Optimal antichains and ideals in Macaulay posets

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Abstract

A ranked poset P is a Macaulay poset if there exists a linear ordering $\preceq$ of its elements such that for any i and any subset F of the i–th level N_i the shadow of the $|F|$ smallest elements of N_i w.r.t. $\preceq$ is contained in the set of the smallest $|\Delta(F)|$ elements of N_{i-1}, where $\Delta(F)$ denotes the shadow of F. We consider the following three optimization problems for P: *(i)* Find the maximum weight of an ideal of given size. *(ii)* Find the minimum weight of an ideal generated by an antichain of given size. *(iii)* Find the maximum number of maximal elements of an ideal of given size. We provide solutions for a class of Macaulay posets which includes all examples known so far.

1 Introduction and Notations

In this paper we will study several optimization problems for posets which were solved for Boolean lattices first. It is our goal to work out the essential "axioms" that enable the solution of the problems for a larger class of ranked posets. We will present a survey of those results which are also contained in the monograph [10]. Detailed proofs will be given only if they are new and, in particular, not contained in [10]. Special attention is paid to the 0–1 subword order defined below.

Let $(P, \leq)$ (briefly, P) be a partially ordered set (briefly, poset). It is called *ranked* if there is a function $r : P \to \mathbb{N}$ such that $r(p) = 0$ for some minimal element p of P and $r(q) = r(p) + 1$ whenever $q \gtrdot p$ (*q covers p*), i.e., $q > p$ and there is no v with $p < v < q$. The *rank of P* is defined by $r(P) := \max\{r(p) : p \in P\}$ ($r(P) = \infty$ is allowed). We speak of a *graded poset* if any two minimal elements have the same rank and any two maximal elements have the same rank. The set $N_i = N_i(P) := \{p \in P : r(p) = i\}$ is said to be the *i–th level of P*. Throughout we suppose that N_i is finite for each i.

Often we will work with the *dual* P^* of P, i.e., the poset on the same ground set (set of elements) with $p \leq_{P^*} q$ iff $p \geq_P q$. If $r(P) < \infty$, then the dual of a ranked poset is ranked. Obviously, this is not true if $r(P) = \infty$. But if we allow that the rank function maps P into $\{0, -1, -2, \dots\}$ and demand, as previously, that $r_{P^*}(q) = r_{P^*}(p) + 1$ whenever $q \gtrdot_{P^*} p$, then turning to the dual is possible. In this paper we will work with such generalized rank functions of P^* if $r(P) = \infty$.

A subset A of P is called an *antichain* if $p, q \in A, p \leq q$ imply $p = q$, and a subset I of P is called an *ideal* (resp. *filter*) if $q \in I, p \leq q$ (resp. $p \geq q$) imply $p \in I$. The *width $d(P)$ of P* is the largest size of an antichain in P. If A is an antichain, then, obviously, the set $\mathcal{I}(A) := \{p \in P :$ there is some $q \in A$ with $p \leq q\}$ is an ideal – the *ideal generated by A*. Conversely, if I is an ideal then the set $\max(I) := \{p \in I :$ there is no $q \in I$ with $p < q\}$ is an antichain – the *set of maximal elements of I*. In these cases, we have $\max(\mathcal{I}(A)) = A$ and $\mathcal{I}(\max(I)) = I$.

Let $w : \{0, \dots, r(P)\} \to \mathbb{R}$ be any function. For $F \subseteq P$ we set

$$w(F) := \sum_{p \in F} w(r(p)).$$

We will address the following three problems[1].

Problem 1. Find $\max\{w(I) : I \text{ is an ideal}, |I| = m\}$ for $m = 1, \dots, |P|$.

Problem 2. Find $\min\{w(\mathcal{I}(A)) : A \text{ is an antichain}, |A| = m\}$ for $m = 1, \dots, d(P)$.

Problem 3. Find $\max\{|\max(I)| : I \text{ is an ideal}, |I| = m\}$ for $m = 1, \dots, |P|$.

We will provide solutions if w is increasing and (for Problems 2,3) nonnegative and if P satisfies several axioms which we will describe in the following.

Let $\preceq$ be any new linear order on the ranked poset P. If F is a subset of P and m a natural number, then let $\mathcal{C}(m, F)$ (resp. $\mathcal{L}(m, F)$) be the set of the first (resp. last) m elements of F w.r.t. $\preceq$. For $F \subseteq N_i$ we abbreviate $\mathcal{C}(|F|, N_i)$ by $\mathcal{C}(F)$ (and, analogously, $\mathcal{L}(|F|, N_i)$ by $\mathcal{L}(F)$). For a still shorter notation we will often write just $\mathcal{C}F$ (resp. $\mathcal{L}F$). Exchanging $F \subseteq N_i$ by $\mathcal{C}F$ is called *compression*, and F is *compressed* if $F = \mathcal{C}F$. In this case, the set F is also called an *initial segment* whereas a *final segment* is a subset F of N_i with $\mathcal{L}F = F$. A *segment* in N_i is a set of elements of N_i which are consecutive w.r.t. $\preceq$ (restricted to N_i).

We define the *lower* (resp. *upper*) *shadow* of an element p of P by $\Delta(p) := \{q \in P : q \lessdot p\}$ (resp. $\nabla(p) := \{q \in P : q \gtrdot p\}$). The *shadow* of a subset F of P is the union of the shadows of its elements and is denoted by $\Delta(F)$ (resp. $\nabla(F)$).

The ranked poset P is said to be a *Macaulay poset* if there exists a linear order $\preceq$ of its elements such that

$$\Delta(\mathcal{C}F) \subseteq \mathcal{C}(\Delta(F)) \text{ for all } F \subseteq N_i \text{ and all } i \in \{1, \dots, r(P)\}. \qquad (1)$$

Though there may exist several linear orders such that (1) holds, we assume in the following that with a Macaulay poset always some fixed linear order $\preceq$

[1] We are grateful to G. Ziegler for stimulating the study of Problem 3

satisfying (1) is given, i.e., we consider Macaulay posets as triples $(P, \leq, \preceq)$ and speak of a *Macaulay poset* P *with the associated linear order* $\preceq$. It is a simple exercise to verify that (1) holds iff

$$\begin{gathered} |\Delta(\mathcal{C}F)| \leq |\Delta(F)| \text{ and } \Delta(\mathcal{C}F) = \mathcal{C}(\Delta(\mathcal{C}F)) \\ \text{for all } F \subseteq N_i \text{ and all } i \in \{1, \ldots, r(P)\}, \end{gathered} \tag{2}$$

i.e., iff compressed subsets have minimum shadows among all subsets of the same level and of the same size, and the shadows of compressed subsets are compressed. The notion "Macaulay poset" stems from the fact that in 1927 Macaulay [21] found the first example of such a poset.

2 Examples of Macaulay posets

In the following we define the ground sets, the partial orders and the associated linear orders of some Macaulay posets.

Example 1. The poset $Mon(n)$ of monomials in n variables.

Elements:	Monomials $p = x_1^{a_1} \cdots x_n^{a_n}$, where $a_i \in \mathbb{N}$ for all i.
Partial order:	$p \leq p'$ iff p divides p', i.e., $a_i \leq a_i'$ for all i.
Linear order:	$p \prec p'$ iff $a_l > a_l'$, where l is the smallest index i with $a_i \neq a_i'$.

The linear order in this example is called the *lexicographic order*. To be precise, Macaulay and in a simplified way Sperner [23] proved (2) for the dual of $Mon(n)$, but this implies (2) also for $Mon(n)$ which can be checked easily.

In general we have, as observed e.g. by Bezrukov in [4]:

Proposition 1 *If P is a Macaulay poset, then so is its dual P^*.*

In this and in the following "duality propositions", the associated linear order for P^* is the dual of the associated linear order for P. The next example is given by the well–known Kruskal–Katona theorem found by Schützenberger [22] (proof incomplete), Kruskal [15], Harper [12] (not explicitly mentioned) and Katona [13].

Example 2. The Boolean lattice B_n.

Elements:	Subsets X of $[n] := \{1, \dots, n\}$.
Partial order:	$X \le X'$ iff $X \subseteq X'$.
Linear order:	$X \prec_{rl} X'$ iff $\max\{X \setminus X'\} < \max\{X' \setminus X\}$ (where $\max\{\emptyset\} := 0$).

The linear order $\preceq_{rl}$ is called the *reverse lexicographic order.* Using the representation of subsets by means of characteristic vectors, we see that Example 2 is a special case of the next example due to Clements and Lindström [6]:

Example 3. Chain products $S(k_1, \dots, k_n)$, where $k_1 \ge \dots \ge k_n$.

Elements:	n-tuples $\boldsymbol{a} = (a_1, \dots, a_n)$, where $a_i \in \{0, \dots, k_i\}$ for all i.
Partial order:	$\boldsymbol{a} \le \boldsymbol{a'}$ iff $a_i \le a'_i$ for all i.
Linear order:	$\boldsymbol{a} \prec \boldsymbol{a'}$ iff $a_l < a'_l$, where l is the largest index i with $a_i \ne a'_i$.

Note that Example 1 also is a special case of Example 3. Another generalization of the Boolean lattice was studied by Lindström [19] ($k_1 = \dots = k_n = 2$), Leeb [18], Bezrukov [3] ($k_1 = \dots = k_n$) and Leck [16]:

Example 4. Star products $T(k_1, \dots, k_n)$, where $k_1 \le \dots \le k_n$.

Elements:	n-tuples $\boldsymbol{a} = (a_1, \dots, a_n)$, where $a_i \in \{k_n - k_i, k_n - k_i + 1, \dots, k_n\}$ for all i.
Partial order:	$\boldsymbol{a} \le \boldsymbol{a'}$ iff $a_i = a'_i$ or $a'_i = k_n$ for all i.
Linear order:	Let $\boldsymbol{a}(j) := \{i \in [n] : a_i = j\}$. $\boldsymbol{a} \prec \boldsymbol{a'}$ iff $\boldsymbol{a}(l) \succ_{rl} \boldsymbol{a'}(l)$, where l is the smallest number j with $\boldsymbol{a}(j) \ne \boldsymbol{a'}(j)$.

An at first glance different poset was investigated by Frankl, Füredi, and Kalai [11], London [20] ($k_n - k_1 \le 1$) and Engel [10]:

Example 5. The poset of colored subsets $Col(k_1, \dots, k_n)$, where $k_1 \leq \dots \leq k_n$.

Let $A_i := \{i, n+i, \dots, (k_i - 1)n + i\}, i = 1, \dots, n$, and $A := \cup_{i=1}^n A_i$.

Elements: Subsets X of A, where $|X \cap A_i| \leq 1$ for all i.
Partial order: $X \leq X'$ iff $X \subseteq X'$.
Linear order: $X \prec_{rl} X'$ iff $\max\{X \setminus X'\} < \max\{X' \setminus X\}$
(where $\max\{\emptyset\} := 0$).

In fact, Example 5 is (using Proposition 1) the "dual" of Example 4 since $T(k_1, \dots, k_n)$ and $Col(k_1, \dots, k_n)$ are dually isomorphic. The isomorphism φ is given by

$$\varphi(X) = \boldsymbol{a} \text{ where } a_j := \begin{cases} k_n - 1 - \frac{x-j}{n} & \text{if } \exists x \in X : x \equiv j \pmod n, \\ k_n & \text{otherwise.} \end{cases}$$

Before giving the last example, we will introduce a new linear order $\preceq_{vip}$ on the Boolean lattice B_n. Let $X \prec_{vip} X'$ iff $|X| < |X'|$ or ($|X| = |X'|$ and $X' \prec_{rl} X$). We use the index *vip* since the *vip-order* provides a solution of a vertex isoperimetric problem: For $X \subseteq [n]$ let $\Gamma^B(X) := \{Y \subseteq [n] : |(Y \setminus X) \cup (X \setminus Y)| \leq 1\}$ and for $F \subseteq B_n$ let $\Gamma^B(F) := \cup_{X \in F} \Gamma^B(X)$. Thus $\Gamma^B(F)$ consists of F and the "vertex boundary" of F. The following classical result is due to Harper [12] (for a short proof cf. [10]).

Theorem 1 *We have*

$$|\Gamma^B(F)| \geq |\Gamma^B(\mathcal{C}_{vip}(m, B_n))| = |\Gamma^B(\mathcal{L}_{vip}(m, B_n))| \text{ for all } F \subseteq B_n,\ |F| = m,$$

i.e., taking the first or last m elements in the vip-order minimizes $|\Gamma^B(F)|$ over all m–element families in B_n.

If we use characteristic vectors (i.e., elements of $S(1, \dots, 1)$) for subsets of $[n]$ we have $\boldsymbol{a} \preceq_{vip} \boldsymbol{a}'$ iff $r(\boldsymbol{a}) < r(\boldsymbol{a}')$ or ($r(\boldsymbol{a}) = r(\boldsymbol{a}')$ and $\boldsymbol{a}' \preceq \boldsymbol{a}$). The following example was studied by Danh and Daykin [7, 8], Ahlswede and Cai [1], and Bezrukov [5].

Example 6. The 0–1 subword order $SO(2)$.

Elements:	Words $a_1 \ldots a_k$ with letters $a_i \in \{0,1\}$ for all i, $k = 0, 1, \ldots$.
Partial order:	$a_1 \ldots a_k \leq a'_1 \ldots a'_l$ iff $k \leq l$ and there are $1 \leq i_1 < \cdots < i_k \leq l$ such that $a_1 = a'_{i_1}, \ldots, a_k = a'_{i_k}$.
Linear order:	$a_1 \ldots a_k \prec a'_1 \ldots a'_l$ iff $(a_1, \ldots, a_k) \prec_{vip} (a'_1, \ldots, a'_l)$ for $k = l$ and $(\underbrace{0, \ldots, 0}_{l-k}, a_1, \ldots, a_k) \preceq_{vip} (a'_1, \ldots, a'_l)$ for $k < l$.

Note that we have here $\boldsymbol{a} \lessdot \boldsymbol{a'}$ iff $\boldsymbol{a}$ can be obtained from $\boldsymbol{a'}$ by deleting exactly one letter. Restricted to one level, the given linear order is the *vip* order, and the interaction between the levels is arranged such that we obtain a rank greedy Macaulay poset (to be defined in the next section).

Since this example is not very well–known (and not contained in [10]) we will provide a simple proof of (2) which is based on a numerical approach of Ahlswede and Cai [1]. More exactly, we will prove (2) for $SO(2)^*$ which implies (2) for $SO(2)$ by Proposition 1. We briefly denote the word $\underbrace{i \ldots i}_{n}$ by $\boldsymbol{i}_n$. Since words can be considered as abbreviations of tuples each element $a_1 \ldots a_n$ of $SO(2)$ is also an element of $S(\underbrace{1, \ldots, 1}_{n})$ and can be interpreted as an element of B_n as well. In order to avoid confusion between the rank functions in $SO(2)$ and in the Boolean lattice we write $r^B(\boldsymbol{a})$ for the number of ones in $\boldsymbol{a}$ which is the rank of $\boldsymbol{a}$ in the corresponding Boolean lattice.

Here and in the following we need the notion of the new shadow. Let P be a ranked poset and $\preceq$ a new linear order on P. The *new lower* (and *upper*) *shadow* of an element p of P are defined, respectively, by

$$\Delta_{new}(p) := \{q \in P : q \lessdot p \text{ and there is no } v \prec p \text{ with } q \lessdot v\},$$
$$\nabla_{new}(p) := \{q \in P : q \gtrdot p \text{ and there is no } v \succ p \text{ with } q \gtrdot v\}.$$

In contrast to the shadows of elements, the new shadows of elements are always pairwise disjoint. Again the new shadow of a subset of P is defined to be the union of the new shadows of its elements.

Now let us look at $SO(2)$ together with the linear order given above.

Lemma 1 *Let $\boldsymbol{a} = a_1 \dots a_n \in SO(2)$, $n \geq 1$. Then*

$$\Delta_{new}(\boldsymbol{a}) = \begin{cases} \emptyset & \text{if } a_1 = 1, \\ \{a_2 \dots a_n\} & \text{if } a_1 = 0. \end{cases}$$

Proof. Let $\boldsymbol{b} \in \Delta(\boldsymbol{a})$. If $\boldsymbol{b}$ can be obtained from $\boldsymbol{a}$ by deleting a one, then $\boldsymbol{b} \lessdot \boldsymbol{b}0 \prec \boldsymbol{a}$, hence $\boldsymbol{b} \notin \Delta_{new}(\boldsymbol{a})$. Thus suppose that $\boldsymbol{b} = a_1 \dots a_{j-1} a_{j+1} \dots a_n$ where $a_j = 0$ ($j \in \{1, \dots, n\}$). Without loss of generality we may assume that $a_{j-1} = 1$ for $j > 1$ (we may always delete the first zero in the corresponding "block" of zeros). If $j > 1$, then $\boldsymbol{b} \lessdot 0a_1 \dots a_{j-2} 1 a_{j+1} \dots a_n \prec a_1 \dots a_{j-1} 0 a_{j+1} \dots a_n = \boldsymbol{a}$, hence $\boldsymbol{b} \notin \Delta_{new}(\boldsymbol{a})$. If $j = 1$, then $\boldsymbol{b} = a_2 \dots a_n \in \Delta_{new}(\boldsymbol{a})$. Indeed, let $\boldsymbol{b} \lessdot a_2 \dots a_{l-1} \epsilon a_l \dots a_n =: \boldsymbol{c}, l \in \{2, \dots, n+1\}$, where for $l > 2$ w.l.o.g. $a_{l-1} \neq \epsilon$. If $\epsilon = 1$ then, clearly, $\boldsymbol{c} \succ \boldsymbol{a}$. If $\epsilon = 0$ and $l = 2$, then $\boldsymbol{c} = \boldsymbol{a}$, and if $\epsilon = 0$ and $l > 2$, then $\boldsymbol{c} = a_2 \dots a_{l-1} 0 a_l \dots a_n \succ a_1 \dots a_{l-2} 1 a_l \dots a_n = \boldsymbol{a}$. ∎

Lemma 2 (a) *Let $\boldsymbol{a} = \mathbf{1}_n$. Then*

$$\nabla_{new}(\boldsymbol{a}) = \{0\mathbf{1}_n, 10\mathbf{1}_{n-1}, \mathbf{1}_2 0\mathbf{1}_{n-2}, \dots, \mathbf{1}_n 0, \mathbf{1}_{n+1}\}, \ \text{i.e., } |\nabla_{new}(\boldsymbol{a})| = n+2.$$

(b) *Let $\boldsymbol{a} = \mathbf{1}_l 0 a_{l+2} \dots a_n$, where $l \in \{0, \dots, n-1\}$, and let $\boldsymbol{\alpha} := a_{l+2} \dots a_n$. Then*

$$\nabla_{new}(\boldsymbol{a}) = \{0\mathbf{1}_l 0\boldsymbol{\alpha}, 10\mathbf{1}_{l-1}0\boldsymbol{\alpha}, \dots, \mathbf{1}_{l-1}010\boldsymbol{\alpha}, \mathbf{1}_l 00\boldsymbol{\alpha}\}, \ \text{i.e., } |\nabla_{new}(\boldsymbol{a})| = l+1.$$

Proof. Case (a) is trivial, so we study case (b). Let $\boldsymbol{b} \in \nabla(\boldsymbol{a})$. If $\boldsymbol{b}$ can be obtained from $\boldsymbol{a}$ by inserting a one then we may delete any zero in $\boldsymbol{b}$ to obtain an element $\boldsymbol{c}$ with $\boldsymbol{b} \gtrdot \boldsymbol{c} \succ \boldsymbol{a}$, hence $\boldsymbol{b} \notin \nabla_{new}(\boldsymbol{a})$. Thus suppose that $\boldsymbol{b} = a_1 \dots a_h 0 a_{h+1} \dots a_n$ ($h \in \{0, \dots, n\}$) where w.l.o.g. $a_h = 1$ if $h > 0$. If $h > l$ (i.e., we inserted a zero after a one not belonging to the first block of ones), then $\boldsymbol{b} \gtrdot \mathbf{1}_l a_{l+2} \dots 10 a_{h+1} \dots a_n \succ \mathbf{1}_l 0 a_{l+2} \dots 1 a_{h+1} \dots a_n = \boldsymbol{a}$, hence $\boldsymbol{b} \notin \nabla_{new}(\boldsymbol{a})$. If $h \leq l$ (i.e., $\boldsymbol{b}$ is any element of the RHS of the claim), then $\boldsymbol{b} \in \nabla_{new}(\boldsymbol{a})$. Indeed, let $\boldsymbol{c}$ be any word which can be obtained from $\boldsymbol{b}$ by deleting any letter ϵ. If $\epsilon = 1$ then, clearly, $\boldsymbol{c} \prec \boldsymbol{a}$. In the case $\epsilon = 0$ we have $\boldsymbol{c} = \boldsymbol{a}$ if the previously inserted zero is deleted and $\boldsymbol{c} \prec \boldsymbol{a}$ if a zero right to the inserted zero is deleted. ∎

For subsets F, G of a poset P we use the notation $F \prec G$ if $p \prec q$ for all $p \in F$ and all $q \in G$.

Lemma 3 *Let $\boldsymbol{a}, \boldsymbol{a'} \in N_n$. If $\boldsymbol{a} \prec \boldsymbol{a'}$, then $\nabla_{new}(\boldsymbol{a}) \prec \nabla_{new}(\boldsymbol{a'})$.*

Proof. We may suppose that $\boldsymbol{a'} \neq \mathbf{1}_n$ since this case is trivial. If $r^B(\boldsymbol{a}) > r^B(\boldsymbol{a'})$ then, by Lemma 2, evidently $\nabla_{new}(\boldsymbol{a}) \prec \nabla_{new}(\boldsymbol{a'})$. Thus we may supppose that both numbers are equal. Let

$$\boldsymbol{a} = \mathbf{1}_l 0 a_{l+2} \dots a_n \text{ and } \boldsymbol{a'} = \mathbf{1}_{l'} 0 a'_{l'+2} \dots a'_n.$$

Let i be the largest index with $a_i \neq a'_i$. By supposition, $a_i = 1, a'_i = 0$. If $i \geq \max\{l+2, l'+2\}$ then by Lemma 2, $\nabla_{new}(\boldsymbol{a}) \prec \nabla_{new}(\boldsymbol{a'})$. The only case remaining is $i = l'+1$ and $l' > l$ (recall that $r^B(\boldsymbol{a}) = r^B(\boldsymbol{a'})$). The greatest element of $\nabla_{new}(\boldsymbol{a})$ w.r.t. $\preceq$ is $\boldsymbol{b} := \mathbf{1}_l 00 a_{l+2} \dots a_n$ and the smallest element of $\nabla_{new}(\boldsymbol{a'})$ is $\boldsymbol{b'} := 0\mathbf{1}_{l'} 0 a'_{l'+2} \dots a'_n$. Obviously, $\boldsymbol{b} \prec \boldsymbol{b'}$. ■

Because each element in level N_{i+1} appears exactly once in the new upper shadow of an element of N_i we obtain as an immediate consequence:

Lemma 4 *The upper shadow of a final segment of N_i in $SO(2)$ is a final segment of N_{i+1}.*

Now we look at the Boolean lattice B_n realized as $S(1, \dots, 1)$. We denote the corresponding (new) shadow operators by $\Delta^B, \Delta^B_{new}, \nabla^B, \nabla^B_{new}$.

Lemma 5 (a) *Let $\boldsymbol{a} = \mathbf{1}_n$. Then*

$$\Delta^B_{new}(\boldsymbol{a}) = \{(0, \mathbf{1}_{n-1}), (1, 0, \mathbf{1}_{n-2}), \dots, (\mathbf{1}_{n-1}, 0)\}, \text{ i.e., } |\Delta^B_{new}(\boldsymbol{a})| = n.$$

(b) *Let $\boldsymbol{a} = (\mathbf{1}_l, 0, a_{l+2}, \dots, a_n)$ where $l \in \{0, \dots, n-1\}$ and let $\boldsymbol{\alpha} := (a_{l+2}, \dots, a_n)$. Then*

$$\Delta^B_{new}(\boldsymbol{a}) = \{(0, \mathbf{1}_{l-1}, 0, \boldsymbol{\alpha}), (1, 0, \mathbf{1}_{l-2}, 0, \boldsymbol{\alpha}), \dots, (\mathbf{1}_{l-1}, 0, 0, \boldsymbol{\alpha})\}$$

i.e., $|\Delta^B_{new}(\boldsymbol{a})| = l$.

Proof. Case (a) is trivial, hence we look at Case (b). Let $\boldsymbol{b} \in \Delta^B(\boldsymbol{a})$. Then $\boldsymbol{b}$ can be obtained from $\boldsymbol{a}$ by replacing some $a_h = 1$ by 0. If $h > l$ (i.e., the replacement is not contained in the first block), then

$$\begin{array}{rll} \boldsymbol{b} & \lessdot & (\mathbf{1}_l, 1, a_{l+2}, \ldots, a_{h-1}, 0, a_{h+1}, \ldots, a_n) \\ & \preceq_{rl} & (\mathbf{1}_l, 0, a_{l+2}, \ldots, a_{h-1}, 1, a_{h+1}, \ldots, a_n) \quad = \quad \boldsymbol{a}, \end{array}$$

hence $\boldsymbol{b} \notin \Delta^B_{new}(\boldsymbol{a})$. If $h \le l$ (i.e., $\boldsymbol{b}$ is any element of the RHS of the claim), then $\boldsymbol{b} \in \Delta^B_{new}(\boldsymbol{a})$. Indeed, let $\boldsymbol{c} \in \nabla^B(\boldsymbol{b})$. If $\boldsymbol{c} \ne \boldsymbol{a}$, then $\boldsymbol{c}$ can be obtained from $\boldsymbol{b}$ by replacing some $b_g = 0$ by 1, where $g > l$. But then $\boldsymbol{c} \succ_{rl} (\boldsymbol{a})$. ■

By Lemmas 2 and 5, we have:

Lemma 6 *For $\boldsymbol{a} \in SO(2)$,*

$$|\nabla_{new}(\boldsymbol{a})| = |\Delta^B_{new}(\boldsymbol{a})| + \begin{cases} 2 & \text{if } \boldsymbol{a} = \mathbf{1}_n \text{ for some } n, \\ 1 & \text{otherwise.} \end{cases}$$

Since, for every m, $\mathcal{L}_{vip}(m, B_n)$ is a filter in B_n we may prove immediately by induction on m:

Lemma 7 *For $1 \le m \le 2^n$, we have*

$$|\Gamma^B(\mathcal{L}_{vip}(m, B_n))| = \sum_{\boldsymbol{a} \in \mathcal{L}_{vip}(m, B_n)} |\Delta^B_{new}(\boldsymbol{a})| + 1.$$

Lemmas 6 and 7 imply an important relation between the Boolean lattices and the 0–1 subword order:

Theorem 2 *Let F be a final segment in the n–th level of $SO(2)$. Then*

$$|\nabla(F)| = |\Gamma^B(F)| + |F|.$$

Proof. We have

$$|\nabla(F)| = \sum_{\boldsymbol{a} \in F} |\nabla_{new}(\boldsymbol{a})| = \sum_{\boldsymbol{a} \in F} |\Delta^B_{new}(\boldsymbol{a})| + |F| + 1 = |\Gamma^B(F)| + |F|.$$

■

In addition, we need an estimation for Γ^B which is related to results of Eckhoff, Wegner [9] and Katona [14].[2] Let

$$G(n,m) := |\Gamma^B(\mathcal{C}_{vip}(m, B_n))|.$$

Lemma 8 *If $m_0, m_1 \leq 2^{n-1}$, then*

$$G(n, m_0 + m_1) \leq \max\{m_0, G(n-1, m_1)\} + \max\{m_1, G(n-1, m_0)\}.$$

Proof. For a set $F \subseteq B_{n-1}$, let

$$(F, l) := \{(a_1, \ldots, a_{n-1}, l) : (a_1, \ldots, a_{n-1}) \in F\}.$$

Let $F'_i := \mathcal{C}_{vip}(m_i, B_{n-1})$ and let $F_i := (F'_i, i)$, $i = 0, 1$. Then, by Theorem 1,

$$\begin{aligned} G(n, m_0 + m_1) \leq & |\Gamma^B(F_0 \cup F_1)| \\ = & |(F'_0, 1) \cup (\Gamma^B(F'_1), 1) \cup (F'_1, 0) \cup (\Gamma^B(F'_0), 0)| \\ \leq & \max\{m_0, G(n-1, m_1)\} + \max\{m_1, G(n-1, m_0)\} \end{aligned}$$

since $(F'_0, 1)$ and $(\Gamma^B(F'_1), 1)$ as well as $(F'_1, 0)$ and $(\Gamma^B(F'_0), 0)$ are related by inclusion. ■

Theorem 3 *For all $F \subseteq N_n(SO(2))$ and all $n \in \mathbb{N}$,*

$$|\nabla(\mathcal{L}F)| \leq |\nabla(F)|.$$

Proof. We proceed by induction on n. The cases $n = 0$ and $n = 1$ are trivial, thus let us look at the step $n-1 \to n$. We classify the given F w.r.t. the last coordinate, i.e., we write

$$F = (F'_0, 0) \cup (F'_1, 1).$$

Let $m_i := |F'_i|, i = 0, 1$ (note that $m_0, m_1 \leq 2^{n-1}$). Obviously,

$$\nabla(F) = (F, 0) \cup (\nabla(F'_0), 0) \cup (F, 1) \cup (\nabla(F'_1), 1).$$

[2]We are thanking Roger Labahn for proposing a symmetric formulation of Lemma 8.

Moreover, by the induction hypothesis, Lemma 8, and Theorem 2,

$$\begin{aligned}
|\nabla(F)| &\geq \max\{|F|, |\nabla(F_0')|\} + \max\{|F|, |\nabla(F_1')|\} \\
&\geq \max\{|F|, |\nabla(\mathcal{L}F_0')|\} + \max\{|F|, |\nabla(\mathcal{L}F_1')|\} \\
&= \max\{|F|, |\Gamma^B(\mathcal{L}F_0')| + |F_0'|\} + \max\{|F|, |\Gamma^B(\mathcal{L}F_1')| + |F_1'|\} \\
&= \max\{m_0 + m_1, G(n-1, m_0) + m_0\} \\
&\quad + \max\{m_0 + m_1, G(n-1, m_1) + m_1\} \\
&= \max\{m_1, G(n-1, m_0)\} + \max\{m_0, G(n-1, m_1)\} + m_0 + m_1 \\
&\geq G(n, m_0 + m_1) + m_0 + m_1 \\
&= |\Gamma^B(\mathcal{L}F)| + |F| \\
&= |\nabla(\mathcal{L}F)|.
\end{aligned}$$

■

3 Solution of Problem 1 for a class of Macaulay posets

We call a Macaulay poset P *rank greedy* if the associated linear order $\preceq$ is a linear extension of the ordering $\leq$ of P, i.e., if

$$p \leq q \text{ implies } p \preceq q \tag{3}$$

and if

$$\Delta(p) \prec q, \;\; r(p) > r(q) \text{ imply } p \prec q. \tag{4}$$

It is obvious that any Macaulay poset can be made rank greedy by rearranging the linear order $\preceq$, where the new linear order $\preceq'$ can be obtained from $\preceq$ iteratively: $\preceq'$ starts with the smallest element w.r.t. $\preceq$, and if we already have some initial part of $\preceq'$, then as the next element take the smallest w.r.t. $\preceq$ of maximum rank among those whose shadows are already completely in that initial part of $\preceq'$.

It is an easy exercise to verify the following proposition (see [10]):

Proposition 2 *If P is a rank greedy Macaulay poset, then so is its dual P^*.*

This proposition is equivalent to the following statement: For Macaulay posets the conditions (3) and (4) are equivalent to conditions (3) and

$$q \prec \nabla(p),\ r(p) < r(q) \text{ imply } q \prec p \tag{5}$$

(see the proof of Proposition 8.3.1 in [10]). For example, in [10] it is shown that the Macaulay posets from Examples 1 to 5 are rank greedy.

Theorem 4 *The 0–1 subword order $SO(2)$ is rank greedy.*

Proof. It is easy to check that $\preceq$ from Example 6 is a linear extension of $\leq$. Let us verify (5), thus let

$$\boldsymbol{b} \prec \nabla(\boldsymbol{a}) \text{ and } r(\boldsymbol{a}) < r(\boldsymbol{b}).$$

From Lemmas 2 and 4 one can easily derive that the smallest element of $\nabla(\boldsymbol{a})$ w.r.t. $\preceq$ is $\boldsymbol{a'} := 0\boldsymbol{a}$. Hence we have $\boldsymbol{b} \prec \boldsymbol{a'}$ and thus $\boldsymbol{b} \prec \boldsymbol{a}$. ■

The importance of rank greedy Macaulay posets is given by the next theorem.

Theorem 5 *Let P be a rank greedy Macaulay poset and let $w : \{0, \dots, r(P)\} \to \mathbb{R}$ be increasing. Then, for $m = 1, \dots, |P|$,*

$$w(\mathcal{C}(m, P)) = \max\{w(I) : \ I \text{ is an ideal, } |I| = m\}.$$

The proof and some historical remarks as well as applications to edge isoperimetric problems are contained in [10]. Here we only mention Ahlswede and Katona [2] who studied B_n and Bezrukov [5] who formulated this theorem in an equivalent way. For $SO(2)$ this result gives the trivial set $\{\epsilon, 0, 00, 000, 0000, \dots\}$. But a restriction to the first, say $n+1$, levels of $SO(2)$ is possible because of the following trivial observation.

Proposition 3 *If P is a rank greedy Macaulay poset, then so is the poset induced by $\cup_{i=0}^{n} N_i$ where $0 \leq n \leq r(P)$.*

4 Solution of Problems 2 and 3 for a class of Macaulay posets

The Macaulay poset is said to be *little–submodular* if

$$|\Delta(\mathcal{C}F_1)|+|\Delta(\mathcal{C}F_2)| \geq |\Delta(\mathcal{C}(F_1\cap F_2))|+|\Delta(\mathcal{C}(F_1\cup F_2))| \text{ for all } F_1, F_2 \subseteq N_i$$
$$\text{with } F_1\cap F_2=\emptyset \text{ or } F_1\cup F_2 = N_i \text{ and for all } i.$$

It is called *additive* if

$$|\Delta_{new}(F_1)| \geq |\Delta_{new}(F_2)| \geq |\Delta_{new}(F_3)| \text{ for all segments } F_1, F_2, F_3 \subseteq N_i$$
$$\text{with } |F_1| = |F_2| = |F_3| \text{ where } F_1 \text{ is initial and } F_3 \text{ is final and for all } i.$$

From [10] we know:

Proposition 4 *Let P be a Macaulay poset. Then it is little–submodular iff it is additive.*

Proposition 5 *Let P be a finite graded Macaulay poset. Then P is little–submodular iff so is its dual P^*.*

The Macaulay poset P is called *shadow increasing* if

$$|\Delta(\mathcal{C}F_1)| \geq |\Delta(\mathcal{C}F_2)| \text{ for all } F_1 \subseteq N_{i+1}, F_2 \subseteq N_i \text{ with } |F_1| = |F_2| \text{ and for all } i.$$

In [10] the following is shown:

Proposition 6 *Let P be a graded shadow increasing Macaulay poset. Then P is rank unimodal and*

$$|\Delta(\mathcal{C}F_1)| \geq |\Delta(\mathcal{C}F_2)| \textit{ for all } F_1 \subseteq N_u, F_2 \subseteq N_l \textit{ with } |F_1| = |F_2| \textit{ and } l < u.$$

In Figure 1, there is given an example of a graded shadow increasing Macaulay poset whose dual is not shadow increasing.

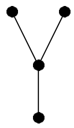

Figure 1

Later we will need the following property. The Macaulay poset is called *final shadow increasing* if

$$|\Delta_{new}(\mathcal{L}F_1)| \geq |\Delta_{new}(\mathcal{L}F_2)| \text{ for all } i, F_1 \subseteq N_{i+1}, F_2 \subseteq N_i \text{ with } |F_1| = |F_2|. \tag{6}$$

Proposition 7 *Let P be a graded, little–submodular, shadow increasing Macaulay poset. If P^* is shadow increasing, then P is final shadow increasing and, moreover,*

$$|\Delta_{new}(\mathcal{L}F_1)| \geq |\Delta_{new}(\mathcal{L}F_2)| \textit{ for all } F_1 \subseteq N_u, F_2 \subseteq N_l \textit{ with } |F_1| = |F_2|,\ l < u. \tag{7}$$

Proof. Let $F_1 \subseteq N_{i+1}$ and $F_2 \subseteq N_i$ with $|F_1| = |F_2|$.

Case 1. $F_1 = N_{i+1}$. Then by supposition and Proposition 4

$$|\Delta_{new}(\mathcal{L}F_1)| = |\Delta(\mathcal{C}F_1)| \geq |\Delta(\mathcal{C}F_2)| = |\Delta_{new}(\mathcal{C}F_2)| \geq |\Delta_{new}(\mathcal{L}F_2)|.$$

Case 2. $F_1 \neq N_{i+1}$. Let $G_j := \Delta_{new}(\mathcal{L}F_j)$, $j = 1, 2$. Then it is easy to see that the sets $G_j, j = 1, 2$, are initial segments in P^* and that

$$\mathcal{L}F_j \supseteq \nabla(G_j),\ j = 1, 2.$$

Let e be the last element of $N_i \setminus G_1$ w.r.t. $\preceq$. This element exists since $\mathcal{L}F_1 \neq N_{i+1}$ and thus $\Delta_{new}(\mathcal{L}F_1) \neq N_i$. Clearly, $G_1 \cup \{e\}$ is initial in P^* and

$$\mathcal{L}F_1 \subsetneq \nabla(G_1 \cup \{e\}).$$

Assume that the claim is false, i.e., $|G_1| < |G_2|$. Since P^* is shadow increasing, we obtain

$$|F_1| = |\mathcal{L}F_1| < |\nabla(G_1 \cup \{e\})| \leq |\nabla(G_2)| \leq |\mathcal{L}F_2| = |F_2|,$$

a contradiction. Now the inequality (7) follows from an iterated application of (6) noting that P is rank unimodal by Proposition 6. ■

In Figure 2, there is given an example of a graded shadow increasing Macaulay poset such that P^* is shadow increasing but P is not final shadow increasing.

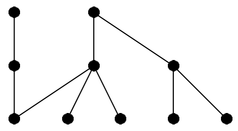

Figure 2

The dual of the poset given in Figure 1 is an example of a graded little–submodular Macaulay poset P such that P^* is shadow increasing but P is not final shadow increasing. For the sake of completeness, we show that the reverse direction is true under a weaker supposition.

Proposition 8 *Let P be a Macaulay poset. If P is final shadow increasing, then P^* is shadow increasing.*

Proof. Assume the contrary. Then there exist some final segments $G_2 \subseteq N_i, G_1 \subseteq N_{i+1}$ of same size such that for $F_j := \nabla(G_j)$, $j = 1, 2$,

$$|F_1| > |F_2|.$$

Obviously, $G_j \subseteq \Delta_{new}(F_j)$ and the sets $F_j, j = 1, 2$, are final segments. Let a be the first element of F_1 w.r.t. $\preceq$. Because a is related to some element of G_1 we have

$$\Delta_{new}(F_1 \setminus \{a\}) \subsetneq G_1.$$

Noting that $F_1 \setminus \{a\}$ is a final segment, too, and that P is final shadow increasing we derive

$$|\Delta_{new}(F_2)| \geq |G_2| = |G_1| > |\Delta_{new}(F_1 \setminus \{a\})| \geq |\Delta_{new}(F_2)|,$$

a contradiction. ■

Essential for our optimization problems is the following definition. The Macaulay poset P is called *weakly shadow increasing* if

$$|\Delta(F_1)| \geq |\Delta_{new}(F_2)| \text{ for all segments } F_1 \subseteq N_u, F_2 \subseteq N_l \text{ with } |F_1| = |F_2|$$
$$\text{such that } F_1 \text{ is initial and } F_1 \cup F_2 \text{ is an antichain, and for all } l \leq u.$$

In Figure 3, there is given an example of a weakly shadow increasing Macaulay poset such that its dual is not weakly shadow increasing.

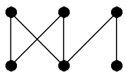

Figure 3

Proposition 9 *Let P be a graded, little–submodular, shadow increasing Macaulay poset. Then P is weakly shadow increasing.*

Proof. Let F_1 and F_2 be as in the definition of a weakly shadow increasing Macaulay poset and let F_2' be an initial segment of N_l of size $|F_2|$. Then by Propositions 4 and 6,

$$|\Delta(F_1)| \geq |\Delta(F_2')| \geq |\Delta_{new}(F_2)|.$$

■

In Figure 4, an example of a Macaulay poset P is given such that P and P^* are weakly shadow increasing but P is not rank unimodal.

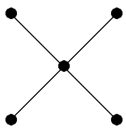

Figure 4

It is known (cf. [10, 17]) that the posets from Examples 1 to 5 are little–submodular and shadow increasing, hence, by Proposition 9, they are are weakly shadow increasing. Now let us look at the 0–1 subword order. We call a Macaulay poset *shadow symmetric* if

$$|\Delta(\mathcal{C}F)| = |\Delta(\mathcal{L}F)| \text{ for all } F \subseteq N_i \text{ and all } i.$$

The poset $SO(2)$ is shadow symmetric since the operation of exchanging the letters 0 and 1, obviously, defines an automorphism and replaces initial segments by final segments.

Theorem 6 *Let P be a Macaulay poset. If P is shadow symmetric, then it is little–submodular.*

Proof. Let $F_1, F_2 \subseteq N_i$.

Case 1. $F_1 \cap F_2 = \emptyset$. Then $\mathcal{C}F_1$ and $\mathcal{L}F_2$ are disjoint, i.e., $|F_1 \cup F_2| = |\mathcal{C}F_1 \cup \mathcal{L}F_2|$. Consequently,

$$\begin{aligned}|\Delta(\mathcal{C}F_1)| + |\Delta(\mathcal{C}F_2)| &= |\Delta(\mathcal{C}F_1)| + |\Delta(\mathcal{L}F_2)| \\ &= |\Delta(\mathcal{C}F_1) \cup \Delta(\mathcal{L}F_2)| + |\Delta(\mathcal{C}F_1) \cap \Delta(\mathcal{L}F_2)| \\ &\geq |\Delta(\mathcal{C}F_1 \cup \mathcal{L}F_2)| \geq |\Delta(\mathcal{C}(F_1 \cup F_2))| + 0.\end{aligned}$$

Case 2. $F_1 \cup F_2 = N_i$. Then $\mathcal{C}F_1 \cup \mathcal{L}F_2 = N_i$ and $|\mathcal{C}F_1 \cap \mathcal{L}F_2| = |F_1| + |F_2| - |N_i| = |F_1 \cap F_2|$. Consequently,

$$\begin{aligned}|\Delta(\mathcal{C}F_1)| + |\Delta(\mathcal{C}F_2)| &= |\Delta(\mathcal{C}F_1 \cup \mathcal{L}F_2)| + |\Delta(\mathcal{C}F_1) \cap \Delta(\mathcal{L}F_2)| \\ &\geq |\Delta(N_i)| + |\Delta(\mathcal{C}F_1 \cap \mathcal{L}F_2)| \\ &\geq |\Delta(\mathcal{C}(F_1 \cup F_2))| + |\Delta(\mathcal{C}(F_1 \cap F_2))|.\end{aligned}$$

■

Theorem 7 *The 0–1 subword order $SO(2)$ is shadow increasing.*

Proof. Let $F_1 \subseteq N_{n+1}$ and $F_2 \subseteq N_n$ be initial segments of same size, $n \geq 1$ ($n = 0$ is trivial). As in the proof of Theorem 3 we write F_1 in the form

$$F_1 = (F_0^1, 0) \cup (F_1^1, 1)$$

where $F_i^1 := \{a_1 \dots a_{n-1} : a_1 \dots a_{n-1} i \in F_1\}, i = 0, 1$. Obviously, the sets F_i^1 are initial segments in N_{n-1}, $i = 0, 1$. Because the new shadow only depends on the first coordinate (see Lemma 1) we have

$$|\Delta(F_1)| = |\Delta_{new}(F_1)| = |\Delta_{new}(F_0^1)| + |\Delta_{new}(F_1^1)| = |\Delta(F_0^1)| + |\Delta(F_1^1)|. \tag{8}$$

Since $|F_2| = |F_1| = |F_0^1| + |F_1^1|$ we may partition F_2 into two sets F_2' and F_2'' of size $|F_0^1|$ and $|F_1^1|$, respectively. By the little–submodularity of $SO(2)$ (Theorem 6),

$$|\Delta(\mathcal{C}F_2')| + |\Delta(\mathcal{C}F_2'')| \geq |\Delta(\mathcal{C}F_2)|,$$

i.e.,

$$|\Delta(F_0^1)| + |\Delta(F_1^1)| \geq |\Delta(F_2)|. \tag{9}$$

From (8) and (9) the claim follows. ■

Unfortunately, $SO(2)^*$ is not shadow increasing (take the one element sets $\{1 \dots 1\}$). Moreover, it is *shadow decreasing*:

Theorem 8 *Let $F_1 \subseteq N_n$ and $F_2 \subseteq N_{n+1}$ be final segments of same size in $SO(2)$, $n \geq 0$. Then*

$$|\nabla(F_1)| \leq |\nabla(F_2)|.$$

Proof. We write F_2 in the form $F_2 = (F_0^2, 0) \cup (F_1^2, 1)$. Obviously, the sets F_i^2 are final segments in N_n. Using Lemma 2 we have

$$\begin{aligned}
|\nabla(F_2)| &= |\nabla_{new}(F_0^2, 0)| + |\nabla_{new}(F_1^2, 1)| \\
&= |\nabla_{new}(F_0^2 \setminus \{\mathbf{1}_n\})| + |\nabla_{new}(\mathbf{1}_n 0)| \\
&\quad + |\nabla_{new}(F_1^2 \setminus \{\mathbf{1}_n\})| + |\nabla_{new}(\mathbf{1}_{n+1})| \\
&= |\nabla_{new}(F_0^2)| - (n+2) + (n+1) + |\nabla_{new}(F_1^2)| - (n+2) + (n+3) \\
&= |\nabla_{new}(F_0^2)| + |\nabla_{new}(F_1^2)| \\
&= |\nabla(F_0^2)| + |\nabla(F_1^2)|.
\end{aligned}$$

Now the proof can be completed exactly as the proof of Theorem 7 (note Proposition 5). ■

Because of this bad situation we introduced the notion of a weakly shadow increasing Macaulay poset. Fortunately we have:

Theorem 9 *The dual $SO(2)^*$ of the 0–1 subword order is weakly shadow increasing.*

Proof. Let $F_1 \subseteq N_l$ be a final segment of $SO(2)$ and $F_2 \subseteq N_u$ a segment with the properties $l \leq u$, $|F_1| = |F_2|$ and

$$\boldsymbol{a} \not\leq \boldsymbol{b} \text{ for all } \boldsymbol{a} \in F_1 \text{ and } \boldsymbol{b} \in F_2. \tag{10}$$

We have to show that

$$|\nabla(F_1)| \geq |\nabla_{new}(F_2)|.$$

We prove this inequality by induction on $m := |F_1| = |F_2|$. If $m = 1$, then $F_1 = \{\mathbf{1}_l\}$. The element $\boldsymbol{b}$ of F_2 cannot start with l ones because otherwise $\mathbf{1}_l \leq \boldsymbol{b}$. By Lemma 2,

$$|\nabla(F_1)| = |\nabla_{new}(\mathbf{1}_l)| = l + 2 > l \geq |\nabla_{new}(\boldsymbol{b})| = |\nabla_{new}(F_2)|.$$

Now we look at the step $< m \rightarrow m$, where $m > 1$. We partition N_n, $n \in \{l, u\}$, into the Boolean levels

$$N^B_{n,j} := \{\boldsymbol{a} \in N_n : r^B(\boldsymbol{a}) = j\}, j \in \{0, \ldots, n\}.$$

Let

$$h := \min\{j : N^B_{l,j} \cap F_1\} \neq \emptyset.$$

We have $h < l$ since $m > 1$. Further let

$$A := N^B_{l,h} \cap F_1.$$

By the definition of $\preceq$ for $SO(2)$, the set A is an initial segment in the Boolean level $N^B_{l,h}$ (with the linear order $\preceq_{rl}$). Let

$$k := \max\{j : N^B_{u,j} \cap F_2\} \neq \emptyset.$$

We have $k \leq h$ because we could otherwise delete in any element $\boldsymbol{b}$ of $N^B_{u,k} \cap F_2$ $u - l$ letters beginning with the zeros to obtain an element $\boldsymbol{a}$ in some $N^B_{l,j}$ where $h < j$. But then $\boldsymbol{a} \in F_1$ and $\boldsymbol{a} \leq \boldsymbol{b}$ in contrast to (10). Let

$$B := N^B_{u,k} \cap F_2.$$

Obviously, B is a final segment in the Boolean level $N^B_{u,k}$.

Case 1. $|B| \leq |A|$. Let A' be the set of the first $|B|$ elements of F_1. We have $A' \subseteq A$, and A' is a segment in $N^B_{l,h}$. Let

$$A'' := \{\boldsymbol{a}\mathbf{0}_{u-l} : \boldsymbol{a} \in A'\}.$$

Then A'' is a segment in $N^B_{u,h}$. Moreover, since $\mathbf{1}_l \notin A'$, we have by Lemma 2

$$|\nabla_{new}(A')| = |\nabla_{new}(A'')|. \tag{11}$$

Lemma 6 yields

$$|\nabla_{new}(A'')| = |\Delta^B_{new}(A'')| + |A''| = |\Delta^B_{new}(A'')| + |B|, \tag{12}$$

$$|\nabla_{new}(B)| = |\Delta^B_{new}(B)| + |B|. \tag{13}$$

If we apply Proposition 7 to the Boolean lattice B_u and the levels $N^B_{u,k}$, $N^B_{h,k}$ and use the additivity of B_n, then we obtain

$$|\Delta^B_{new}(A'')| \geq |\Delta^B_{new}(\mathcal{L}A'')| \geq |\Delta^B_{new}(B)|. \tag{14}$$

From (11) to (14) we derive

$$|\nabla_{new}(A')| \geq |\nabla_{new}(B)|.$$

By the induction hypothesis,

$$|\nabla(F_1 \setminus A')| \geq |\nabla_{new}(F_2 \setminus B)|.$$

Thus

$$|\nabla(F_1)| = |\nabla(F_1 \setminus A')| + |\nabla_{new}(A')| \geq |\nabla_{new}(F_2 \setminus B)| + |\nabla_{new}(B)| = |\nabla_{new}(F_2)|.$$

Case 2. $|B| > |A|$. Let B' be the set of the last $|A|$ elements of F_2. We have $B' \subseteq B$, and B' is a segment in $N^B_{u,k}$. Let

$$A' := \{\boldsymbol{a}\mathbf{0}_{u-l} : \boldsymbol{a} \in A\}.$$

Then A' is an initial segment in $N^B_{u,h}$. As in Case 1, we have

$$|\nabla_{new}(A)| = |\nabla_{new}(A')| = |\Delta^B_{new}(A')| + |A|,$$

$$|\nabla_{new}(B')| = |\Delta^B_{new}(B')| + |A|.$$

Moreover, since the Boolean lattice B_u is shadow increasing and additive,

$$|\Delta^B_{new}(A')| = |\Delta^B(A')| \geq |\Delta^B(\mathcal{C}B')| \geq |\Delta^B_{new}(B')|.$$

Thus

$$|\nabla_{new}(A)| \geq |\nabla_{new}(B')|$$

which, as in Case 1, yields

$$|\nabla(F_1)| \geq |\nabla_{new}(F_2)|.$$

■

The importance of weakly shadow increasing Macaulay posets is given by the next results.

Theorem 10 *Let P and P^* be weakly shadow increasing Macaulay posets and let $w : \{0, \ldots, r(P)\} \to \mathbb{R}_+$ be increasing. Let m be a positive integer not greater than the width of P. Then there are integers i and $1 \leq a \leq |N_i|$ such that the antichain*

$$\mathcal{C}(a, |N_i|) \cup (N_{i-1} \setminus \Delta(\mathcal{C}(a, |N_i|)))$$

has size m and generates an ideal of minimum weight in the class of all antichains of size m.

Hints for Proof. The proof is the same as the proof of Theorem 8.3.5 in [10]. Though there it was supposed that P and P^* are graded, little–submodular and shadow increasing Macaulay posets, a thorough inspection of the proof shows that the essential inequalities

$$|\Delta(X')| \geq |\Delta_{new}(X)|, \; |\Delta_{new}(Y)| \leq |\Delta(F_u)| \text{ as well as } |\nabla(F_l)| \geq |\nabla_{new}(X)|$$

follow already from the suppositions of the present theorem. ■

Corollary 1 *Let P and P^* be weakly shadow increasing Macaulay posets which are not antichains. Then P and P^* are graded and have the Sperner property, i.e., $d(P) = \max\{|N_j| : j \in \{0, \ldots, r(P)\}$.*

Proof. First we show that P (and thus P^*) is graded. Assume that there is a maximal element $p \in N_i$ such that $i < r(P)$. From the suppositions one can easily derive that there must be some $j > i$ such that for the first element q of N_j the inequality $|\Delta(q)| \geq 1$ holds. Thus we know that there is some last element v of $\Delta(q)$ w.r.t. $\preceq$. Let w be the last element of N_i. Since P^* is a Macaulay poset, $0 = |\nabla(p)| \geq |\nabla(w)|$. Obviously, $\{v, w\}$ is an antichain, but $0 = |\nabla(w)| < |\nabla_{new}(v)|$, a contradiction to the fact that P^* is weakly shadow increasing. In the same way it follows that each minimal element of P has rank 0, hence P is graded.

Now we prove the Sperner property. Let m be the width of P (and P^*). By Theorem 10, there exists an antichain F of size m such that for some i, $F \subseteq N_{i-1} \cup N_i, A := F \cap N_i$ is an initial segment, and $B := F \cap N_{i-1}$ is a final segment. In particular, $\Delta(A) = N_{i-1} \setminus B, \Delta_{new}(N_i \setminus A) = B$. It is sufficient to show that $m \leq \max\{|N_{i-1}|, |N_i|\}$. Assume the contrary. Then $|\Delta(A)| < |A|$ since otherwise $|F| = |A| + |B| \leq |\Delta(A)| + |B| \leq |N_{i-1}|$. Analogously, $|\nabla(B)| < |B|$.

Claim. $|A| > |N_i \setminus A|$ and $|B| > |N_{i-1} \setminus B|$.

Proof of Claim. By duality, it is enough to assume that $|A| \leq |N_i \setminus A|$. Let A' be the set of the first $|A|$ elements of $N_i \setminus A$ and let $F' := (F \cup A') \setminus \Delta_{new}(A')$. Then, obviously, F' is an antichain, and since P is weakly shadow increasing,

$$|F'| = |F| + |A'| - |\Delta_{new}(A')| \geq |F| + |A| - |\Delta_{new}(A)| > |F| = m,$$

a contradiction to the definition of the width of P. Thus the proof of the claim is complete.

We continue with the proof of the corollary. Let A' be the set of the first $|N_i \setminus A|$ elements of $|A|$. Then

$$|N_{i-1} \setminus B| = |\Delta(A)| \geq |\Delta(A')| \geq |\nabla_{new}(N_i \setminus A)| = |B| > |N_{i-1} \setminus B|,$$

a contradiction. ■

Theorem 11 *The numbers i and a in Theorem 10 are uniquely determined. We have $i = \min\{j : m \leq |N_j|\}$ and $a = \min\{b : b + |N_{i-1}| - |\Delta(\mathcal{C}(b, |N_i|))| = m\}$.*

Hint for Proof. The proof is the same as the proof of Theorem 8.3.6 in [10] (note that in the proof of Claim 1 one only needs the Sperner property of $P_{[0,i']}$ and not the rank unimodality). ∎

Theorem 12 *Let P and P^* be weakly shadow increasing Macaulay posets. Let $0 < m < |P|$ and $m = \sum_{j=0}^{i} |N_j| + a$, where $0 \le a < |N_{i+1}|$. Then $N_0 \cup \cdots \cup N_i \cup \mathcal{C}(a, |N_{i+1}|)$ has the maximum number of maximal elements in the class of all ideals F in P of size m.*

Proof. Let $\mathcal{I}(m)$ (resp. $\mathcal{I}_1(m)$) be the set of all ideals (resp. compressed ideals) in P of size m (an ideal F is compressed if $F_j := F \cap N_j$ is compressed for every j).

Claim 1. $\max\{|\max(F)| : F \in \mathcal{I}(m)\} = \max\{|\max(F)| : F \in \mathcal{I}_1(m)\}$.

Proof of Claim 1. The relation "$\ge$" is trivial, thus we prove "$\le$". Let $F \in \mathcal{I}(m)$. It is well–known (cf. [10, Theorem 8.2.1]) that the compression F_1 of F is an ideal as well (a compressed ideal). Thus $F_1 \in \mathcal{I}_1(m)$. Obviously, (with $f_j := |F \cap N_j|$)

$$|\max(F)| = \sum_{j=0}^{r(P)} (f_j - |\Delta(F_{j+1})|) \le \sum_{j=0}^{r(P)} (f_j - |\Delta(\mathcal{C}F_{j+1})|) = |\max(F_1)|$$

which proves the claim.

Thus we may restrict the investigations to compressed ideals. Obviously, we can generally assume that $m \ge W_0$ since otherwise the claim in the theorem is trivial. Let $\mathcal{I}_2(m)$ be the set of compressed ideals F in P of size m for which $F_0 = N_0$.

Claim 2. $\max\{|\max(F)| : F \in \mathcal{I}_1(m)\} = \max\{|\max(F)| : F \in \mathcal{I}_2(m)\}$.

Proof of Claim 2. Again the relation "$\ge$" is trivial, and we prove "$\le$". Let F be an optimal ideal of $\mathcal{I}_1(m)$. If $F_0 = N_0$ we are done, thus let $F_0 \subsetneq N_0$. We delete step by step maximal elements of F which are the last ones in their levels w.r.t. $\preceq$ and add consecutively w.r.t. $\preceq$ the elements after F_0 in N_0 to F (if there are still ones). Here the number of maximal elements does not decrease, and in each step we obtain a set from $\mathcal{I}_1(m)$. Since $m \geq W_0$ the procedure terminates in an ideal $F_2 \in \mathcal{I}_2(m)$, and clearly $|\max(F_1)| \leq |\max(F_2)|$ which proves the claim.

Thus we may restrict the investigations to compressed ideals with a full lower level. For $F \in \mathcal{I}_2(m)$ let

$$l = l(F) := \max\{j : F_j = N_j\} \text{ and } u = u(F) := \max\{j : F_j \neq \emptyset\}.$$

Let $\mathcal{I}_3(m)$ be the set of those optimal ideals F from $\mathcal{I}_2(m)$ for which $u(F) - l(F)$ is minimum. Obviously, it is enough to prove the next claim:

Claim 3. For $F \in \mathcal{I}_3(m)$, $u(F) - l(F) \leq 1$.

Proof of Claim 3. Assume that $u(F) - l(F) \geq 2$.

Case 1. $|F_u| \leq |N_{l+1}| - |F_{l+1}|$. Let G be the family which can be obtained from F by deleting F_u and adding the next $|F_u|$ elements in N_{l+1} after F_{l+1}. Note that $|G_{l+1} \setminus F_{l+1}| = |F_u|$. Obviously, $G \in \mathcal{I}_2(m)$. Since P is weakly shadow increasing,

$$|\Delta(F_u)| \geq |\nabla_{new}(G_{l+1} \setminus F_{l+1})|.$$

Consequently,

$$|\max(G)| = |\max(F)| - |F_u| + |F_u| + |\Delta(F_u)| - |\Delta_{new}(G_{l+1} \setminus F_{l+1})| \geq |\max(F)|.$$

Thus G is optimal, but $u(G) - l(G) < u(F) - l(F)$. This is a contradiction to $F \in \mathcal{I}_3(m)$.

Case 2. $|F_u| > |N_{l+1}| - |F_{l+1}|$. Let $H_l := N_l \setminus \Delta(F_{l+1})$ and $H_{l+1} := \nabla(H_l)$. Let $S := N_{l+1} \setminus (F_{l+1} \cup H_{l+1})$. Let F' be the family which can be obtained from F by deleting the last $|S|$ elements from F_u and adding S. Obviously, $|\max(F')| \geq |\max(F)| - |S| + |S|$, and thus $F' \in \mathcal{I}_3(m)$. If $|F'_u| \leq |N_{l+1}| - |F'_{l+1}|$, then we have a contradiction by Case 1, thus let $|F'_u| > |N_{l+1}| - |F'_{l+1}|$. By construction,

$$\Delta(F_{l+1}) = \Delta(F'_{l+1}) \text{ and } \nabla(H_l) = N_{l+1} \setminus F'_{l+1}.$$

Let T be the set of the last $|\nabla(H_l)|$ elements of F'_u and let G be the family which can be obtained from F' by deleting T and adding $N_{l+1} \setminus F'_{l+1}$. Obviously, $G \in \mathcal{I}_2(m)$ and $u(G) - l(G) < u(F) - l(F)$. It remains to prove that

$$|\max(G)| \geq |\max(F')| \tag{15}$$

since then we have the desired contradiction to $F' \in \mathcal{I}_3(m)$. We have

$$|\max(G)| = |\max(F')| - |T| + |T| + |\Delta_{new}(T)| - |H_l|.$$

Thus (15) is equivalent to

$$|\Delta_{new}(T)| \geq |H_l|.$$

Assume the contrary, $|\Delta_{new}(T)| < |H_l|$. Note that

$$|\Delta(\mathcal{C}T)| \geq |\Delta_{new}(N_{l+1} \setminus F'_{l+1})| = |H_l|$$

since P is weakly shadow increasing. Hence $|\Delta(F'_u)| \geq |H_l|$. Let U be the set of the last $|H_l|$ elements of $\Delta(F'_u)$. Then $\nabla_{new}(U) \supsetneq T$ since otherwise $\Delta_{new}(T) \supseteq \Delta_{new}(T \cap \nabla_{new}(U)) \supseteq U$, i.e., $|\Delta_{new}(T)| \geq |H_l|$ in contrast to our assumption. Consequently, $|\Delta_{new}(U)| > |T|$. Moreover, since P^* is weakly shadow increasing $|\nabla(H_l)| \geq |\nabla_{new}(U)|$. This yields the contradiction

$$|\nabla(H_l)| > |T| = |\nabla(H_l)|.$$

■

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