A CONJECTURE ON ORTHOGONAL DOUBLE COVERS BY PATHS

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Abstract. An Orthogonal Double Cover (ODC) of the complete graph is a set of graphs such that every two of them share exactly one edge and every edge of the complete graph belongs to exactly two of the graphs. We consider the case when all graphs are paths and introduce a construction that we conjecture to work in all finite fields of order 1 mod 4.

1. Introduction

Let $\mathcal{C}$ be a collection of n spanning subgraphs (called *pages*) of K_n, the complete graph on n vertices. We call $\mathcal{C}$ an *Orthogonal Double Cover* (ODC) if

1. every edge of K_n is an edge in exactly two of the pages,
2. any two pages share exactly one edge.

If all pages in $\mathcal{C}$ are isomorphic to a graph G then $\mathcal{C}$ is said to be an ODC by G. Clearly, G must have exactly $n-1$ edges then.

The first class of graphs for which the existence of an ODC was considered are almost-hamiltonian cycles, that means graphs consisting of a cycle of length $n-1$ and one isolated vertex [1, 5, 6, 7]. Although progress on the existence problem has been made it is far from being solved completely.

The study of ODCs by hamiltonian paths is motivated by the following result.

Theorem 1.1. (Heinrich and Nonay [4]) *If there exists an ODC of K_n by an hamiltonian path, then there exists an ODC of K_{4n} by an almost-hamiltonian cycle.*

But for which values of n does an ODC of K_n by a path exist? This question was studied in several papers but has been answered for some special classes of values n [3, 2, 4, 8, 9] only.

One promising approach uses ODCs with an additional property. An ODC by a path is called 2-*colorable* if the edges of each page of the ODC can be colored alternately such that each edge of the complete graph is covered twice by the same color (the example in Figure 1 has

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this property). 2-colorable ODCs are particularly interesting because of the following multiplication theorem.

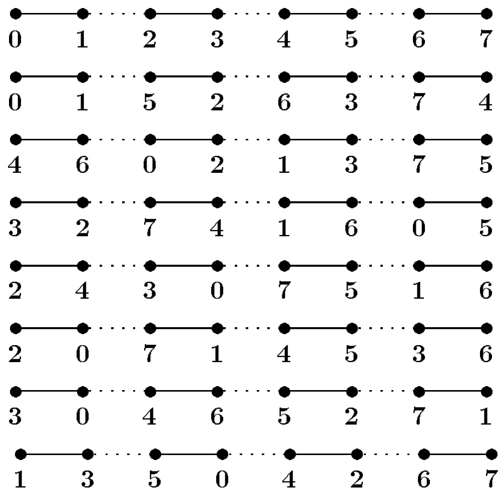

FIGURE 1. A 2–colorable ODC by P_8

Theorem 1.2. (Horton and Nonay [8]) *Suppose that for $m \geq 3$ there exists an ODC of K_m by a path. If for some prime power $n \geq 5$ there is a 2-colorable ODC of K_n by a path, then there is an ODC of K_{nm} by a path.*

The above theorem makes it desirable to have a direct construction of 2–colorable ODCs, in particular when the number of vertices is a prime. Unfortunately, a simple counting argument implies that for $n \equiv 3(\mod 4)$ there is no 2-colorable ODC of K_n by a path. In section 3 we will introduce a construction that we conjecture to yield 2-colorable ODCs of K_n by a path for all prime powers $n \equiv 1(\mod 4)$.

Most of the known ODCs are *group-generated.* In the sequel, let $\mathcal{A}$ be an additive group $\mathcal{A}$ of order n. An ODC $\mathcal{C}$ of K_n is said to be generated by $\mathcal{A}$ if every page of $\mathcal{C}$ can be derived from a given graph G with the vertices labeled by the elements of $\mathcal{A}$ by adding some group-element to every vertex of G. A graph G with this property is called an *ODC–starter* with respect to $\mathcal{A}$.

Consider the complete graph on the vertex set $\mathcal{A}$. Define the *length* ℓ of an edge xy by $\ell = \{x-y, y-x\}$. Then two edges of the same length ℓ have *distance* k, if either adding or substracting k to the vertices of one of the edges gives the other one. Clearly, if two edges have distance k then they also have distance $-k$. An ODC–starters with respect to $\mathcal{A}$ can be characterized as below.

Lemma 1.3. (cf. [8]) *Let G be a graph on the vertex set $\mathcal{A}$. The graph G is an ODC–starter with respect to $\mathcal{A}$ if and only if the following conditions are satisfied:*

1. *For every element $a \in \mathcal{A}$ of order 2 the graph G contains exactly one edge e with $a \in \ell(e)$.*
2. *For every nonzero-element $a \in \mathcal{A}$ of order $\neq 2$ the graph G contains exactly two edges e_1 and e_2 with $a \in \ell(e_1)$ and $a \in \ell(e_2)$.*
3. *The set of distances in G is $\mathcal{A} \setminus \{0\}$.*

Clearly, an ODC–starter which is a path generates a 2–colorable ODC iff there is an odd number of edges on the path inbetween every two edges of the same length.

Example 1.4. The path $0, 1, 4, 6, 2, 3, 8, 5, 7$ is an ODC–Starter with respect to $\mathbb{Z}_9$. The resulting ODC is not 2–colorable.

Example 1.5. The path $0, 3, 4, 2, 1$ is an ODC–Starter with respect to $\mathbb{Z}_5$, which generates a 2–colorable ODC.

2. ODCs from half–starters

In [3], ODCs by paths are constructed using half–starters. A *half–starter* in the additive abelian group $\mathcal{A}$ of order $4n+1$ is a set of unordered pairs $H = \{\{s_i, t_i\} : 1 \leq i \leq 2n\}$ which satisfies:

1. $\{s_i : 1 \leq i \leq 2n\} \cup \{t_i : 1 \leq i \leq 2n\} = \mathcal{A} \setminus \{0\}$,
2. $\{\pm(s_i - t_i) : 1 \leq i \leq 2n\}$ contains exactly half the nonzero elements of $\mathcal{A}$ and each such element occurs exactly twice.
3. Consider the pairs of H as edges. Then the set of distances of edges of the same length contains exactly half the nonzero elements of $\mathcal{A}$.

A *translate* $H + a$ of the half–starter H is the set $\{\{s_i + a, t_i + a\} : 1 \leq i \leq 2n\}$ for some $a \in \mathcal{A}$. Since $(s_i + a) - (t_i + a) = s_i - t_i$ the translation does not change the sets in the half–starter conditions 2 and 3.

We call two half–starters H_1 and H_2 *supplementary*, if the union of the sets arising from half–starter condition 2 is $\mathcal{A} \setminus \{0\}$ and so is the union of the sets arising from condition 3. This means that $H_1 \cup H_2$ (considered as a graph) is an ODC–starter with respect to $\mathcal{A}$. Now the following fact is obvious.

Proposition 2.1. *If H_1 and H_2 are supplementary half–starters, then the union of any translates of H_1 and H_2 is an ODC–starter. If this ODC–starter is a path, then the resulting ODC is 2–colorable.*

Example 2.2. The path $0, 3, 4, 2, 1$ in example 1.5 consists of the half–starter $\{\{1, 2\}, \{3, 4\}\}$ and the translate $\{\{0, 3\}, \{2, 4\}\}$.

For the rest of this section assume that $\mathcal{A}$, together with some multiplication, is an abelian ring. For $t \in \mathcal{A}$ the *multiple* $t \cdot H$ of H is the set $\{\{t \cdot s_i, t \cdot t_i\} : 1 \leq i \leq 2n\}$. The following statement is easy to verify.

Proposition 2.3. *Let H be a half–starter with respect to $\mathcal{A}$. If $m \in \mathcal{A}$ is of multiplicative order $4n$, then also $m \cdot H$ is a half–starter.*

3. The conjecture

Let us now consider half–starters in finite fields. Be $\mathcal{F}$ a finite field of order $4n+1$. We define $H_1 = \{\{\omega^{2k}, \omega^{2k+1}\} : 0 \leq k \leq 2n-1\}$ and $H_2 = \{\{\omega^{2k+1}, \omega^{2k+2}\} : 0 \leq k \leq 2n-1\}$, where ω is some primitive element of $\mathcal{F}$. Since $\{\omega^k, \omega^{k+1}\}$ has the same two lengths as $\{\omega^{2n+k}, \omega^{2n+k+1}\}$ and the distance of these two edges is $\{\pm\omega^k(-\omega - 1)\}$ it follows that H_1 and H_2 are supplementary half–starters. The resulting ODC–starter $H_1 \cup H_2$ is an almost–hamiltonian graph [1].

We now investigate the graphs $H_1 \cup (H_2 + t)$. Clearly, all such unions are ODC–starters and all of them consist of disjoint cycles and a path. But when is the entire graph $H_1 \cup (H_2 + t)$ a path? Computer tests provided the result below for initial values of n.

Lemma 3.1. *For every finite field $\mathcal{F}$ of order $4n + 1 < 2000$ there exists a primitive element ω such that $H_1(\omega) \cup (H_2(\omega) + 1)$ is a path.*

Example 3.2.

- In GF(13) $H_1(\omega) \cup (H_2(\omega) + 1)$ is a path for the primitive elements $\omega = 2, 7$,
- in GF(29) for $\omega = 3, 8, 10, 11, 14, 19, 26, 27$,
- in GF(37) for $\omega = 5, 15, 17, 18, 24, 35$,
- in GF(101) for $\omega = 8, 38, 50, 55, 90, 99$.

In general, we conjecture the following.

Conjecture 3.3. *For every finite field $\mathcal{F}$ of order $4n + 1$ there exists a primitive element ω such that $H_1(\omega) \cup (H_2(\omega) + 1)$ is a path.*

It turns out that the structure of the graph $H_1 \cup (H_2 + t)$ depends only on the choice of the primitive element ω, not on the particular translation t.

Lemma 3.4. *The graphs $H_1 \cup (H_2 + t)$ with $t \in \mathcal{F} \setminus \{0\}$ are mutually isomorphic.*

Proof. We show that for any choice of t the resulting graph is isomorphic to $H_1 \cup (H_2 + 1)$. Clearly, it holds $\omega \cdot H_1 = H_2$ and $\omega \cdot H_2 = H_1$. So for a quadratic element $t \in \mathcal{F} \setminus \{0\}$ it follows that $t \cdot H_1 = H_1$ and $t \cdot H_2 = H_2$, and for *a* non–quadratic it is $t \cdot H_1 = H_2$ and $t \cdot H_2 = H_1$. Therefore

$$H_2 + t = t \cdot (t^{-1} \cdot H_2 + 1) = \begin{cases} t \cdot (H_2 + 1) & \text{if } t \text{ is a quadratic residue,} \\ t \cdot (H_1 + 1) & \text{else.} \end{cases}$$

It yields

$$\begin{aligned} H_1 \cup (H_2 + t) &= t \cdot (t^{-1} \cdot H_1 \cup (t^{-1} \cdot H_2 + 1)) \\ &= \begin{cases} t \cdot (H_1 \cup (H_2 + 1)) & \text{if } t \text{ is a quadratic residue,} \\ t \cdot (H_2 \cup (H_1 + 1)) & \text{else.} \end{cases} \end{aligned}$$

This proves that $H_1 \cup (H_2 + t)$ is isomorphic to $H_1 \cup (H_2 + 1)$ if t is a quadratic residue. Since $H_2 \cup (H_1 + 1) = (H_1 \cup (H_2 - 1)) + 1$ and -1 is a quadratic residue the claim also holds for quadratic non-residues. $\square$

We do not know for which primitive elements ω the graph $H_1(\omega) \cup (H_2(\omega) + 1)$ is a path. But with ω also the primitive element ω^{-1} forms half–starters that give a path.

Lemma 3.5. *The graph* $H_1(\omega^{-1}) \cup (H_2(\omega^{-1}) + 1)$ *is isomorphic to* $H_1(\omega) \cup (H_2(\omega) + 1)$.

Proof. Consider $\{\omega^{2k}, \omega^{2k+1}\} \in H_1(\omega)$. With $\delta := \omega^{-1}$ and $\delta^{4n} = 1$ it is

$$\{\omega^{2k}, \omega^{2k+1}\} = \left\{\frac{1}{\omega^{-2k}}, \frac{1}{\omega^{-2k-1}}\right\} = \{\delta^{4n-2k}, \delta^{4n-2k-1}\} \in H_2(\delta).$$

So $H_1(\omega) = H_2(\delta)$ and therefore $H_2(\omega) = H_1(\delta)$. Since -1 is a quadratic residue it yields $-H_i = H_i$, $i = 1, 2$. It follows that

$$\begin{aligned} -(H_1(\omega) \cup (H_2(\omega) + 1)) + 1 &= (-H_1(\omega) + 1) \cup -H_2(\omega) \\ &= (H_1(\omega) + 1) \cup H_2(\omega) \\ &= (H_2(\delta) + 1) \cup H_1(\delta) \end{aligned}$$

which proves the lemma. $\square$

4. Conclusion

We introduced a construction that yields 2-colorable ODCs of K_n by P_n for all prime powers $n < 2000$ with $n \equiv 1 (\mod 4)$. We conjecture that our construction gives such ODCs for all $n \equiv 1 (\mod 4)$.

In combination with Theorem 1.2 and Theorem 1.1 the following is established: If all prime factors of n are smaller than 2000 and congruent 1 mod 4, then there exist an ODC of K_n by a hamiltonian path and an ODC of K_{4n} by an almost-hamiltonian cycle.

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