Another generalization of Lindström's theorem on subcubes of a cube

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Abstract

We consider the poset $P(N; A_1, A_2, \ldots, A_m)$ consisting of all subsets of a finite set N which do not contain any of the A_i's, where the A_i's are mutually disjoint subsets of N. The elements of P are ordered by inclusion. We show that P belongs to the class of Macaulay posets, i.e. we show a Kruskal-Katona type theorem for P. For the case that the A_i's form a partition of N, the dual P^* of P became known as the orthogonal product of simplices. Since the property of being a Macaulay poset is preserved by turning to the dual, we show in particular that orthogonal products of simplices are Macaulay posets. Besides, we prove that the posets P and P^* are additive.

Keywords: Macaulay posets, shadow minimization, Kruskal-Katona theorem, orthogonal product of simplices

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1 Introduction

We study the poset $P(N; A_1, A_2, \ldots, A_m)$ of all subsets of a finite set N which do not contain any of the non-empty, pairwise disjoint subsets $A_1, A_2, \ldots, A_m \subset N$. The elements of P are ordered by inclusion. Our main result says that P belongs to the class of Macaulay posets, i.e. we prove an analogue of the Kruskal-Katona theorem [7, 8] for P. This generalizes a result from [10], where the case $m = 2$ is covered. The proof given in this paper does apply to the case $m \geq 3$, exclusively,

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$m = 2$ required a special treatment. In this sense, we continue the work begun in [10].

In order to define what we mean by a Macaulay poset, we need to introduce a few notions. Let $(P, \leq)$ be a poset. We use the notation $x \lessdot y$ to indicate that $x \neq y$ and that $x \leq z \leq y$ implies $z \in \{x, y\}$, in this case we say that y *covers* x. Furthermore, we assume P to be ranked, i.e. there is a function $r: \ P \to \mathbb{N}$ such that $x \lessdot y$ implies $r(y) = r(x) + 1$, where $\mathbb{N}$ denotes the set of natural numbers (including 0). The *rank* of P is the number $r(P) := \max\{r(x) \mid x \in P\}$, and for $i = 0, 1, \ldots, r(P)$ the i-th *level* of P is the set $N_i(P) := \{x \in P \mid r(x) = i\}$ which sometimes is also denoted by just P_i. For $X \subseteq P$ we use the notation $N_i(X) = X \cap P_i$. The *shadow* of an element $x \in P$ is the set $\Delta(x) := \{y \in P \mid y \leq x \text{ and } r(y) = r(x) - 1\}$, and the shadow of a subset $X \subseteq P$ is $\Delta(X) := \bigcup_{x \in X} \Delta(x)$.

Consider a linear ordering $\prec$ of the elements of P. For $X \subseteq P_i$ let $C(X)$ denote the set of the first $|X|$ elements of P_i w.r.t. $\prec$. The set $C(X)$ is called the *compression* of X, and if $X = C(X)$ holds, then X is called *compressed*. For $\emptyset \subset X \subseteq P$ and $1 \leq m \leq |X|$, the set of the first m elements of X w.r.t. $\prec$ is denoted by $C(m, X)$. If necessary, we will write C_P instead of C.

The poset P is said to be a *Macaulay poset* if the ordering $\prec$ can chosen such that for all $i \in \{1, 2, \ldots, r(P)\}$ and all $X \subseteq P_i$ the following inclusion holds:

$$\Delta(C(X)) \subseteq C(\Delta(X)) \quad . \tag{1.1}$$

In this case, we also say that $(P, \leq, \prec)$ is a *Macaulay structure*.

It is well-known that (1.1) holds for all i and $X \subseteq P_i$ if and only if for $i \in \{1, 2, \ldots, r(P)\}$ and $X \subseteq P_i$ the two conditions

$$|\Delta(C(X))| \leq |\Delta(X)| \tag{1.2}$$

and

$$C(\Delta(C(X))) = \Delta(C(X)) \tag{1.3}$$

are satisfied (cf. [2, 4]). By (1.2), compressed subsets have minimum-sized shadow among all subsets of the same level with fixed cardinality. That means, the solutions to the *Shadow Minimization Problem (SMP)* form a *nested structure* since $C(m, P_i) \subset C(m+1, P_i)$ for $1 \leq m < |P_i|$. By (1.3), shadows of compressed subsets are compressed as well. Therefore, we speak of the *continuity* of the solutions to the SMP.

The dual of $(P, \leq)$ is the poset $(P, \leq^*)$ with $x \leq^* y$ whenever $y \leq x$ holds in P. If it is clear what $\leq$ is, then we will briefly denote $(P, \leq^*)$ by P^*. Obviously,

P^* is ranked with the rank-function defined by $r^*(x) = r(P) - r(x)$ for $x \in P$. Let further be $\prec^*$ be the reverse of $\prec$, i.e. we have $x \prec^* y$ whenever $y \prec x$. It is not hard to show that $(P, \leq^*, \prec^*)$ is a Macaulay structure if and only if $(P, \leq, \prec)$ is a Macaulay structure (see [2] or [4] for proof).

From now on let $P = P(N; A_1, A_2, \ldots, A_m)$. Throughout this paper we use the notations $A_0 := N \setminus (A_1 \cup A_2 \cup \cdots \cup A_m)$ and $k_j = |A_j|$ for $j = 0, 1, \ldots, m$. Clearly, P can be represented as the cartesian product $B_{k_0} \times B'_{k_1} \times \cdots \times B'_{k_m}$, where by B_n we denote the Boolean lattice of order n and by B'_n the Boolean lattice of order n without its maximal element. Hence, P^* can be seen as the product $B_{k_0} \times (B'_{k_1})^* \times \cdots \times (B'_{k_m})^*$, where $(B'_n)^*$ stands for a Boolean lattice of order n without its minimal element.

For number of special choices of the parameters, P (or, equivalently, P^*) have been considered w.r.t. the property of being a Macaulay poset:

In the special case $k_0 = 0$, $k_1 = k_2 = \cdots = k_m = 2$ the poset P^* is isomorphic to the poset formed by all subcubes of an m-cube ordered by inclusion. In this case, a linear ordering $\prec$ for which (1.1) holds has been introduced by Lindström [12]. (The proof of (1.1) given in [12], however, contains a gap, as the author himself pointed out later.) His result has been generalized to cartesian powers of stars by Leeb [11] and, independently, by Bezrukov [1]. Essentially the same, but in the dual version, has been found in [5]. The *colored complexes* introduced there are cartesian products of stars of *almost* equal size. This case, however, is somehow covered by the result for powers of stars because colored complexes occur as left-compressed ideals there, as one can easily derive from the definition of the corresponding ordering $\prec$. The observation that colored complexes are the duals of the star powers in [11, 1] is due to Engel [4]. Finally, it has been shown in [9] that products of stars of arbitrary sizes are Macaulay posets. To avoid confusion, we recommend the study of chapter 8 of [4], where all these results are summarized in a much more detailed way. Not recorded there is a very recent generalization of the Leeb-Bezrukov result: Bezrukov and Elsässer [3] proved that powers of spiders are Macaulay posets.

Sali [14] investigated P^* in the special case $m = 2$, $k_0 = 0$. He interpreted P^* as the poset of all submatrices of a matrix ordered by containment. Furthermore, he conjectured a Kruskal-Katona type theorem to hold and suggested a linear order. His conjecture was proved in [10], also if the condition $k_0 = 0$ is omitted. In the case $m \geq 2$, $k_0 = 0$ (that is, if the "forbidden" A_i's form a partition of N), the poset P^* became known as the *orthogonal product of simplices* the study of which has been suggested by Harper. Two of his former PhD students provided essential contributions: Moghadam [13] settled the SMP in several special cases, and Vasta [15] gave a solution to the related *Maximum Rank Ideal Problem (MRI)*.

For an overview we refer to the forthcoming monograph [6]. Finally, note that the main theorem of this paper and a general theorem of Engel [4] on the *Maximum Weight Ideal Problem* immediately yield a generalization of the mentioned result by Vasta. (Vasta's result covers the case of that the weight function is equal to the rank function.)

2 The main result

We will now introduce a linear ordering $\prec$ of the elements of P. Throughout we assume that N consists of positive integers, where $n := |N| = k_0 + k_1 + \cdots + k_m$. Hence, $r(P) = n - m$. If we just want to indicate that we are considering the poset P with these parameters, then instead of $P(N; A_1, \ldots, A_m)$ we will sometimes use the more abstract notation $P(n; k_1, \ldots, k_m)$. Furthermore, we suppose $2 \leq k_1 \leq k_2 \leq \cdots \leq k_m$, and that for $j = 1, 2, \ldots, m$ the smallest element of A_j is greater than the largest element of A_{j-1}, i.e.

$$A_j = \{a_j^1, a_j^2, \ldots, a_j^{k_j}\}$$

with $a_j^1 < a_j^2 < \cdots < a_j^{k_j}$ for $j = 0, 1, \ldots, m$ and $a_{j-1}^{k_{j-1}} < a_j^1$ for $j = 1, 2, \ldots, m$. Sometimes (when considering subposets of P) we could possibly run into the case $k_1 = 1$. For this case, note that then P is equal to $P(N \setminus \{A_1\}; A_2, \ldots, A_m)$.

For $F \in P$ and $j = 1, 2, \ldots, m$ we define $a_j(F) := \max(A_j \setminus F)$ and

$$A(F) := \{a_1(F), a_2(F), \ldots, a_m(F)\} \quad .$$

Furthermore, for $A \in A_1 \times A_2 \times \cdots \times A_m$ we use the notation

$$B(A) := \{F \in P \mid A(F) = A\} \quad .$$

Note that, if $A = \{a_1^{t_1}, a_2^{t_2}, \ldots, a_m^{t_m}\}$, then the elements of $B(A)$ form a Boolean lattice of order $k_0 + \sum_{j=1}^{m}(t_j - 1)$ as a subposet of P.

Our definition of $\prec$ involves the *reverse-lexicographic ordering* which for any $F, G \subseteq N$ is given by

$$F \prec_{r\ell} G \iff \max(F \setminus G) < \max(G \setminus F) \quad .$$

Now we can establish the ordering $\prec$ on P by the following two conditions:

(1) $F \prec G$ whenever $A(F) \neq A(G)$ and $\min[A(F) \setminus A(G)] > \min[A(G) \setminus A(F)]$,

(2) $F \prec G$ whenever $A(F) = A(G)$ and $F \prec_{r\ell} G$.

If it is necessary to indicate that we are considering the ordering $\prec$ *on* P, then we will use the notation $\prec_P$.

The main result of this paper is the following theorem which says that the triple $(P, \subseteq, \prec)$ is a Macaulay structure.

Theorem 2.1 *The inclusion* $\Delta(C(\mathcal{F})) \subseteq C(\Delta(\mathcal{F}))$ *holds for all* $\mathcal{F} \subseteq P_i$ *and all* $i \in \{1, 2, \ldots, n-m\}$ *w.r.t. the linear ordering* $\prec$.

The above result has been conjectured in an equivalent form by Moghadam [13]. Therefore, the problem of proving such a theorem is also referred to as *Moghadam's problem*, for instance in [15].

3 Some preparations

In the sequel, we will make use of a few more definitions and notations. A *segment* $\mathcal{S}$ is a subset of some level P_i of P which consists of elements that are consecutive w.r.t. $\prec$. In particular, we call $\mathcal{S}$ an *initial segment* resp. *final segment* if it consists of the first resp. last elements of P_i w.r.t. $\prec$. More generally, if $\mathcal{S}_1, \mathcal{S}_2 \subseteq P_i$ are segments, we say that $\mathcal{S}_1$ is an initial (resp. final) segment of $\mathcal{S}_2$ if $\mathcal{S}_1$ consists of the first (resp. last) elements of $\mathcal{S}_2$. The *new-shadow* $\Delta_{new}(F)$ of an element $F \in P_i$ is the set of all members of $\Delta(F)$ which are not contained in the shadow of any element of P_i preceding F in the ordering $\prec$. The new-shadow of a subset $\mathcal{F} \subseteq P_i$ is defined by $\Delta_{new}(\mathcal{F}) := \bigcup_{F \in \mathcal{F}} \Delta_{new}(F)$. Finally, for two subsets $\mathcal{F}, \mathcal{G} \subseteq P$ we use the notation $\mathcal{F} \prec \mathcal{G}$ to indicate that the last element of $\mathcal{F}$ precedes the first element of $\mathcal{G}$ in the order $\prec$.

The following observation is immediately from the definition of $\prec$ and the well-known fact that (1.3) holds for Boolean lattices w.r.t. the reverse-lexicographic order (see [4] for instance).

Proposition 3.1 *Let* $i \in \{1, 2, \ldots, n-m\}$ *and* $A \in A_1 \times A_2 \times \cdots \times A_m$ *such that* $N_{i-1}(B(A))$ *and* $N_i(B(A))$ *are both non-empty. Furthermore, let* $\mathcal{S}$ *be an initial segment of* $N_i(B(A))$.

(a) $\Delta_{new}(N_i(B(A))) = N_{i-1}(B(A))$ *holds.*

(b) $\Delta_{new}(\mathcal{S})$ *is an initial segment of* $N_{i-1}(B(A))$.

As a corollary we obtain that (1.3) is satisfied for P w.r.t. $\prec$.

Corollary 3.2 *The equation* $C(\Delta(C(\mathcal{F}))) = \Delta(C(\mathcal{F}))$ *is satisfied for all* $\mathcal{F} \subseteq P_i$ *and all* $i \in \{1, 2, \ldots, n-m\}$.

The next observation is also easy to verify. It will be important in proving Theorem 2.1, in particular for the partial compression in Section 5.

Proposition 3.3 *Let* A *be a singleton subset of* A_0 *or one of the sets* $A_1, A_2, \ldots, A_m$. *Furthermore, let* $P' := P(N \setminus A; A_1, A_2, \ldots, A_m)$ *if* A *is a singleton subset of* A_0, *and let* $P' := P(N \setminus A_j; A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_m)$ *if* $A = A_j$ *with* $j \in \{1, 2, \ldots, m\}$. *Suppose further that* $F, G \in P$ *such that* $F \cap A = G \cap A$. *Then* $F \prec_P G$ *holds if and only if* $(F \setminus A) \prec_{P'} (G \setminus A)$.

For the forthcoming inductions in Sections 4 and 5, the partition of P which we are going to introduce now is crucial. For $\ell = 1, 2, \ldots, k_1$ let the subset $\mathcal{B}(\ell) \subseteq P$ be defined by

$$\mathcal{B}(\ell) := \bigcup_{\substack{A \in A_1 \times \cdots \times A_m, \\ a_1^\ell \in A}} B(A) \quad .$$

In other words, $\mathcal{B}(\ell)$ is the collection of all $F \in P$ such that the greatest element of A_1 which is *not* contained in F is a_1^ℓ. Clearly,

$$P = \mathcal{B}(k_1) \cup \mathcal{B}(k_1 - 1) \cup \cdots \cup \mathcal{B}(1)$$

is a partition of P. From the definitions of p and $\prec$ we obtain the following proposition.

Proposition 3.4 *The sets* $\mathcal{B}(\ell)$ $(\ell = 1, 2, \ldots, k_1)$ *have the following properties:*

(a) $\mathcal{B}(k_1) \prec \mathcal{B}(k_1 - 1) \prec \cdots \prec \mathcal{B}(1)$,

(b) $P \setminus \mathcal{B}(1)$ *is equal to* $P' = P(N;\ A_1 \setminus \{a_1^{k_1}\}, A_2, \ldots, A_m)$. *Furthermore, for* $F, G \in P'$ *we have* $F \prec_P G$ *if and only if* $F \prec_{P'} G$.

(c) $P \setminus \mathcal{B}(k_1)$ *consists of all* $F \cup \{a_1^{k_1}\}$ *with*

$$F \in P' = P(N \setminus \{a_1^{k_1}\};\ A_1 \setminus \{a_1^{k_1}\}, A_2, \ldots, A_m).$$

Furthermore, for $F, G \in P'$ *we have* $(F \cup \{a_1^{k_1}\}) \prec_P (G \cup \{a_1^{k_1}\})$ *if and only if* $F \prec_{P'} G$.

Sometimes it will be necessary to refine the above partition, i.e. to consider a partition of $\mathcal{B}(\ell)$. Let $\ell \in \{1, 2, \dots, k_1\}$. For $h = 1, 2, \dots, k_2$ define

$$\mathcal{B}(\ell, h) := \bigcup_{\substack{A \in A_1 \times \cdots \times A_m, \\ a_1^\ell, a_2^h \in A}} B(A) \quad .$$

That means, $\mathcal{B}(\ell, h)$ consists of all $F \in P$ with $\max(A_1 \setminus F) = a_1^\ell$ and $\max(A_2 \setminus F) = a_2^h$. Hence,

$$\mathcal{B}(\ell) = \mathcal{B}(\ell, k_2) \cup \mathcal{B}(\ell, k_2 - 1) \cup \cdots \cup \mathcal{B}(\ell, 1)$$

is a partition of $\mathcal{B}(\ell)$. This partition has the following properties.

Proposition 3.5 *Let $\ell \in \{1, 2, \dots, k_1\}$.*

(a) $\mathcal{B}(\ell, k_2) \prec \mathcal{B}(\ell, k_2 - 1) \prec \cdots \prec \mathcal{B}(\ell, 1)$,

(b) $\mathcal{B}(\ell) \setminus \mathcal{B}(\ell, 1)$ *consists of all* $F \cup \{a_1^{\ell+1}, a_1^{\ell+2}, \dots, a_1^{k_1}\}$ *with*

$$F \in P' = P\left(N \setminus \{a_1^\ell, a_1^{\ell+1}, \dots, a_1^{k_1}\};\ A_2 \setminus \{a_2^1\}, A_3, \dots, A_m\right) \quad .$$

Furthermore, for $F, G \in P'$ we have

$$\left(F \cup \{a_1^{\ell+1}, a_1^{\ell+2}, \dots, a_1^{k_1}\}\right) \prec_P \left(G \cup \{a_1^{\ell+1}, a_1^{\ell+2}, \dots, a_1^{k_1}\}\right)$$

if and only if $F \prec_{P'} G$.

(c) $\mathcal{B}(\ell) \setminus \mathcal{B}(\ell, k_2)$ *consists of all* $F \cup \{a_1^{\ell+1}, a_1^{\ell+2}, \dots, a_1^{k_1}\} \cup \{a_2^{k_2}\}$ *with*

$$F \in P' = P\left(N \setminus (\{a_1^\ell, a_1^{\ell+1}, \dots, a_1^{k_1}\} \cup \{a_2^{k_2}\});\ A_2 \setminus \{a_2^{k_2}\}, A_3, \dots, A_m\right) \quad .$$

Furthermore, for $F, G \in P'$ we have

$$\left(F \cup \{a_1^{\ell+1}, a_1^{\ell+2}, \dots, a_1^{k_1}\} \cup \{a_2^{k_2}\}\right) \prec_P \left(G \cup \{a_1^{\ell+1}, a_1^{\ell+2}, \dots, a_1^{k_1}\} \cup \{a_2^{k_2}\}\right)$$

if and only if $F \prec_{P'} G$.

Propositions 3.5 (b) and (c) imply an important observation.

Corollary 3.6 *For $\ell = 1, 2, \dots, k_1 - 1$ the subposets $\mathcal{B}(\ell + 1) \setminus \mathcal{B}(\ell + 1, k_2)$ and $\mathcal{B}(\ell) \setminus \mathcal{B}(\ell, 1)$ are both isomorphic to $P(n - k_1 + \ell - 1;\ k_2 - 1, k_3, \dots, k_m)$.*

We conclude the section with another statement we will need in proving the main theorem.

Lemma 3.7 *Let $i \in \{1, 2, \ldots, n-m\}$, and let $F \in P_i$.*

(a) Suppose that $k_0 = 0$ and that $F \notin \mathcal{B}(1)$. Then $\Delta_{new}(F) \neq \emptyset$ holds if and only if $a_1^1 \in F$.

(b) Suppose that $k_0 \geq 1$. Then $\Delta_{new}(F) \neq \emptyset$ holds if and only if $a_0^1 \in F$.

Proof. (a) Let $k_0 = 0$, and let $H \in P_{i-1} \setminus \mathcal{B}(1)$. The assertion is implied by the following statement which is easily verified: The first element of P_i w.r.t. $\prec$ which contains H as a subset is $H \cup \{a_1^1\}$.

(b) If $k_0 \geq 1$, then the claim follows by the simple observation that for every $H \in P_{i-1}$ with $a_0^1 \in H$ the first element of P_i w.r.t. $\prec$ which contains H as a subset is $H \cup \{a_0^1\}$. ∎

4 Additivity of the shadow-function

In this section we will establish the additivity of the so-called shadow-function on the level P_i. In general, this is a useful property in many applications (see [4]). In particular, we will make use of it in the proof of Theorem 2.1.

Let $i \in \{1, 2, \ldots, n-m\}$. The *shadow-function* sf_i is defined for $t = 1, 2, \ldots, |P_i|$ by

$$\mathrm{sf}_i(t) := |\Delta(C(t, P_i))| \quad .$$

The function sf_i is called *little-submodular* if for all $1 \leq t_1, t_2 \leq |P_i|$ the inequality

$$\mathrm{sf}_i(t_1) + \mathrm{sf}_i(t_2) \geq \begin{cases} \mathrm{sf}_i(t_1 + t_2) & \text{if } t_1 + t_2 \leq |P_i|, \\ |P_{i-1}| + \mathrm{sf}_i(t_1 + t_2 - |P_i|) & \text{if } t_1 + t_2 > |P_i| \end{cases}$$

holds. sf_i is said to be *additive* if the inequality

$$|\Delta(\mathcal{S}_1)| \geq |\Delta_{new}(\mathcal{S}_2)| \geq |\Delta_{new}(\mathcal{S}_3)| \tag{4.1}$$

is satisfied for all segments $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \subseteq P_i$ with $|\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3|$, where $\mathcal{S}_1$ is initial and $\mathcal{S}_3$ is final.

The following observation is due to Engel [4] (of course, not only for the poset P).

Lemma 4.1 *Let $i \in \{1, 2, \ldots, n-m\}$. The shadow-function sf_i is additive if and only if it is little-submodular.*

Theorem 4.2 *The shadow function* sf_i *is additive for all* $i \in \{1, 2, \ldots, n-m\}$.

Proof. Let $i \in \{1, 2, \ldots, n-m\}$, and let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \subseteq P_i$ be like above. We have to show that (4.1) holds.

We proceed by double induction on m and $s := |\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3|$. If $m = 0$, then P is a Boolean lattice and Theorem 4.2 is well-known (see [4] for instance). For $m = 1$ Theorem 4.2 has been proven to be true in [10]. Hence, we assume $m \geq 2$ and that the assertion holds for all P' with $m' < m$. If $s = 1$, then $|\Delta(\mathcal{S}_1)| = i$, $|\Delta_{new}(\mathcal{S}_3)| = 0$, and (4.1) is satisfied thereby. Consequently, we assume that $s \geq 2$ and that (4.1) holds for segments of cardinality $s' < s$.

Furthermore, without loss of generality we can assume that either $k_1 = 2$ or that the assertion is true for all P' with $m' = m$ and $k_1' < k_1$.

1. We first show $|\Delta(\mathcal{S}_1)| \geq |\Delta_{new}(\mathcal{S}_2)|$. By the induction hypothesis (induction on s), it suffices to find a final segment $\mathcal{S}_1'$ of $\mathcal{S}_1$ and an initial or final segment $\mathcal{S}_2'$ of $\mathcal{S}_2$ such that $|\mathcal{S}_1'| = |\mathcal{S}_2'|$ and $|\Delta_{new}(\mathcal{S}_1')| \geq |\Delta_{new}(\mathcal{S}_2')|$. Hence, we can assume $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.

If $\mathcal{S}_2 \cap \mathcal{B}(1) = \emptyset$, then we are done by Proposition 3.4 (b) and by the choice of k_1. Hence, we suppose that $\mathcal{S}_2 \cap \mathcal{B}(1) \neq \emptyset$. If $\mathcal{S}_2 \not\subseteq \mathcal{B}(1)$, then $\mathcal{S}_2' := \mathcal{S}_2 \setminus \mathcal{B}(1)$ is an initial segment of $\mathcal{S}_2$ and a final segment of $N_i(P \setminus \mathcal{B}(1))$. Now, again by Proposition 3.4 (b) and the choice of k_1, the inequality $|\Delta_{new}(\mathcal{S}_1')| \geq |\Delta_{new}(\mathcal{S}_2')|$ holds, where $\mathcal{S}_1'$ consists of the last $|\mathcal{S}_2'|$ elements of $\mathcal{S}_1$ w.r.t. $\prec$. Therefore, we can assume $\mathcal{S}_2 \subseteq \mathcal{B}(1)$. Similarly, if $\mathcal{S}_1 \not\subseteq \mathcal{B}(k_1)$, then put $\mathcal{S}_1' := \mathcal{S}_1 \setminus \mathcal{B}(k_1)$ and $\mathcal{S}_2' := C(|\mathcal{S}_1'|, \mathcal{S}_2)$. Now we are done by Proposition 3.4 (c) and the choice of k_1. Consequently, we also suppose that $\mathcal{S}_1 \subseteq \mathcal{B}(k_1)$.

Assume now that $k_1 \geq 3$. If $s \leq |P_i \setminus (\mathcal{B}(k_1) \cup \mathcal{B}(1))|$, then let $\mathcal{S} := C(s, P_i \setminus \mathcal{B}(k_1))$. By the minimality of k_1 and Proposition 3.4 (b) we have $|\Delta(\mathcal{S}_1)| \geq |\Delta_{new}(\mathcal{S})|$, and by the minimality of k_1 and Proposition 3.4 (c) we have $|\Delta_{new}(\mathcal{S})| \geq |\Delta_{new}(\mathcal{S}_2)|$. Consequently, we are done in this case. If $s > |P_i \setminus (\mathcal{B}(k_1) \cup \mathcal{B}(1))|$, then put $\mathcal{S} := P_i \setminus (\mathcal{B}(k_1) \cup \mathcal{B}(1))$. Further let $\mathcal{S}_1'$ be the final segment of $\mathcal{S}_1$ of size $|\mathcal{S}|$, and let $\mathcal{S}_2' := C(|\mathcal{S}|, \mathcal{S}_2)$. By the minimality of k_1 and Proposition 3.4 (b) we have $|\Delta_{new}(\mathcal{S}_1')| \geq |\Delta_{new}(\mathcal{S})|$, and by the minimality of k_1 and Proposition 3.4 (c) we have $|\Delta_{new}(\mathcal{S})| \geq |\Delta_{new}(\mathcal{S}_2')|$. Again, this yields the claim.

Finally, assume that $k_1 = 2$. If $\mathcal{S}_2 \not\subseteq \mathcal{B}(1,1)$, then put $\mathcal{S}_2' := \mathcal{S}_2 \setminus \mathcal{B}(1,1)$, and let $\mathcal{S}_1'$ be the final segment of size $|\mathcal{S}_2'|$ of $\mathcal{S}_1$. By Corollary 3.6, for the final segment $\mathcal{S}_2''$ of $N_i(\mathcal{B}(2))$ of size $|\mathcal{S}_2'|$ the equality $|\Delta_{new}(\mathcal{S}_2'')| = |\Delta_{new}(\mathcal{S}_2')|$ holds. By Proposition 3.4 (b) and the induction hypothesis (induction on m), we have $|\Delta_{new}(\mathcal{S}_1')| \geq |\Delta_{new}(\mathcal{S}_2'')|$ which implies the claim. Therefore, we can assume $\mathcal{S}_2 \subseteq \mathcal{B}(1,1)$. Similarly, if $\mathcal{S}_1 \not\subseteq \mathcal{B}(2, k_2)$, then put $\mathcal{S}_1' := \mathcal{S}_1 \setminus \mathcal{B}(2, k_2)$

and $\mathcal{S}_2' := C(|\mathcal{S}_1'|, \mathcal{S}_2)$. By Corollary 3.6, $|\Delta_{new}(\mathcal{S}_1'')| = |\Delta_{new}(\mathcal{S}_1')|$ holds, where $\mathcal{S}_1'' := C(|\mathcal{S}_1'|, \mathcal{B}(1))$. By Proposition 3.4 (c) and the induction hypothesis we have $|\Delta_{new}(\mathcal{S}_1'')| \geq |\Delta_{new}(\mathcal{S}_2')|$, which implies the claim. Consequently, we assume $\mathcal{S}_1 \subseteq \mathcal{B}(2, k_2)$. If $s \leq s' := |N_i(\mathcal{B}(2) \setminus \mathcal{B}(2, k_2))|$, then, by Propositions 3.4 (b),(c), Corollary 3.6, and the induction hypothesis, we have

$$|\Delta(\mathcal{S}_1)| \geq |\Delta_{new}(\mathcal{S})| = |\Delta_{new}(\mathcal{S}')| \geq |\Delta_{new}(\mathcal{S}_2)| \quad ,$$

where $\mathcal{S} := C(s, N_i(\mathcal{B}(2) \setminus \mathcal{B}(2, k_2)))$ and $\mathcal{S}' := C(s, N_i(\mathcal{B}(1)))$. If $s > s'$, then, again by Propositions 3.4 (b),(c), Corollary 3.6, and the induction hypothesis, we have

$$|\Delta(\mathcal{S}_1')| \geq |N_{i-1}(\mathcal{B}(2) \setminus \mathcal{B}(2, k_2))| = |N_{i-1}(\mathcal{B}(1) \setminus \mathcal{B}(1, 1))| \geq |\Delta_{new}(\mathcal{S}_2')| \quad ,$$

where $\mathcal{S}_1'$ is the final segment of $\mathcal{S}_1$ of size s' and $\mathcal{S}_2' := C(s', \mathcal{S}_2)$. This implies the assertion.

2. The inequality $|\Delta_{new}(\mathcal{S}_2)| \geq |\Delta_{new}(\mathcal{S}_3)|$ can be shown in an absolutely analogous way. ■

Finally, let us remark that Macaulay posets for which each of the functions sf_i $(i = 1, 2, \ldots, r(P))$ is additive are called additive as well. According to Engel [4], for Macaulay posets additivity is preserved when turning to the dual. So, as an application of the above theorem and Theorem 2.1, we obtain that, in particular, orthogonal products of simplices are additive.

5 Proof of Theorem 2.1

According to the introduction and by Lemma 3.2, it remains to show that the inequality $|\Delta(C(\mathcal{F}))| \leq |\Delta(\mathcal{F})|$ is satisfied for all $\mathcal{F} \subseteq P_i$ with $i \in \{1, 2, \ldots, n - m\}$.

We proceed by induction on m. If $m \in \{0, 1\}$, then the assertion is implied directly by the Kruskal-Katona Theorem [7, 8]. For $m = 2$, Theorem 2.1 has been established in [10]. Hence, we assume that $m \geq 3$ and that the claim is true for all P' with $m' < m$. Furthermore, without loss of generality we can assume that $k_1 = 2$ or that the assertion holds for all P' with $m' = m$ and $k_1' < k_1$. Also without loss of generality, we can assume that $k_0 = 0$ or that the assertion is true for all P' with $m' = m$, $k_1' = k_1$ and $k_0' < k_0$. (So, in fact, we are going to run a triple induction.)

Next, we define the *partial compression* operators C_A. Let A be a either a singleton subset of A_0 (if $k_0 \geq 1$) or one of the sets $A_1, A_2, \ldots, A_m$. For $\mathcal{F} \in P_i$

and $S \subseteq A$ define

$$\mathcal{F}(S) := \{F \in \mathcal{F} \mid F \cap A = S\}$$

and

$$\overline{\mathcal{F}}(S) := \{F \setminus S \mid F \in \mathcal{F}(S)\} \quad .$$

Further, let

$$P' := \begin{cases} P(N \setminus A;\, A_1, A_2, \ldots, A_m) & \text{if } A \subseteq A_0, \\ P(N \setminus A_j;\, A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_m) & \text{if } A \in \{A_1, A_2, \ldots, A_m\}. \end{cases}$$

Hence, $\overline{\mathcal{F}}(S) \subseteq P'_{i-|S|}$. Finally, put

$$C_A(\mathcal{F}(S)) := \left\{S \cup G \mid G \in C_{P'}\left(\overline{\mathcal{F}}(S)\right)\right\}$$

and

$$C_A(\mathcal{F}) := \bigcup_{S \subseteq A} C_A(\mathcal{F}(S)) \quad .$$

Lemma 5.1 *Let A be either a singleton subset of A_0 or $A \in \{A_1, A_2, \ldots, A_m\}$. The inclusion $\Delta(C_A(\mathcal{F})) \subseteq C_A(\Delta(\mathcal{F}))$ holds for all $i \in \{1, 2, \ldots, n-m\}$ and all $\mathcal{F} \subseteq P_i$.*

Proof. By the above definition of C_A, we have $\Delta(C_A(\mathcal{F})) = \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_1 = \bigcup_{S \subseteq A} \left\{G \cup H \mid G \in \Delta(S),\ H \in C_{P'}\left(\overline{\mathcal{F}}(S)\right)\right\}$$

and

$$\mathcal{F}_2 = \bigcup_{S \subseteq A} \left\{S \cup G \mid G \in \Delta\left(C_{P'}\left(\overline{\mathcal{F}}(S)\right)\right)\right\} \quad .$$

On the other hand,

$$\Delta(\mathcal{F}) = \bigcup_{S \subseteq A} \left(\left\{G \cup H \mid G \in \Delta(S),\ H \in \overline{\mathcal{F}}(S)\right\} \cup \left\{S \cup G \mid G \in \Delta\left(\overline{\mathcal{F}}(S)\right)\right\}\right)$$

holds. Hence, $\mathcal{F}_1 \subseteq C_A(\Delta(\mathcal{F}))$ and $\mathcal{F}_3 \subseteq C_A(\Delta(\mathcal{F}))$, where

$$\mathcal{F}_3 = \bigcup_{S \subseteq A} \left\{S \cup G \mid G \in C_{P'}\left(\Delta\left(\overline{\mathcal{F}}(S)\right)\right)\right\} \quad .$$

Now $\mathcal{F}_2 \subseteq \mathcal{F}_3$ since, by the above assumptions, (1.1) holds for P' and $\prec$. This implies the claim. ■

Let us fix now a family $\mathcal{F} \subseteq P_i$ with the property that $|\Delta(\mathcal{F})| \leq |\Delta(\mathcal{F}')|$ holds for all $\mathcal{F}' \subseteq P_i$ with $|\mathcal{F}'| = |\mathcal{F}|$. We have to show

$$|\Delta(\mathcal{F})| \geq |\Delta(C(\mathcal{F}))| \quad . \tag{5.1}$$

According to Proposition 3.3, the partial compression operators C_A work *from right to left*, i.e. for all A like in Lemma 5.1 there is a bijection $\varphi_A :\ \mathcal{F} \to C_A(\mathcal{F})$ such that $\varphi_A(F) \preceq F$ for all $F \in \mathcal{F}$. Consequently, by Lemma 5.1, without loss of generality we can assume

$$C_A(\mathcal{F}) = \mathcal{F} \ \text{ for all } \ A \in \left\{\{a_0^1\}, \{a_0^2\}, \ldots, \{a_0^{k_0}\}, A_1, A_2, \ldots, A_m\right\} \quad . \tag{5.2}$$

If $\mathcal{F} \cap \mathcal{B}(1) = \emptyset$ or $N_i(\mathcal{B}(k_1)) \subseteq \mathcal{F}$, then (5.1) follows by Propositions 3.4 (b),(c) and the induction hypothesis (for $k_1 = 2$) resp. the choice of k_1 (for $k_1 \geq 3$). Hence, from now on we assume

$$\mathcal{F} \cap \mathcal{B}(1) \neq \emptyset \tag{5.3}$$

and

$$N_i(\mathcal{B}(k_1)) \nsubseteq \mathcal{F} \quad . \tag{5.4}$$

If $i < k_2$, then we have $P_i = P_i'$, where $P' = P(n; a_1)$. Clearly, the elements of A_1 are not the greatest elements of the ground set N since $m \geq 3$. Therefore, the ordering $\prec_P$ on P_i is different from the order $\prec_{P'}$ on P_i'. Nevertheless, it is an easy exercise to show that

$$|\Delta(C_P(t, P_i))| = |\Delta(C_{P'}(t, P_i'))| \tag{5.5}$$

holds for all $t \in \{1, 2, \ldots, |P_i|\}$. (In fact, on both sides of the equation the bound given by the Kruskal-Katona theorem is attained.) To show the equation we can argue like this: If $t \leq |N_i(P \setminus \mathcal{B}(1))|$, then (5.5) holds by $C_P(t, P_i) \subseteq P \setminus \mathcal{B}(1)$ and $C_{P'}(t, P_i') \subseteq N_i(P(n, a_1 - 1))$ together with Proposition 3.4 (b) and the choice of k_1. Similarly, using Proposition 3.4 (c), the equation (5.5) holds also for $t > |N_i(P \setminus \mathcal{B}(1))|$. Hence, we suppose

$$i \geq k_2 \quad . \tag{5.6}$$

Let F be the first element of $N_i(\mathcal{B}(1))$ w.r.t. $\prec$. By (5.3) and (5.2) with $A = A_1$, we have

$$F \in \mathcal{F} \quad . \tag{5.7}$$

If $k_0 \geq 1$, then, by (5.6), $a_0^1 \in F$. By (5.2) with $A = \{a_0^1\}$ and Lemma 3.7 (b), we obtain then $\Delta(N_i(P \setminus \mathcal{B}(1))) \subseteq \Delta(\mathcal{F})$. Without loss of generality, we can assume then that $N_i(P \setminus \mathcal{B}(1)) \subset \mathcal{F}$ which contradicts (5.4). In the sequel, we therefore suppose

$$k_0 = 0 \quad . \tag{5.8}$$

Let G denote the last element of $N_i(\mathcal{B}(k_1))$ w.r.t. $\prec$.

Lemma 5.2 *$G \notin \mathcal{F}$ holds.*

Proof. Assume the contrary, i.e. $G \in \mathcal{F}$. We construct a contradiction to (5.4) by showing $H \in \mathcal{F}$ for all $H \in N_i(\mathcal{B}(k_1))$.

We proceed by induction on $k := k_2 - |H \cap A_2|$. By Lemma 3.5 (a) and (5.6), we have $G \in \mathcal{B}(k_1, 1)$, i.e. $A_2 \setminus \{a_2^1\} \subset G$. By (5.2) with $A = A_2$, this implies $N_i(\mathcal{B}(k_1, 1)) \subseteq \mathcal{F}$. If $k = 1$ and $H \notin N_i(\mathcal{B}(k_1, 1))$, then $H \in \mathcal{F}$ follows by $(H \setminus \{a_2^1\}) \cup \{a\} \in \mathcal{B}(k_1, 1)$ and (5.2) with $A = A_1$, where $\{a\} = A_2 \setminus H$. Hence, we are done for $k = 1$.

Suppose that $k \geq 2$, and that $H' \in \mathcal{F}$ for all H' with $k' < k$. Put $a := \max(A_2 \setminus H)$. By (5.6), there is an $a_j \in H \cap A_j$ for some $j \in \{1, 2, \ldots, m\} \setminus \{2\}$. By Proposition 3.5 (a), $H \prec H' := (H \setminus \{a_j\}) \cup \{a\}$. Now the claims follows from the induction hypothesis and (5.2) for $A = A_p$ with $p \in \{1, 2, \ldots, m\} \setminus \{2, j\}$. (Such a p exists because of $m \geq 3$.) ■

Case 1. Suppose that $k_1 = 2$.

First, note that $\mathcal{F} \cap \mathcal{B}(1, 1) = \emptyset$ because of Lemma 5.2 and (5.2) with $A = A_2$. Hence,

$$\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$$

with $\mathcal{F}_1 \subseteq \mathcal{B}(2, k_2)$, $\mathcal{F}_2 \subseteq \mathcal{B}(2) \setminus \mathcal{B}(2, k_2)$, $\mathcal{F}_3 \subseteq \mathcal{B}(1) \setminus \mathcal{B}(1, 1)$.

By the definition of $\prec$ and (5.6), $F = \{a_1^2\} \cup (A_2 \setminus \{a_2^{k_2}\}) \cup H$, where H consists of the smallest $i - k_2$ elements of $(A_3 \setminus \{a_3^{k_3}\}) \cup \cdots \cup (A_m \setminus \{a_m^{k_m}\})$. Using (5.7) and (5.2) with $A = A_3$, this implies $F' := \{a_1^1\} \cup (A_2 \setminus \{a_2^1\}) \cup H \in \mathcal{F}$. Obviously, $F' \in \mathcal{B}(2, 1)$. From this and (5.2) with $A = A_1$, it follows that all elements of $N_i(\mathcal{B}(2) \setminus \mathcal{B}(2, 1))$ which contain a_1^1 are in $\mathcal{F}$. Now by (5.8) and Lemma 3.7 (a), $N_{i-1}(\mathcal{B}(2) \setminus \mathcal{B}(2, 1)) \subseteq \Delta(\mathcal{F})$ holds. Therefore, without loss of generality we can assume that $N_i(\mathcal{B}(2) \setminus \mathcal{B}(2, 1)) \subseteq \mathcal{F}$. Consequently,

$$\mathcal{F}_1 = \mathcal{B}(2, k_2)$$

and

$$\mathcal{B}(2) \setminus (\mathcal{B}(2, k_2) \cup \mathcal{B}(2, 1)) \subseteq \mathcal{F}_2 \quad .$$

By (5.2) with $A = A_2$, we know that $\mathcal{F}_2 \cap \mathcal{B}(2, 1)$ is an initial segment of $N_i(\mathcal{B}(2, 1))$. Thereby, in total $\mathcal{F}_2$ is an initial segment of $N_i(\mathcal{B}(2) \setminus \mathcal{B}(2, k_2))$. On the other hand, again by (5.2) with $A = A_2$, the family $\mathcal{F}_3$ is an initial segment of $N_i(\mathcal{B}(1) \setminus \mathcal{B}(1, 1))$. Now (5.1) is implied by Corollary 3.6 together with Theorem 4.2 and Lemma 4.1.

Case 2. Suppose that $k_1 \geq 3$.

Here we show that $F \in \mathcal{F}$ implies $G \in \mathcal{F}$, and we are done by (5.7) and Lemma 5.2. By the definition of $\prec$, we know that $F = (A_1 \setminus \{a_1^1\}) \cup H$, where H consists of the $i - k_1 + 1$ smallest elements of $(A_2 \setminus \{a_2^{k_2}\}) \cup \dots \cup (A_m \setminus \{a_m^{k_m}\})$. Now (5.6) and (5.2) with $A = A_m$ imply the existence of an $F' = \{a_1^{k_1}\} \cup (A_2 \setminus \{a_2^1\}) \cup H' \in \mathcal{F}$ such that $H' \cap A_m = H \cap A_m$ and $a_1^2 \notin H'$. Clearly, $H' \notin \mathcal{B}(k_1)$ and therefore $G \prec H'$. Together with (5.2) for $A = A_2$ this yields $G \in \mathcal{F}$.

This concludes the proof.

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