ell$ for $k \in J_1$. Suppose that the components of $\vec\delta$ and $\vec\delta'$ satisfy the inequalities $1-\mu_j < \delta'_j \le \delta_j < 1$ for $j \in \tilde J$ and $\max(1-\mu_j, 0) < \delta'_j \le \delta_j < 1$ for $j \notin \tilde J$. If there are no eigenvalues of the pencil $\mathfrak{A}$ on the lines $\operatorname{Re}\lambda = -\beta + 1/2$ and $\operatorname{Re}\lambda = -\beta' + 1/2$, then*

$$u = \sum_{\nu=1}^{N} \sum_{j=1}^{I_\nu} \sum_{s=0}^{\kappa_{\nu,j}-1} c_{\nu,j,s}\, \rho^{\lambda_\nu} \sum_{\sigma=0}^{s} \frac{1}{\sigma!}\, (\log\rho)^\sigma\, u^{(\nu,j,s-\sigma)}(\omega) + w, \tag{4.5}$$

where $w \in \mathcal{W}^2_{\beta',\vec\delta'}(\mathcal{K}; \tilde J)^\ell$ is a solution of problem (1.1)–(1.3), λ_ν are the eigenvalues of the pencil $\mathfrak{A}$ between the lines $\operatorname{Re}\lambda = -\beta + 1/2$ and $\operatorname{Re}\lambda = -\beta' + 1/2$ and $u^{(\nu,j,s)}$ are eigenvectors and generalized eigenvectors corresponding to the eigenvalue λ_ν.

Proof: In the case $\vec\delta = \vec\delta'$ the theorem can be proved in the same way as for smooth Ω (cf. [6, Th.1.2], [7, Th.6.1.4]), since the spectra of the pencils $\mathfrak{A}$ and $\mathfrak{A}_{\vec\delta}$ coincide.

Let $\vec\delta \ne \vec\delta'$. Since $V^0_{\beta',\vec\delta'}(\mathcal{K}) \subset V^0_{\beta',\vec\delta}(\mathcal{K})$ and $V^{l-1/2}_{\beta',\vec\delta'}(\Gamma_j) \subset V^{l-1/2}_{\beta',\vec\delta}(\Gamma_j)$ for $\delta'_j \le \delta_j$, we obtain (4.5) with $w \in W^2_{\beta',\vec\delta}(\mathcal{K})^\ell$. We have to show that $w \in \mathcal{W}^2_{\beta',\vec\delta'}(\mathcal{K}; \tilde J)^\ell$. Let ζ_k be as in the proof of Lemma 4.2 and $\eta_k = \zeta_{k-1} + \zeta_k + \zeta_{k+1}$. Furthermore, we set $\tilde\zeta_k(x) = \zeta_k(2^k x)$, $\tilde\eta_k(x) = \eta_k(2^k x)$, and $v(x) = w(2^k x)$. The support of $\tilde\zeta_k$ is contained in $\{x : 1/2 < |x| < 2\}$. Therefore, due to Theorem 2.1 and the analogous result for the Dirichlet and mixed problems, we have $\tilde\zeta_k v \in \mathcal{W}^2_{\beta',\vec\delta'}(\mathcal{K};\tilde J)^\ell$ and

$$\begin{aligned}\|\tilde\zeta_k v\|^2_{\mathcal{W}^2_{\beta',\vec\delta'}(\mathcal{K};\tilde J)^\ell} &\le c\Big(\|\tilde\eta_k Lv\|^2_{V^0_{\beta',\vec\delta'}(\mathcal{K})^\ell} + \sum_{j\in J_0}\|\tilde\eta_k v\|^2_{V^{3/2}_{\beta',\vec\delta'}(\Gamma_j)^\ell}\\ &\quad + \sum_{j\in J_1}\|\tilde\eta_k Bv\|^2_{V^{1/2}_{\beta',\vec\delta'}(\Gamma_j)^\ell} + \|\tilde\eta_k v\|^2_{\mathcal{W}^2_{\beta',\vec\delta}(\mathcal{K};\tilde J)^\ell}\Big)\end{aligned}$$

with a constant c independent of k. Multiplying this inequality by $2^{2k(\beta'-2)+3}$ and substituting $2^k x = y$, we obtain the same estimate with $\tilde\zeta_k,\tilde\eta_k$ instead of ζ_k,η_k for w. Now the assertion follows from the equivalence of the norm in $\mathcal{W}^l_{\beta,\vec\delta}(\mathcal{K};\tilde J)$ with the norm

$$\|u\| = \Big(\sum_{k=-\infty}^{+\infty}\|\zeta_k u\|^2_{\mathcal{W}^l_{\beta,\vec\delta}(\mathcal{K},\tilde J)}\Big)^{1/2}$$

and the analogous result for the trace spaces. ■

The following statement is an analogon to Theorem 2.1.

Lemma 4.3 *Let $u \in \mathcal{W}^2_{\beta,\vec\delta}(\mathcal{K};\tilde J)^\ell$ be a solution of problem (1.1)–(1.3) with $(\rho\partial_\rho)^\nu f \in V^0_{\beta,\vec\delta}(\mathcal{K})^\ell$ for $\nu = 0,1,\dots,k$, $(\rho\partial_\rho)^\nu g_j \in V^{3/2}_{\beta,\vec\delta}(\Gamma_j)^\ell$ for $j\in J_0$ and $\nu = 0,\dots,k$, $(\rho\partial_\rho)^\nu g_j \in V^{1/2}_{\beta,\vec\delta}(\Gamma_j)^\ell$ for $j\in J_1$ and $\nu=0,1,\dots,k$. Suppose that the components of $\vec\delta$ satisfy the inequalities $1-\mu_j < \delta_j < 1$ for $j\in\tilde J$, $\max(1-\mu_j,0)<\delta_j<1$ for $j\notin\tilde J$ and that the line $\operatorname{Re}\lambda = -\beta+1/2$ is free of eigenvalues of the pencil $\mathfrak{A}$. Then $(\rho\partial_\rho)^\nu u \in \mathcal{W}^2_{\beta,\vec\delta}(\mathcal{K};\tilde J)^\ell$ for $\nu=1,\dots,k$ and*

$$\begin{aligned}\sum_{\nu=0}^k\|(\rho\partial_\rho)^\nu u\|_{\mathcal{W}^2_{\beta,\vec\delta}(\mathcal{K};\tilde J)^\ell} &\le c\sum_{\nu=0}^k\Big(\|(\rho\partial_\rho)^\nu f\|_{V^0_{\beta,\vec\delta}(\mathcal{K})^\ell} + \sum_{j\in J_0}\|(\rho\partial_\rho)^\nu g_j\|_{V^{3/2}_{\beta,\vec\delta}(\Gamma_j)^\ell}\\ &\quad + \sum_{j\in J_0}\|(\rho\partial_\rho)^\nu g_j\|_{V^{1/2}_{\beta,\vec\delta}(\Gamma_j)^\ell}\Big).\end{aligned}$$

Proof: We set $u_t(x) = \dfrac{u(x)-u(tx)}{1-t}$, where t is an arbitrary real number $1/2<t<1$. It can be easily verified that

$$\begin{aligned}&Lu_t(x) = f_t(x) + (1+t)f(tx) \ \text{ in } \mathcal{K},\\ &u_t(x) = (g_j)_t(x) \ \text{ on } \Gamma_j \text{ for } j\in J_0, \quad Bu_t(x) = (g_j)_t(x) + g_j(tx) \ \text{ on } \Gamma_j \text{ for } j\in J_1.\end{aligned}$$

Furthermore, $u_t(x) \to \sum_{j=1}^3 x_j\partial_{x_j}u(x) = \rho\partial_\rho u(x)$ as $t\to 1$. By Theorem 4.1, we have

$$\begin{aligned}\|u_t\|_{\mathcal{W}^2_{\beta,\vec\delta}(\mathcal{K};\tilde J)^\ell} &\le c\Big(\|f_t\|_{V^0_{\beta,\vec\delta}(\mathcal{K})^\ell} + (1+t)\,\|f(t\cdot)\|_{V^0_{\beta,\vec\delta}(\mathcal{K})^\ell} + \sum_{j\in J_0}\|(g_j)_t\|_{V^{3/2}_{\beta,\vec\delta}(\Gamma_j)^\ell}\\ &\quad + \sum_{j\in J_1}\|(g_j)_t\|_{V^{1/2}_{\beta,\vec\delta}(\Gamma_j)^\ell} + \sum_{j\in J_1}\|g_j(t\cdot)\|_{V^{1/2}_{\beta,\vec\delta}(\Gamma_j)^\ell}\Big).\end{aligned} \tag{4.6}$$

Using the equality

$$f_t(x) = \int_0^1\sum_{j=1}^r x_j(\partial_{x_j}f)\big((t+\tau-t\tau)x\big)\,d\tau = \int_0^1(\rho\partial_\rho f)\big((t+\tau-t\tau)x\big)\,d\tau,$$

it can be easily shown that

$$\|f_t\|_{V^0_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \le c\, \|\rho\partial_\rho f\|_{V^0_{\beta,\vec{\delta}}(\mathcal{K})^\ell}$$

with c independent of t. Analogously,

$$\|(g_j)_t\|_{V^{l-1/2}_{\beta,\vec{\delta}}(\Gamma_j)^\ell} \le c\, \|\rho\partial_\rho g_j\|_{V^{l-1/2}_{\beta,\vec{\delta}}(\Gamma_j)^\ell}\,.$$

For the proof of the last inequality one can use the equivalence of the norm in $V^{l-1/2}_{\beta,\delta}(\Gamma_j)$ with the second norm in (4.4) and an expression analogous to (2.9) for the norm of $\zeta_k g_j$. Consequently, from (4.6) it follows that $\rho\partial_\rho u \in \mathcal{W}^2_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$. Repeating this procedure, we obtain $(\rho\partial_\rho u)^\nu \in \mathcal{W}^2_{\beta,\vec{\delta}}(\mathcal{K};\tilde{J})^\ell$ for $\nu = 2,\dots,k$ together with the desired estimate. ■

4.3. Existence of weak solutions to the Neumann problem

In this and in the following two subsections we restrict ourselves to the Neumann problem, i.e., $J_0 = \emptyset$.

Let $V^1_\beta(\mathcal{K}) = W^1_{\beta,\vec{0}}(\mathcal{K})$ be the space with the norm (1.11). From Hardy's inequality it follows that $V^1_0(\mathcal{K})^\ell$ coincides with $\mathcal{H}$. By $V^{-1}_{-\beta}(\mathcal{K})$ we denote the dual space of $V^1_\beta(\mathcal{K})$ with respect to the scalar product in $L_2(\mathcal{K})$. Let ζ_k be smooth functions depending only on $\rho = |x|$ satisfying (4.2). It can be easily shown (see [7, Sect.6.1]) that the norm in $V^{\pm 1}_\beta(\mathcal{K})$ is equivalent to

$$\|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{V^{\pm1}_\beta(\mathcal{K})} \Big)^{1/2}. \tag{4.7}$$

We consider weak solutions $u \in V^1_\beta(\mathcal{K})^\ell$ of problem (1.1), (1.3). Obviously, the sesquilinear form $b_{\mathcal{K}}(\cdot,\cdot)$ is continuous on $V^1_\beta(\mathcal{K})^\ell \times V^1_{-\beta}(\mathcal{K})^\ell$. Consequently, it generates a linear and continuous operator $\mathcal{A}_\beta \,:\, V^1_\beta(\mathcal{K})^\ell \to V^{-1}_\beta(\mathcal{K})^\ell$ by the equality

$$\big(\mathcal{A}_\beta u, v\big)_{\mathcal{K}} = b_{\mathcal{K}}(u,v), \qquad u \in V^1_\beta(\mathcal{K})^\ell,\ v \in V^1_{-\beta}(\mathcal{K})^\ell.$$

Lemma 4.4 *For every $u \in V^1_\beta(\mathcal{K})^\ell$ the inequality*

$$\|u\|_{V^1_\beta(\mathcal{K})^\ell} \le c\, \Big(\|\mathcal{A}_\beta u\|_{V^{-1}_\beta(\mathcal{K})^\ell} + \|u\|_{V^0_{\beta-1}(\mathcal{K})^\ell} \Big) \tag{4.8}$$

is satisfied.

Proof: Let $\zeta_k = \zeta_k(\rho)$ be smooth functions satisfying (4.2), and let $\eta_k = \zeta_{k-1} + \zeta_k + \zeta_{k+1}$. We show that

$$\|\zeta_k u\|^2_{V^1_\beta(\mathcal{K})^\ell} \le c\, \|\zeta_k \mathcal{A}_\beta u\|^2_{V^{-1}_\beta(\mathcal{K})^\ell} + \varepsilon\, \|\eta_k u\|^2_{V^1_\beta(\mathcal{K})^\ell} + C(\varepsilon)\, \|\eta_k u\|^2_{V^0_{\beta-1}(\mathcal{K})^\ell}\,, \tag{4.9}$$

where c, ε and $C(\varepsilon)$ depend only on the constants c_0, c_1, c_2 in (4.2) and ε can be chosen arbitrarily small.

Let first $k = 0$. Integrating by parts, we get

$$a(\zeta_0 u, v) = \big(\zeta_0 \mathcal{A}_\beta u, v\big)_{\mathcal{K}} + c_1(u,v) - c_2(u,v),$$

where

$$c_1(u,v) = \int_{\mathcal{K}} \sum_{i,j=1}^3 A_{i,j}\, (\partial_{x_i}\zeta_0)\, u \cdot \partial_{x_j}\overline{v}\, dx, \qquad c_2(u,v) = \int_{\mathcal{K}} \sum_{i,j=1}^3 A_{i,j} (\partial_{x_j}\zeta_0)\, \partial_{x_i} u \cdot \overline{v}\, dx.$$

Since problem (1.7) is uniquely solvable in $\mathcal{H} = V_0^1(\mathcal{K})^\ell$ for arbitrary $F \in V_0^{-1}(\mathcal{K})^\ell$, we obtain

$$\|\zeta_0 u\|_{V_0^1(\mathcal{K})^\ell} \le c \Big(\|\zeta_0 \mathcal{A}_\beta u\|_{V_0^{-1}(\mathcal{K})^\ell} + \sup_{\|v\|_{V_0^1(\mathcal{K})^\ell}=1} |c_1(u,v) - c_2(u,v)| \Big).$$

Here

$$|c_1(u,v)| \le c\, \|\eta_0 u\|_{L_2(\mathcal{K})^\ell}\ \|v\|_{V_0^1(\mathcal{K})^\ell}$$

and

$$c_2(u,v) = -\int_{\mathcal{K}} \sum_{i,j=1}^{3} A_{i,j} u\, \partial_{x_i}(\overline{v}\partial_{x_j}\zeta_0)\, dx + \int_{\partial\mathcal{K}\backslash\mathcal{S}} \sum_{i,j=1}^{3} A_{i,j}(\partial_{x_j}\zeta_0)\, n_i\, u\cdot \overline{v}\, d\sigma.$$

The last equality implies

$$\begin{aligned}
|c_2(u,v)| &\le c \Big(\|\eta_0 u\|_{L_2(\mathcal{K})^\ell}\ \|v\|_{V_0^1(\mathcal{K})^\ell} + \|\eta_0 u\|_{L_2(\partial\mathcal{K}\backslash\mathcal{S})^\ell}\ \|v\|_{L_2(\partial\mathcal{K}\backslash\mathcal{S})^\ell} \Big) \\
&\le c \Big(\|\eta_0 u\|_{L_2(\mathcal{K})^\ell} + \|\eta_0 u\|_{W^{3/4}(\mathcal{K})^\ell} \Big) \|v\|_{V_0^1(\mathcal{K})^\ell} \\
&\le \Big(\varepsilon\, \|\eta_0 u\|_{V_0^1(\mathcal{K})^\ell} + C(\varepsilon)\, \|\eta_0 u\|_{L_2(\mathcal{K})^\ell} \Big) \|v\|_{V_0^1(\mathcal{K})^\ell}.
\end{aligned}$$

Therefore,

$$\sup_{\|v\|_{V_0^1(\mathcal{K})^\ell}=1} \big|c_2(u,v)\big| \le \varepsilon\, \|\eta_0 u\|_{V_0^1(\mathcal{K})^\ell} + C(\varepsilon)\, \|\eta_0 u\|_{L_2(\mathcal{K})^\ell}$$

which implies (4.9) for $k = 0$. By means of the transformation $x = 2^k y$, we obtain (4.9) with the same constants c, ε and $C(\varepsilon)$ for $k \neq 0$. Summing up in (4.9) and using the equivalence of the norms in $V_\beta^{\pm 1}(\mathcal{K})$ to (4.7), we obtain (4.8). ■

Theorem 4.3 *Suppose that there are no eigenvalues of the pencil $\mathfrak{A}$ on the line* $\mathrm{Re}\,\lambda = -\beta - 1/2$. *Then the operator $\mathcal{A}_\beta$ is an isomorphism.*

Proof: Let u be an arbitrary vector-function from $V_\beta^1(\mathcal{K})^\ell$. Since $V_\beta^1(\mathcal{K}) \subset W^0_{\beta-1,(\varepsilon-1)\vec{1}}(\mathcal{K})$, where ε is an arbitrarily small positive number (see Lemma 4.1) and $\vec{1} = (1,\dots,1)$, the vector function

$$w \stackrel{def}{=} \rho^{2(\beta-1)} \prod_j \big(\frac{r_j}{\rho}\big)^{2(\varepsilon-1)} u$$

belongs to $W^0_{1-\beta,(1-\varepsilon)\vec{1}}(\mathcal{K})^\ell$. From the absence of eigenvalues of the pencil $\mathfrak{A}$ on the line $\mathrm{Re}\,\lambda = -\beta - 1/2$ it follows that the line $\mathrm{Re}\,\lambda = \beta - 1/2$ is also free of eigenvalues. Consequently, by Theorem 4.1, there exists a solution $v \in W^2_{1-\beta,(1-\varepsilon)\vec{1}}(\mathcal{K})^\ell$ of the problem

$$Lv = w \ \text{ in } \mathcal{K}, \qquad Bv = 0 \ \text{ on } \partial\mathcal{K}\backslash\mathcal{S}$$

which satisfies the inequality

$$\|v\|_{W^2_{1-\beta,(1-\varepsilon)\vec{1}}(\mathcal{K})^\ell} \le c\, \|w\|_{W^0_{1-\beta,(1-\varepsilon)\vec{1}}(\mathcal{K})^\ell} \le c'\, \|u\|_{W^0_{\beta-1,(\varepsilon-1)\vec{1}}(\mathcal{K})^\ell} \tag{4.10}$$

with a constant c independent of u. This implies

$$\begin{aligned}
\|u\|^2_{W^0_{\beta-1,(\varepsilon-1)\vec{1}}(\mathcal{K})^\ell} &= \int_{\mathcal{K}} u\, \overline{w}\, dx = \int_{\mathcal{K}} u\, \overline{Lv}\, dx = b_{\mathcal{K}}(u,v) = \big(\mathcal{A}_\beta u, v\big)_{\mathcal{K}} \\
&\le c\, \|\mathcal{A}_\beta u\|_{V_\beta^{-1}(\mathcal{K})^\ell}\ \|v\|_{V^1_{-\beta}(\mathcal{K})^\ell} \le c\, \|\mathcal{A}_\beta u\|_{V_\beta^{-1}(\mathcal{K})^\ell}\ \|v\|_{W^2_{1-\beta,(1-\varepsilon)\vec{1}}(\mathcal{K})^\ell} \\
&\le c\, \|\mathcal{A}_\beta\|_{V_\beta^{-1}(\mathcal{K})^\ell}\ \|u\|_{W^0_{\beta-1,(\varepsilon-1)\vec{1}}(\mathcal{K})^\ell}.
\end{aligned}$$

From the last inequality we conclude that

$$\|u\|_{V^0_{\beta-1}(\mathcal{K})^\ell} \le c\,\|u\|_{W^0_{\beta-1,(\varepsilon-1)\vec{1}}(\mathcal{K})^\ell} \le c\,\|\mathcal{A}_\beta u\|_{V_\beta^{-1}(\mathcal{K})^\ell}.$$

This estimate together with Lemma 4.4 yields

$$\|u\|_{V^1_\beta(\mathcal{K})^\ell} \le c\,\|\mathcal{A}_\beta u\|_{V_\beta^{-1}(\mathcal{K})^\ell}. \tag{4.11}$$

Therefore, the kernel of $\mathcal{A}_\beta$ is trivial and its image is closed.

We prove that for every $F \in V_\beta^{-1}(\mathcal{K})^\ell$ there exists a solution of the equation $\mathcal{A}_\beta u = F$. Let $\{f_k\}_{k\ge 0} \subset C_0^\infty(\overline{\mathcal{K}})^\ell$ be a sequence which converges to F in $V_\beta^{-1}(\mathcal{K})^\ell$. By Theorem 4.1, for every k there exists a solution $u_k \in W^2_{\beta+1,(1-\varepsilon)\vec{1}}(\mathcal{K})^\ell \subset V^1_\beta(\mathcal{K})^\ell$ of the problem $Lu_k = f_k$ in $\mathcal{K}$, $Bu_k = 0$ on $\partial\mathcal{K}\backslash\mathcal{S}$. Since, according to (4.11),

$$\|u_k - u_l\|_{V^1_\beta(\mathcal{K})^\ell} \le c\,\|f_k - f_l\|_{V_\beta^{-1}(\mathcal{K})^\ell}$$

with a constant c independent of k and l, the functions u_k form a Cauchy sequence in $V^1_\beta(\mathcal{K})^\ell$. Its limit u is the solution of the equation $\mathcal{A}_\beta u = F$. The proof is complete. ■

4.4. Regularity of weak solutions to the Neumann problem

Using Theorem 2.2, we can prove the following theorem.

Theorem 4.4 *Let $u \in V^1_{\beta-l+1}(\mathcal{K})^\ell$ be a solution of the equation $\mathcal{A}_{\beta-l+1}u = F$, where the functional $F \in V^{-1}_{\beta-l+1}(\mathcal{K})^\ell$ has the form*

$$(F,v)_{\mathcal{K}} = \int_{\mathcal{K}} f\cdot\overline{v}\,dx + \sum_{j=1}^n \int_{\Gamma_j} g_j\cdot\overline{v}\,d\sigma, \qquad v \in V^1_{-\beta}(\mathcal{K})^\ell, \tag{4.12}$$

with $f \in W^{l-2}_{\beta,\vec{\delta}}(\mathcal{K})^\ell$, $g_j \in W^{l-3/2}_{\beta,\vec{\delta}}(\Gamma_j)^\ell$, δ_j is not integer, and $\max(l-1-\mu_j,0) < \delta_j < l-1$ for $j = 1,\dots,n$. Then $u \in W^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ and

$$\|u\|_{W^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \le c\Big(\|f\|_{W^{l-2}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} + \sum_{j=1}^n \|g_j\|_{W^{l-3/2}_{\beta,\vec{\delta}}(\Gamma_j)^\ell} + \|u\|_{V^1_{\beta-l+1}(\mathcal{K})^\ell}\Big).$$

Proof: Under our assumptions on F, the vector function u is a solution of problem (1.1), (1.3). We define by $\zeta_k, \eta_k, \tilde{\zeta}_k, \tilde{\eta}_k$ the same functions as in the proof of Theorem 4.2 and set $v(x) = u(2^k x)$. Then, by Theorem 2.2, the vector functions $\zeta_k u$ and $\tilde{\zeta}_k v$ belong to $W^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ for $k = 0, \pm 1, \dots$. Furthermore,

$$\|\tilde{\zeta}_k v\|^2_{W^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \le c\Big(\|\tilde{\eta}_k L v\|^2_{W^{l-2}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} + \sum_{j=1}^n \|\tilde{\eta}_k B v\|^2_{W^{l-3/2}_{\beta,\vec{\delta}}(\Gamma_j)^\ell} + \|\tilde{\eta}_k v\|^2_{V^1_{\beta-l+1}(\mathcal{K})^\ell}\Big).$$

Due to (4.2), the constant c is independent of k. Multiplying the last estimate by $2^{2k(\beta-l)+3}$ and substituting $2^k x = y$, we arrive at the inequality

$$\|\zeta_k u\|^2_{W^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \le c\Big(\|\eta_k L u\|^2_{W^{l-2}_{\beta,\vec{\delta}}(\mathcal{K})^\ell} + \sum_{j=1}^n \|\eta_k B u\|^2_{W^{l-3/2}_{\beta,\vec{\delta}}(\Gamma_j)^\ell} + \|\eta_k u\|^2_{V^1_{\beta-l+1}(\mathcal{K})^\ell}\Big).$$

Using the equivalence of the norms in $W^l_{\beta,\vec{\delta}}(\mathcal{K})$ and $W^{l-1/2}_{\beta,\vec{\delta}}(\Gamma_j)$ with norms analogous to (4.4), we obtain the assertion of the theorem. ■

The following statement is an immediate consequence of Theorems 4.3 and 4.4.

Corollary 4.1 *Suppose that δ_j is not integer,* $\max(l-1-\mu_j,0)<\delta_j<l-1$ *for* $j=1,\dots,n$ *and that the line* $\mathrm{Re}\,\lambda = l-\beta-3/2$ *does not contain eigenvalues of the pencil* $\mathfrak{A}$*. Then the Neumann problem* (1.1), (1.3) *is uniquely solvable in* $W^l_{\beta,\vec\delta}(\mathcal{K})^\ell$ *for arbitrary* $f\in W^{l-2}_{\beta,\vec\delta}(\mathcal{K})^\ell$ *and* $g_j\in W^{l-3/2}_{\beta,\vec\delta}(\Gamma_j)^\ell$, $j=1,\dots,n$.

Furthermore, we get the following generalization of Lemma 4.3.

Corollary 4.2 *Let* $u\in V^1_{\beta-l+1}(\mathcal{K})^\ell$ *be a solution of problem* (1.1),(1.3) *with* $(\rho\partial_\rho)^\nu f\in W^{l-2}_{\beta,\vec\delta}(\mathcal{K})^\ell$, $(\rho\partial_\rho)^\nu g_j\in W^{l-3/2}_{\beta,\vec\delta}(\Gamma_j)^\ell$ *for* $\nu=0,1,\dots,k$, $j=1,\dots,n$*, where the components of* $\vec\delta$ *are not integer and satisfy the inequalities* $\max(l-1-\mu_j,0)<\delta_j<l-1$*. Suppose that there are no eigenvalues of the pencil* $\mathfrak{A}$ *on the line* $\mathrm{Re}\,\lambda=l-\beta-3/2$*. Then* $(\rho\partial_\rho)^\nu u\in W^l_{\beta,\vec\delta}(\mathcal{K})^\ell$ *for* $\nu=0,1,\dots,k$ *and*

$$\sum_{\nu=0}^k \|(\rho\partial_\rho)^\nu u\|_{W^l_{\beta,\vec\delta}(\mathcal{K})^\ell}\le c\sum_{\nu=0}^k\Big(\|(\rho\partial_\rho)^\nu f\|_{W^{l-2}_{\beta,\vec\delta}(\mathcal{K})^\ell}+\sum_{j=1}^n\|(\rho\partial_\rho)^\nu g_j\|_{W^{l-3/2}_{\beta,\vec\delta}(\Gamma_j)^\ell}\Big).$$

Proof: Let first $l-1-\delta_j<1$ and, therefore, $\max(1-\mu_j,0)<\delta_j-l+2<1$ for $j=1,\dots,n$. Then, by Lemma 4.3, $(\rho\partial_\rho)^\nu u\in W^2_{\beta-l+2,\vec\delta-(l-2)\vec 1}(\mathcal{K})^\ell\subset V^1_{\beta-l+1}(\mathcal{K})^\ell$ for $\nu=1,\dots,k$. Using Theorem 4.4 and the equalities

$$L\rho\partial_\rho u=(\rho\partial_\rho+2)Lu,\quad B\rho\partial_\rho u=(\rho\partial_\rho+1)Bu,$$

we obtain $(\rho\partial_\rho)^\nu u\in W^l_{\beta,\vec\delta}(\mathcal{K})^\ell$ for $\nu=1,\dots,k$.

Now let $l-1-\delta_j>1$ for $j=1,\dots,n$. By Theorem 4.4, $u\in W^l_{\beta,\vec\delta}(\mathcal{K})^\ell$ and, consequently, $\rho\partial_\rho u\in W^{l-1}_{\beta-1,\vec\delta}(\mathcal{K})^\ell\subset V^1_{\beta-l+1}(\mathcal{K})^\ell$. Since $L\,\rho\partial_\rho u=\rho\partial_\rho f+2f\in W^{l-2}_{\beta,\vec\delta}(\mathcal{K})^\ell$ and $B\,\rho\partial_\rho u|_{\Gamma_j}=\rho\partial_\rho g_j+g_j\in W^{l-3/2}_{\beta,\vec\delta}(\Gamma_j)^\ell$, it follows from Theorem 4.4 that $\rho\partial_\rho u\in W^l_{\beta,\vec\delta}(\mathcal{K})^\ell$. Analogously, we obtain $(\rho\partial_\rho)^\nu u\in W^l_{\beta,\vec\delta}(\mathcal{K})^\ell$ for $\nu=2,\dots,k$.

Finally, let $l-1-\delta_j>1$ for some, but not all, j. Then let $\psi_1,\dots,\psi_n$ be smooth functions on $\overline\Omega$ such that $\psi_j\ge 0$, $\psi_j=1$ near $M_j\cap S^2$, and $\sum\psi_j=1$. We extend ψ_j to $\mathcal{K}$ by the equality $\psi_j(x)=\psi_j(x/|x|)$. Then $\partial_x^\alpha\psi_j(x)\le c\,|x|^{-|\alpha|}$. Consequently, the assumptions of the corollary are satisfied for $\psi_j u$, and from what has been shown above it follows that $(\rho\partial_\rho)^\nu\psi_j u\in W^l_{\beta,\vec\delta}(\mathcal{K})^\ell$ for $j=1,\dots,n$. This completes the proof. ■

Corollary 4.3 *Let* $u\in V^1_\beta(\mathcal{K})^\ell$ *be a solution of the equation* $\mathcal{A}_\beta u=F$*, where* $F\in V^{-1}_\beta(\mathcal{K})^\ell\cap V^{-1}_{\beta'}(\mathcal{K})^\ell$*, If there are no eigenvalues of the pencil* $\mathfrak{A}$ *on the lines* $\mathrm{Re}\,\lambda=-\beta-1/2$ *and* $\mathrm{Re}\,\lambda=-\beta'-1/2$*, then* u *admits the decomposition* (4.5) *with* $w\in V^1_{\beta'}(\mathcal{K})^\ell$.

Proof: By Theorem 4.3, there exists a solution $w\in V^1_{\beta'}(\mathcal{K})^\ell$ of the equation $\mathcal{A}_{\beta'}w=F$. Let χ be a smooth cut off function equal to one near the vertex of $\mathcal{K}$. We assume, without loss of generality, that $\beta'<\beta$. Then $\chi(u-w)\in V^1_\beta(\mathcal{K})^\ell$. Integrating by parts, we obtain

$$b_{\mathcal{K}}\big(\chi(u-w),v\big)=b_{\mathcal{K}}(u-w,\chi v)+\int_{\mathcal{K}}f\cdot\overline v\,dx+\int_{\partial\mathcal{K}\backslash\mathcal{S}}g\cdot\overline v\,d\sigma,$$

for arbitrary $v\in V^1_{-\beta}(\mathcal{K})^\ell$, where

$$f=-\sum_{i,j=1}^3 A_{i,j}\,\big((\partial_{x_j}\chi)\,\partial_{x_i}u+\partial_{x_j}(\partial_{x_i}\chi)u\big),\quad g=\sum_{i,j=1}^3 A_{i,j}\,(\partial_{x_i}\chi)\,n_j u$$

Obviously, $f \in W^0_{\gamma,\vec{\delta}}(\mathcal{K})^\ell$ and $g|_{\Gamma_j} \in W^{1/2}_{\gamma,\vec{\delta}}(\Gamma_j)^\ell$ with arbitrary $\gamma, \vec{\delta}$, $\max(1-\mu_j, 0) < \delta_j < 1$ for $j = 1, \dots, n$. Since $\chi v \in V^1_\beta(\mathcal{K})^\ell \cap V^1_{-\beta'}(\mathcal{K})^\ell$, we have $b_{\mathcal{K}}(u-w, \chi v) = 0$. Consequently, from Theorem 4.4 it follows that $\chi(u-w) \in W^2_{\beta+1,\vec{\delta}}(\mathcal{K})^\ell$. Applying Theorem 4.2, we obtain the decomposition (4.5) for $\chi(u-w)$ with a remainder $w' \in W^2_{\beta'+1,\vec{\delta}}(\mathcal{K})^\ell \subset V^1_{\beta'}(\mathcal{K})^\ell$. Furthermore, since $\beta' < \beta$, the function $(1-\chi)u$ belongs to $V^1_{\beta'}(\mathcal{K})^\ell$. The result follows. ■

Remark 4.1 Let for the Neumann problem in the dihedron $\mathcal{D}_j$ (i.e., in the dihedron which coincides with the cone $\mathcal{K}$ in a neighborhood of the edge point $x^{(j)} = M_j \cap S^2$) the assumptions of Theorem 2.3 be valid. Then in the condition on δ_j in Theorem 4.4 and Corollaries 4.1–4.3 the number $\mu_j = 1$ can be replaced by the real part $\mu_j^{(2)}$ of the first eigenvalue of the pencil $A_j(\lambda)$ on the right of the line $\operatorname{Re}\lambda = 1$. To show this, one has to use Theorem 2.3 instead of Theorem 2.2 in the proof of Theorem 4.4.

Examples. Let us consider, for example, the solution $u \in \mathcal{H} = V^1_0(\mathcal{K})^\ell$ of problem (1.7), where F has the form (4.12) with $f \in W^{l-2}_{\beta,\vec{\delta}}(\mathcal{K})$, $g_j \in W^{l-3/2}_{\beta,\vec{\delta}}(\Gamma_j)$, $l - \beta \geq 1$, δ_j is not integer, $\max(0, l-1-\mu_j) < \delta_j < l-1$. Suppose that, additionally to (1.6), the inequality

$$\sum_{i,j=1}^{3} \big(A_{i,j} f_i \,,\, f_j\big)_{\mathbb{C}^\ell} \geq c \sum_{i=1}^{3} |f_i|^2 \tag{4.13}$$

is satisfied for all $f_1, f_2, f_3 \in \mathbb{C}^\ell$. Then the strip $0 \leq \operatorname{Re}\lambda \leq 1/2$ contains only the eigenvalues $\lambda = 0$ of the pencil $\mathfrak{A}$. The corresponding eigenvectors are constants, while generalized eigenvectors do not exist (see [8, Ch.12]). The same is true, for example, for the Neumann problem to the Lamé system (see [8, Ch.4]). Consequently, there exists a constant vector c such that $u - c \in W^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell$ if $l - \beta - 3/2 < \operatorname{Re}\Lambda_2$, where Λ_2 is the eigenvalue of $\mathfrak{A}$ with smallest positive real part, and δ_j are noninteger numbers such that $\max(0, l-1-\mu_j) < \delta_j < l-1$.

For the Neumann problem to the Laplace equation, we obtain $u \in W^l_{\beta,\vec{\delta}}(\mathcal{K})$ if $-1/2 < l - \beta - 3/2 < \Lambda_2$ and $\max(0, l-1-\pi/\theta_j) < \delta_j < l-1$, where θ_j is the angle at the edge M_j. If $\mathcal{K}$ is convex, then $\Lambda_2 > (\sqrt{5}-1)/2$ (see [4]), and we can choose $l = 3$, $\beta = 1$, $\delta_j = 1 - \varepsilon$ with sufficiently small positive ε. Thus, $u - c \in W^3_{1,(1-\varepsilon)\vec{1}}(\mathcal{K}) \subset W^2_{0,\vec{0}}(\mathcal{K})$ if $f \in W^1_{1,(1-\varepsilon)\vec{1}}(\mathcal{K})$, $g_j \in W^{3/2}_{1,(1-\varepsilon)\vec{1}}(\Gamma_j)$.

In the case of the Lamé system, we make the following assumptions: $-1/2 < l - \beta - 3/2 < \Lambda_2$, $\max(0, l-1-\pi/\theta_j) < \delta_j < l-1$ if $\theta_j < \pi$, and $\max(0, l-1-\xi_+(\theta_j)/\theta_j) < \delta_j < l-1$ if $\theta_j > \pi$, where $\xi_+(\theta)$ is the smallest positive root of (1.8). Furthermore, we assume that the boundary data g_j satisfy the compatibility conditions

$$n_{j+} g_{j-}\big|_{M_j} = n_{j-} g_{j+}\big|_{M_j}$$

where Γ_{j-} and Γ_{j+} are the sides adjacent to the edge M_j. Under these conditions, we get $u \in W^l_{\beta,\vec{\delta}}(\mathcal{K})^3$.

4.5. Estimates for Green's matrix to the Neumann problem

We consider the Neumann problem (1.1), (1.3) with $J_1 = \{1, \dots, n\}$. Suppose that the line $\operatorname{Re}\lambda = -\beta - 1/2$ (and, consequently, also the line $\operatorname{Re}\lambda = \beta - 1/2$) does not contain eigenvalues of the pencil $\mathfrak{A}$. Then the following theorem holds analogously to [12, Th.2.1].

Theorem 4.5 1) *There exists a unique solution $G(x,\xi)$ of the boundary value problem* (1.12), (1.14) *such that the function $x \to \zeta\Big(\frac{|x-\xi|}{r(\xi)}\Big)\, G(x,\xi)$ belongs to the space $V^1_\beta(\mathcal{K})^\ell$ for every fixed $\xi \in \mathcal{K}$ and for every smooth function ζ on $(0,\infty)$ equal to one in $(1,\infty)$ and to zero in $(0, \frac{1}{2})$.*

2) *The function G is infinitely differentiable with respect to $x, \xi \in \overline{\mathcal{K}}\backslash\mathcal{S}$, $x \neq \xi$.*

3) *The function $G(x,\cdot)$ is the unique solution of the problem*

$$L(\partial_\xi)\,\overline{G(x,\xi)} = \delta(x-\xi)\,I_\ell \quad \text{for } x,\xi \in \mathcal{K}, \qquad B(\partial_\xi)\,\overline{G(x,\xi)} = 0 \quad \text{for } x \in \partial\mathcal{K}\backslash\mathcal{S},\ \xi \in \mathcal{K},$$

such that the function $\xi \to \zeta\Big(\frac{|x-\xi|}{r(x)}\Big)\, G(x,\xi)$ belongs to the space $V^1_{-\beta}(\mathcal{K})^{\ell\times\ell}$ for every fixed $x \in \mathcal{K}$.

Theorem 4.6 *Green's function G introduced in Theorem 4.5 satisfies the following estimates for $|x|/2 < |\xi| < 2|x|$:*

$$\begin{aligned} \big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| &\le c\,|x-\xi|^{-1-|\alpha|-|\gamma|} \quad \text{if } |x-\xi| < \min(r(x),r(\xi)), \\ \big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| &\le c\,|x-\xi|^{-1-|\alpha|-|\gamma|} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x-\xi|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|x-\xi|}\Big)^{\delta_{j,\gamma}} \\ &\qquad \text{if } |x-\xi| > \min(r(x),r(\xi)). \end{aligned}$$

Here $\delta_{j,\alpha} = \min(0, \mu_j - |\alpha| - \varepsilon)$ with an arbitrarily small positive ε.

Proof: Since $G(Tx,T\xi) = T^{-1}\,G(x,\xi)$, we may assume, without loss of generality that $|x-\xi| = 1$. Then $3\min(|x|,|\xi|) > |x| + |\xi| > |x-\xi| = 1$. Therefore, we can apply Theorems 2.4 and 2.5 and obtain the desired estimates. ■

For the proof of point estimates for $G(x,\xi)$ in the cases $2\,|x| < |\xi|$ and $|x| > 2\,|\xi|$ we need the following lemma.

Lemma 4.5 *If $u \in W^l_{\beta,\vec{\delta}}(\mathcal{K})$, $\rho\partial_\rho u \in W^l_{\beta,\vec{\delta}}(\mathcal{K})$, $\delta_j \neq l-1$ for $j = 1,\dots,n$, then there is the estimate*

$$\rho^{\beta-l+3/2} \prod_{j=1}^n \big(\frac{r_j}{\rho}\big)^{\max(\delta_j-l+1,0)}\, |u(x)| \le c\,\Big(\|u\|_{W^l_{\beta,\vec{\delta}}(\mathcal{K})} + \|\rho\partial_\rho u\|_{W^l_{\beta,\vec{\delta}}(\mathcal{K})}\Big)$$

with a constant c independent of u and x.

Proof: 1) Applying the estimate

$$\sup_{0<\rho<\infty} |v(\rho)|^2 \le c \int_0^\infty \big(|v(\rho)|^2 + |\rho v'(\rho)|^2\big)\,\frac{d\rho}{\rho}$$

(which is an immediate consequence of Sobolev's lemma) to the function $\rho^{\beta-l+3/2}u(\rho,\omega)$, one obtains

$$\rho^{2(\beta-l)+3}\,|u(\rho,\omega)|^2 \le c \int_0^\infty \rho^{2(\beta-l+1)}\,\big(|u(\rho,\omega)|^2 + |\rho\partial_\rho u(\rho,\omega)|^2\big)\,d\rho. \tag{4.14}$$

Let $\psi_1,\dots,\psi_n$ be smooth functions on $\overline{\Omega}$ such that $\psi_j = 1$ near $M_j \cap S^2$, $\psi_j \ge 0$, and $\sum \psi_j = 1$. Furthermore, let v be an arbitrary function from $W^l_{\vec{\delta}}(\Omega)$. If $\delta_j < l-1$, then $\psi_j v$ is continuous on $\overline{\Omega}$, and the supremum of $\psi_j v$ can be estimated by its norm in $W^l_{\vec{\delta}}(\Omega)$. If $\delta_j > l-1$, then $\psi_j v$ belongs to $V^l_{\vec{\delta}}(\Omega)$ (see, e.g., [7, Th.7.1.1]). Therefore,

$$\big(\frac{r_j}{\rho}\big)^{\delta_j-l+1}\,|\psi_j(\omega)\,v(\omega)| \le c\,\|\psi_j v\|_{W^l_{\vec{\delta}}(\Omega)}$$

(cf. (2.30)). This implies

$$\prod_{j=1}^n \big(\frac{r_j}{\rho}\big)^{\max(\delta_j-l+1,0)}\,|v(\omega)| \le c\,\|v\|_{W^l_{\vec{\delta}}(\Omega)}\,.$$

The last inequality together with (4.14) implies

$$\rho^{2(\beta-l)+3} \prod_{j=1}^{n} \big(\frac{r_j}{\rho}\big)^{2\max(\delta-l+1,0)} |u(\rho,\omega)|^2$$
$$\le c \int_0^\infty \rho^{2(\beta-l+1)} \Big(\|u(\rho,\cdot)\|^2_{W^l_{\vec{\delta}}(\Omega)} + \|\rho\partial_\rho u(\rho,\cdot)\|^2_{W^l_{\vec{\delta}}(\Omega)} \Big)\, d\rho.$$

The result follows. ■

Let again β be a fixed number such that no eigenvalues of the pencil $\mathfrak{A}$ lie on the line $\operatorname{Re}\lambda = -\beta - 1/2$. Furthermore let

$$\Lambda_- < \operatorname{Re}\lambda < \Lambda_+$$

be the widest strip in the complex plane which is free of eigenvalues and contains the line $\operatorname{Re}\lambda = -\beta - 1/2$.

Theorem 4.7 *Let $G(x,\xi)$ be Green's function introduced in Theorem* 4.5. *If $|x| < |\xi|/2$, then*

$$\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \le c\, |x|^{\Lambda_+ - |\alpha| - \varepsilon}\ |\xi|^{-1-\Lambda_+ - |\gamma| + \varepsilon} \prod_{j=1}^{n} \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^{n} \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{\delta_{j,\gamma}},$$

where $\delta_{j,\alpha} = \min(0, \mu_j - |\alpha| - \varepsilon)$ and ε is an arbitrarily small positive number. Analogously, for $|x| > 2|\xi|$ there is the estimate

$$\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \le c\, |x|^{\Lambda_- - |\alpha| + \varepsilon}\ |\xi|^{-1-\Lambda_- - |\gamma| - \varepsilon} \prod_{j=1}^{n} \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^{n} \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{\delta_{j,\gamma}}.$$

Proof: Suppose that $|x| = 1$. We denote by ϕ, ψ smooth functions on $\overline{\mathcal{K}}$ such that $\phi(\eta) = 1$ for $|\eta| \le 1/2$, $\psi = 1$ in a neighborhood of $\operatorname{supp}\phi$, and $\psi(\eta) = 0$ for $|\eta| > 3/4$. Furthermore, let l be an integer, $l > \max\mu_j + 1$. The vector function $\partial_x^\alpha G(x,\cdot)$ is a solution of the problem

$$L(\partial_\xi)\, \partial_x^\alpha \overline{G(x,\xi)} = \partial_x^\alpha \delta(x-\xi)\, I_\ell \ \text{ in } \mathcal{K}, \quad B(\partial_\xi)\, \partial_x^\alpha \overline{G(x,\xi)} = 0 \ \text{ on } \partial\mathcal{K}\backslash\mathcal{S}$$

such that the function $\xi \to \zeta\Big(\frac{|x-\xi|}{r(x)}\Big)\, \partial_x^\alpha G(x,\xi)$ belongs to $V^1_{-\beta}(\mathcal{K})^{\ell\times\ell}$. Here, as in Theorem 4.5, ζ is an arbitrary smooth function on $(0,\infty)$ equal to one in $(1,\infty)$ and to zero in $(0,\frac12)$. In particular, $\psi(\cdot)\partial_x^\alpha G(x,\cdot) \in V^1_{-\beta}(\mathcal{K})^\ell$, $\psi(\cdot)L(\partial_\xi)\partial_x^\alpha\overline{G(x,\cdot)} = 0$ and $\psi(\cdot)B(\partial_\xi)\partial_x^\alpha\overline{G(x,\cdot)}|_{\partial\mathcal{K}\backslash\mathcal{S}} = 0$. Thus, we conclude from Corollaries 4.2 and 4.3 that the functions $\phi(\cdot)\,\partial_\xi^\gamma\partial_x^\alpha G(x,\cdot)$ and $|\xi|\partial_{|\xi|}\phi(\cdot)\,\partial_\xi^\gamma\partial_x^\alpha G(x,\cdot)$ belong to $W^l_{l-1-\beta'+|\gamma|,\vec{\delta}+|\gamma|\vec{1}}(\mathcal{K})^\ell$, where $\beta' < -\Lambda_- - 1/2$ and δ_j are non-integer numbers, $l-1-\mu_j < \delta_j < l-1$ for $j = 1,\dots,n$. The norms of $\phi(\cdot)\,\partial_\xi^\gamma\partial_x^\alpha G(x,\cdot)$ and $|\xi|\partial_{|\xi|}\phi(\cdot)\,\partial_\xi^\gamma\partial_x^\alpha G(x,\cdot)$ can be estimated by the norm of $\psi(\cdot)\partial_x^\alpha G(x,\cdot)$ in $V^1_{-\beta}(\mathcal{K})^\ell$. Hence, by means of Lemma 4.5, we obtain

$$\rho(\xi)^{-\beta'+|\gamma|+1/2} \prod_{j=1}^{n} \Big(\frac{r_j(\xi)}{\rho(\xi)}\Big)^{\max(\delta_j+|\gamma|-l+1,0)} \big|\partial_x^\alpha\partial_\xi^\gamma G(x,\xi)\big| \le c\, \|\psi(\cdot)\,\partial_x^\alpha G(x,\cdot)\|_{V^1_{-\beta}(\mathcal{K})^\ell}, \qquad (4.15)$$

for $|\xi| < 1/2$, where c is independent of x and ξ.

According to Theorem 4.3, the problem

$$b_{\mathcal{K}}(u,v) = (\psi F, v)_{\mathcal{K}}\,, \quad v \in V^1_{-\beta}(\mathcal{K})^\ell,$$

has a unique solution $u \in V^1_\beta(\mathcal{K})^\ell$ for arbitrary $F \in V^{-1}_\beta(\mathcal{K})^\ell$. This solution can be written as

$$u(y) = \big(\psi(\cdot)\, F(\cdot)\,,\, \overline{G(y,\cdot)}\big)_{\mathcal{K}}.$$

Let χ_1, χ_2 be a smooth cut-off function, $\chi_2 = 1$ near x, $\chi_1 = 1$ in a neighborhood of $\operatorname{supp}\chi_2$, $\chi_1(y) = 0$ for $|x-y| > 1/4$. Since $\operatorname{supp}\psi \cap \operatorname{supp}\chi_1 = \emptyset$, we have $\chi_1 Lu = 0$ and $\chi_1 Bu|_{\partial\mathcal{K}\setminus\mathcal{S}} = 0$. Hence, by Corollaries 4.2 and 4.3, we obtain $\chi_2\partial_x^\alpha u \in W^l_{\beta'',\vec{\delta}+|\alpha|\,\vec{1}}(\mathcal{G})^\ell$ and $\chi_2\rho\partial_\rho\partial_x^\alpha u \in W^l_{\beta'',\vec{\delta}+|\alpha|\,\vec{1}}(\mathcal{G})^\ell$ with arbitrary β''. Consequently, Lemma 4.5 yields

$$\prod_{j=1}^n \big(r_j(x)\big)^{\max(\delta_j+|\alpha|-l+1,0)} \big|\partial_x^\alpha u(x)\big| \le c\,\|\chi_1 u\|_{V^1_\beta(\mathcal{K})^\ell} \le c'\,\|F\|_{V^{-1}_\beta(\mathcal{K})^\ell}$$

Thus, the mapping

$$\begin{aligned} V^{-1}_\beta(\mathcal{K})^\ell \ni F \quad \to \quad & \prod_{j=1}^n r_j(x)^{\max(\delta_j+|\alpha|-l+1,0)}\,\partial_x^\alpha u(x) \\ & = \Big(\prod_{j=1}^n r_j(x)^{\max(\delta_j+|\alpha|-l+1,0)}\,\partial_x^\alpha G(x,\cdot)\,\psi(\cdot),\overline{F}\Big)_{\mathcal{K}} \end{aligned}$$

represents a linear and continuous functional on V^{-1}_β with norm independent of x. Therefore, the function

$$\eta \to \prod_{j=1}^n r_j(x)^{\max(\delta_j+|\alpha|-l+1,0)}\,\psi(\eta)\,\partial_x^\alpha G(x,\eta)$$

has a finite norm independent of x in $V^1_{-\beta}(\mathcal{K})^\ell$. This together with (4.15) yields

$$\big|\partial_x^\alpha\partial_\xi^\gamma G(x,\xi)\big| \le c\prod_{j=0}^n r_j(x)^{\min(l-1-\delta_j-|\alpha|,0)}|\xi|^{\beta'-|\gamma|-1/2}\prod_{j=0}^n\Big(\frac{r_j(\xi)}{\rho(\xi)}\Big)^{\min(l-1-\delta_j-|\gamma|,0)}.$$

Setting $\delta_j = l-1-\mu_j+\varepsilon$ and $\beta' = -\Lambda_- - 1/2 - \varepsilon$, we arrive at the desired estimate for $|x| = 1$, $|\xi| < 1/2$. Using the equality $G(Tx,T\xi) = T^{-1}G(x,\xi)$, we obtain this estimate for arbitrary x and $|\xi| < |x|/2$. The proof for the case $|x| < |\xi|/2$ proceeds analogously. ■

Remark 4.2 The estimates in Theorems 4.6 and 4.7 for the derivatives of Green's function can be improved if the direction of the derivatives is tangential to edges. In particular, we have

$$\begin{aligned} \big|\partial_\rho G(x,\xi)\big| &\le c\,|x-\xi|^{-2} \quad \text{if } |x|/2 < |\xi| < 2|x|, \\ \big|\partial_\rho G(x,\xi)\big| &\le c\,|x|^{\Lambda_+-1-\varepsilon}\,|\xi|^{-1-\Lambda_++\varepsilon} \quad \text{if } |x| < |\xi|/2, \\ \big|\partial_\rho G(x,\xi)\big| &\le c\,|x|^{\Lambda_--1+\varepsilon}\,|\xi|^{-1-\Lambda_--\varepsilon} \quad \text{if } |x| > 2|\xi|. \end{aligned}$$

The first estimate follows immediately from Theorems 2.4 and 2.5, while the last two estimates can be proved analogously to Theorem 4.7.

Finally, we consider Green's matrix for the case $\beta = 0$. This means that $G(x,\xi)$ is a solution of problem (1.12), (1.14) such that the function $x \to \zeta\Big(\frac{|x-\xi|}{r(\xi)}\Big)\,G(x,\xi)$ belongs to $\mathcal{H} = V^1_0(\mathcal{K})^\ell$. If condition (4.13) is satisfied, then the strip $-1/2 \le \operatorname{Re}\lambda \le 0$ contains only the eigenvalue $\Lambda_1 = 0$ (see [8, Th.12.3.2,12.3.3]). The eigenvectors corresponding to this eigenvalue are the constant vectors in $\mathbb{C}^\ell$, while generalized eigenvectors do not exist. By [8, Th.4.3.1], the same is true for the Neumann problem to the Lamé system. In this case, we denote by Λ_2 the eigenvalue with smallest positive real part. Using the following lemma, we can improve the estimates in Theorem 4.7.

Lemma 4.6 *Let ϕ, ψ be smooth functions on $\overline{\mathcal{K}}$ with compact supports such that $\phi = 1$ in a neighborhood of the origin and $\psi = 1$ in a neighborhood of* $\operatorname{supp}\phi$. *Furthermore, let $\psi u \in V_0^1(\mathcal{K})^\ell$, $\psi L u = 0$ and $\psi B u|_{\partial\mathcal{K}\setminus\mathcal{S}} = 0$. Suppose that the strip $-1/2 \le \operatorname{Re}\lambda \le 0$ contains only the eigenvalue $\Lambda_1 = 0$, the eigenvectors corresponding to this eigenvalue are the constant vectors in $\mathbb{C}^\ell$ and generalized eigenvectors for Λ_1 do not exist. Then $\phi u = c_0 + v$, where c_0 is a constant vector, $v \in W^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell$, $0 < l-\beta-3/2 < \operatorname{Re}\Lambda_2$, $\max(l-1-\mu_j, 0) < \delta_j < l-1$, δ_j not integer, and*

$$|c_0| + \|v\|_{W^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \le c\, \|\psi u\|_{V_0^1(\mathcal{K})^\ell}\,.$$

Proof: Let χ be a smooth function such that $\chi = 1$ in a neighborhood of $\operatorname{supp}\phi$ and $\psi = 1$ in a neighborhood of $\operatorname{supp}\chi$. Since the derivatives of χ vanish in a neighborhood of the origin and of infinity, we have $L(\chi u) = [L,\chi]u \in W^0_{\beta-l+2,\vec{0}}(\mathcal{K})^\ell$ (here $[L,\chi] = L\chi - \chi L$ is the commutator of L and χ) and $B(\chi u)|_{\Gamma_j} \in W^{1/2}_{\beta-l+2,\vec{0}}(\Gamma_j)^\ell$. Thus, it follows from Theorems 4.2 and 4.4 that $\chi u = c_0 + w$, where $w \in W^2_{\beta-l+2,\vec{\delta}}(\mathcal{K})^\ell$, $0 < l-\beta-3/2 < \operatorname{Re}\Lambda_2$, $\max(0, 1-\mu_j) < \delta_j < 1$,

$$|c_0| + \|v\|_{W^2_{\beta,\vec{\delta}}(\mathcal{K})^\ell} \le c\, \|\psi u\|_{V_0^1(\mathcal{K})^\ell}\,.$$

This implies $L(\phi u) = [L,\phi]\,(w-c_0) \in W^1_{\beta-l+3,\vec{\delta}}(\mathcal{K})^\ell$ and $B(\phi u) = [B,\phi]\,(w-c_0) \in W^{3/2}_{\beta-l+3,\vec{\delta}}(\Gamma_j)^\ell$. Using again Theorems 4.2 and 4.4, we obtain $\phi u = c_0 + v$, where $v \in W^3_{\beta-l+3,\vec{\delta}'}(\mathcal{K})^\ell$, $\max(0, 2-\mu_j) < \delta'_j < 2$. Repeating this argument, we get the same representation with $v \in W^l_{\beta,\vec{\delta}}(\mathcal{K})^\ell$, $\max(l-1-\mu_j, 0) < \delta_j < l-1$. ■

Using the last lemma, we can prove the following statement analogously to Theorem 4.7.

Theorem 4.8 *Let $G(x,\xi)$ be Green's matrix introduced in Theorem 4.5 for $\beta = 0$. Suppose that the strip $-1/2 < \operatorname{Re}\lambda \le 0$ contains only the eigenvalue $\Lambda_1 = 0$, the eigenvectors corresponding to this eigenvalue are the constant vectors in $\mathbb{C}^\ell$ and generalized eigenvectors for Λ_1 do not exist. If $|x| < |\xi|/2$, then*

$$\big|\partial_\xi^\gamma G(x,\xi)\big| \le c\, |\xi|^{-1-|\gamma|} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{\delta_{j,\gamma}},$$

$$\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \le c\, |x|^{\operatorname{Re}\Lambda_2-|\alpha|-\varepsilon}\ |\xi|^{-1-\operatorname{Re}\Lambda_2-|\gamma|+\varepsilon} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{\delta_{j,\gamma}}$$

for $|\alpha| \ne 0$, where $\delta_{j,\alpha} = \min(0, \mu_j - |\alpha| - \varepsilon)$ and ε is an arbitrarily small positive number. Analogously for $|x| > 2|\xi|$ there are the estimates

$$\big|\partial_x^\alpha G(x,\xi)\big| \le c\, |x|^{-1-|\alpha|} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}},$$

$$\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \le c\, |x|^{-1-\operatorname{Re}\Lambda_2-|\gamma|+\varepsilon}\ |\xi|^{\operatorname{Re}\Lambda_2-|\alpha|-\varepsilon} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{\delta_{j,\gamma}}$$

for $|\gamma| \ne 0$.

Remark 4.3 If for the Neumann problem in the dihedron $\mathcal{D}_j$ the assumptions of Theorem 2.3 are valid (i.e., in particular, $\mu_j = 1$ is the eigenvalue of $A_j(\lambda)$ with smallest positive real part), then $G(x,\xi)$ satisfies the estimates in Theorems 4.6–4.8 with $\delta_{j,\alpha} = \min(0, \mu_j^{(2)} - |\alpha| - \varepsilon)$, where $\mu_j^{(2)}$ is the real part of the first eigenvalue of the pencil $A_j(\lambda)$ on the right of the line $\operatorname{Re}\lambda = 1$ (cf. Remark 4.1).

Examples. Let $G(x,\xi)$ be Green's function for problem (1.1), (1.3) such that the function $x \to \zeta\Big(\frac{|x-\xi|}{r(\xi)}\Big)\, G(x,\xi)$ belongs to the space $\mathcal{H} = V_0^1(\mathcal{K})^\ell$ for every fixed $\xi \in \mathcal{K}$ and for every smooth function ζ on $(0,\infty)$ equal to one in $(1,\infty)$ and to zero in $(0,\frac12)$.

1) If $L = -\Delta$, then $G(x,\xi)$ satisfies the estimates in Theorems 4.6 and 4.8 with $\delta_{j,\alpha} = \min(0, \pi/\theta_j - |\alpha| - \varepsilon)$, where θ_j is the angle at the edge M_j and ε is an arbitrarily small positive number. Here Λ_2 is the smallest positive eigenvalue of the pencil $\mathfrak{A}$. Note that the eigenvalues of the pencil $\mathfrak{A}$ are given by $\Lambda_{\pm j} = -1/2 \pm \sqrt{N_j + 1/4}$, where N_j are the eigenvalues of the Beltrami operator on Ω with Neumann boundary conditions.

2) Green's matrix for the Neumann problem to the Lamé system satisfies the estimates in Theorems 4.6 and 4.8, where $\delta_{j,\alpha} = \min(0, \pi/\theta_j - |\alpha| - \varepsilon)$ for $\theta_j < \pi$ and $\delta_{j,\alpha} = \min(0, \xi_+(\theta_j)/\theta_j - |\alpha| - \varepsilon)$ for $\theta_j > \pi$. Here $\xi_+(\theta)$ is the smallest positive root of (1.8).

4.6. Estimates for Green's matrix of the Dirichlet and mixed problems

Let

$$\Gamma^\circ = \bigcup_{j\in J_0} \Gamma_j \quad \text{and} \quad \overset{\circ}{V}{}^1_\beta(\mathcal{K};\Gamma^\circ) = \{u \in V_\beta^1(\mathcal{K}) :\ u = 0 \text{ on } \Gamma^\circ\}.$$

Analogously to Theorem 4.3, it can be proved that the problem

$$b_{\mathcal{K}}(u,v) = (F,v) \ \text{ for all } v \in \overset{\circ}{V}{}^1_{-\beta}(\mathcal{K};\Gamma^\circ)^\ell, \quad u = g_j \text{ on } \Gamma_j \text{ for } j \in J_0$$

has a unique solution $u \in V_\beta^1(\mathcal{K})^\ell$ for arbitrary $F \in (\overset{\circ}{V}{}^1_\beta(\mathcal{K};\Gamma^\circ)^*)^\ell$ and $g_j \in V_\beta^{1/2}(\Gamma_j)^\ell$, $j \in J_0$, if the line $\mathrm{Re}\,\lambda = -\beta - 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$. We call this solution weak solution of problem (1.1)–(1.3).

For weak solutions of the Dirichlet and mixed problems we can prove analogous regularity assertions as for weak solutions of the Neumann problem. In particular, the following statement holds (cf. Corollary 4.2).

Theorem 4.9 *Let $u \in V^1_{\beta-l+1}(\mathcal{K})^\ell$ be a weak solution of problem (1.1)–(1.3) with $(\rho\partial_\rho)^\nu f \in \mathcal{W}^{l-2}_{\beta,\vec\delta}(\mathcal{K};\tilde J)^\ell$ for $\nu = 0,1,\dots,k$, $(\rho\partial_\rho)^\nu g_j \in V^{l-1/2}_{\beta,\vec\delta}(\Gamma_j)^\ell$ for $j \in J_0$, $\nu = 0,1,\dots,k$, $(\rho\partial_\rho)^\nu g_j \in \mathcal{W}^{l-3/2}_{\beta,\vec\delta}(\Gamma_j;\tilde J)^\ell$ for $j \in J_1$, $\nu = 0,\dots,k$. Suppose that the components of $\vec\delta$ satisfy the inequalities $l-1-\mu_j < \delta_j < l-1$ for $j \in \tilde J$, $\max(l-1-\mu_j, 0) < \delta_j < l-1$ for $j \notin \tilde J$ and that there are no eigenvalues of the pencil $\mathfrak{A}$ on the line $\mathrm{Re}\,\lambda = l-\beta-3/2$. Then $(\rho\partial_\rho)^\nu u \in \mathcal{W}^l_{\beta,\vec\delta}(\mathcal{K};\tilde J)^\ell$ for $\nu = 0,1,\dots,k$ and*

$$\begin{aligned}\sum_{\nu=0}^k \|(\rho\partial_\rho)^\nu u\|_{\mathcal{W}^l_{\beta,\vec\delta}(\mathcal{K};\tilde J)^\ell} \ &\le\ c\sum_{\nu=0}^k \Big(\|(\rho\partial_\rho)^\nu f\|_{\mathcal{W}^{l-2}_{\beta,\vec\delta}(\mathcal{K};\tilde J)^\ell} + \sum_{j\in J_0} \|(\rho\partial_\rho)^\nu g_j\|_{V^{l-1/2}_{\beta,\vec\delta}(\Gamma_j)^\ell} \\ &\qquad + \sum_{j\in J_0} \|(\rho\partial_\rho)^\nu g_j\|_{\mathcal{W}^{l-3/2}_{\beta,\vec\delta}(\Gamma_j;\tilde J)^\ell}\Big).\end{aligned}$$

Suppose that there are no eigenvalues of the pencil $\mathfrak{A}$ on the line $\mathrm{Re}\,\lambda = -\beta - 1/2$. We denote by $\Lambda_- < \mathrm{Re}\,\lambda < \Lambda_+$ the widest strip in the complex plane which is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and contains the line $\mathrm{Re}\,\lambda = -\beta - 1/2$. Furthermore, let $G(x,\xi)$ be the unique solution of the boundary value problem (1.12)–(1.14) such that the function $x \to \zeta\big(\frac{|x-\xi|}{r(\xi)}\big)\, G(x,\xi)$ belongs to $V_\beta^1(\mathcal{K})^{\ell\times\ell}$. The following estimates follow immediately from Theorem 2.8.

$$\begin{aligned}\big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \ &\le\ c\,|x-\xi|^{-1-|\alpha|-|\gamma|} \ \text{ if } |x|/2 < |\xi| < 2|x|,\ |x-\xi| < \min(r(x), r(\xi)), \\ \big|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\big| \ &\le\ c\,|x-\xi|^{-1-|\alpha|-|\gamma|} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x-\xi|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|x-\xi|}\Big)^{\delta_{j,\gamma}} \\ &\qquad \text{if } |x|/2 < |\xi| < 2|x|,\ |x-\xi| > \min(r(x), r(\xi)).\end{aligned}$$

Here $\delta_{j,\alpha} = \mu_j - |\alpha| - \varepsilon$ for $j \in \tilde{J}$, $\delta_{j,\alpha} = \min(0, \mu_j - |\alpha| - \varepsilon)$ for $j \notin \tilde{J}$, and ε is an arbitrarily small positive number. Estimates for Green's function in the cases $|x| < |\xi|/2$ and $|x| > 2|\xi|$ can be proved analogously to Theorem 4.7 by means of Theorem 4.9. In the case $|x| < |\xi|/2$ we obtain

$$\left|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\right| \le c\, |x|^{\Lambda_+ - |\alpha| - \varepsilon}\ |\xi|^{-1-\Lambda_+ - |\gamma| + \varepsilon} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{\delta_{j,\gamma}},$$

while for $|x| > 2|\xi|$ there is the estimate

$$\left|\partial_x^\alpha \partial_\xi^\gamma G(x,\xi)\right| \le c\, |x|^{\Lambda_- - |\alpha| + \varepsilon}\ |\xi|^{-1-\Lambda_- - |\gamma| - \varepsilon} \prod_{j=1}^n \Big(\frac{r_j(x)}{|x|}\Big)^{\delta_{j,\alpha}} \prod_{j=1}^n \Big(\frac{r_j(\xi)}{|\xi|}\Big)^{\delta_{j,\gamma}}.$$

References

[1] Agranovich, M. S., Vishik, M. I., *Elliptic problems with parameter and parabolic problems of general type,* Uspekhi Mat. Nauk **19** (1964) 3, 53-161.

[2] Dauge, M., *Problemes de Neumann et de Dirichlet sur un polyedre dans R^3 : regularite dans des espaces de Sobolev L_p*, C. R. Acad. Sci. Paris **307** (1988) 1, 27-32.

[3] Dauge, M., *Neumann problem on polyhedra in L^p Sobolev spaces*, Seminaire "Equations aux derivees partielles" Univ. de Nantes 1 (1988)

[4] Dauge, M., *Neumann and mixed problems on curvilinear polyhedra*, Integr. Equ. Oper. Th. **15** (1992) 227-261.

[5] Grachev, N. V., Maz′ya, V. G., *Estimates for the Neumann problem in a polyhedral cone*, Preprint LiTH-MAT-R-91-28, University of Linköping 1991.

[6] Kondrat′ev, V. A., *Boundary value problems for elliptic equations in domains with conical or angular points,* Trudy Moskov. Mat. Obshch. **16** (1967) 209-292; English transl. in Trans. Moscow Math. Soc. **16** (1967) 227-313.

[7] Kozlov, V. A., Maz′ya, V. G., Roßmann, J., *Elliptic boundary value problems in domains with point singularities,* Mathematical Surveys and Monographs **52**, Amer. Math. Soc., Providence, Rhode Island 1997.

[8] Kozlov, V. A., Maz′ya, V. G., Roßmann, J., *Spectral problems associated with corner singularities of solutions to elliptic equations*, Mathematical Surveys and Monographs **85**, Amer. Math. Soc., Providence, Rhode Island 2001.

[9] Maz′ya, V. G., Plamenevskiĭ, B. A., *L_p estimates of solutions of elliptic boundary value problems in domains with edges*, Trudy Moskov. Mat. Obshch. **37** (1978) 49-93; English transl. in: Trans. Moscow Math. Soc. **37** (1980) 49-97.

[10] Maz′ya, V. G., Plamenevskiĭ, B. A., *Estimates of the Green function and Schauder estimates for solutions of elliptic boundary value problems in a dihedral angle*, Sibirsk. Mat. Zh. **19** (1978) 5, 1065-1082 (in Russian).

[11] Maz′ya, V. G., Plamenevskiĭ, B. A., *Estimates in L_p and Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary,* Math. Nachr. **81** (1978) 25-82; English transl. in: Amer. Math. Soc. Transl., Vol. 123 (1984) 1-56.

[12] Maz′ya, V. G., Plamenevskiĭ, B. A., *On the asymptotics of the fundamental solutions of elliptic boundary value problems in regions with conical points*, Probl. Mat. Anal. **7** (1979) 100-145, English transl. in: Sel. Math. Sov. **4** (1985) 4, 363-397.

[13] Maz′ya, V. G., Plamenevskiĭ, B. A., *The first boundary value problem for the classical equations of mathematical physics on piecewise smooth domains,* Part I: Z. Anal. Anwendungen **2** (1983) 335-359, Part 2: Z. Anal. Anwendungen **2** (1983) 523-551 (in Russian).

[14] Maz′ya, V. G., Roßmann, J., *Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten*, Math. Nachr. **138** (1988) 27-53.

[15] Maz′ya, V. G., Roßmann, J., *On the Agmon-Miranda maximum principle for solutions of elliptic equations in polyhedral and polygonal domains*, Ann. Global Anal. Geom. **9** (1991) 3, 253-303.

[16] Nazarov, S. A., *Estimates near an edge for for the solution of the Neumann problem for an elliptic system*, Vestnik Leningrad Univ. Math. **21**:1 (1988) 52-59.

[17] Nazarov, S. A., *The polynomial property of self-adjoint elliptic boundary value problems and an algebraic description of their attributes*, Uspekhi Mat. Nauk **54** (1999) 5, 77-142; English transl. in: Russian Mat. Surveys **54** (1999) 5, 947-1014.

[18] Nazarov, S. A., Plamenevskiĭ, B. A., *Elliptic problems in domains with piecewise smooth boundaries,* De Gruyter Expositions in Mathematics **13**, Berlin - New York 1994.

[19] Nicaise, S., *Regularity of the solutions of elliptic systems in polyhedral domains*, Bull. Belg. Math. Soc. **4** (1997) 411-429.

[20] Roßmann, J., *The asymptotics of the solutions of linear elliptic variational problems in domains with edges,* Z. Analysis Anwendungen **9** (1990) 3, 565-575.

[21] Roßmann, J., *On two classes of weighted Sobolev-Slobodezkij spaces in a dihedral angle,* Banach Center Publications **27** Part 2 (1992) 399-424.

[22] Solonnikov, V. A., *Estimates of solutions to the Neumann problem for elliptic second order equations in domains with edges on the boundary*, Preprint P-4-83, Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) (in Russian)

[23] Zajaczkowski, W., Solonnikov, V. A., *Neumann problem for second order elliptic equations in domains with edges on the boundary*, Zap. Nauchn. Semin. Leningrad. Otdel. Mat. Inst. Steklova **127** (1983) 7-48; English transl. in: J. Sov. Math. **27** (1984) 2561-2586.