There is no ODC with all pages isomorphic to $C_4 \dot\cup C_3 \dot\cup C_3 \dot\cup v$

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Abstract

Let n be a natural number and $\mathcal{C} = \{P_0, \ldots, P_{n-1}\}$ a collection of spanning subgraphs of K_n, the complete graph on n vertices. $\mathcal{C}$ is called an Orthogonal Double Cover (ODC) if every edge of K_n belongs to exactly two elements of $\mathcal{C}$ and every two elements of $\mathcal{C}$ have exactly one edge in common. Gronau, Mullin and Schellenberg showed that the complete graph K_n has an ODC whose elements consist of cycles of length at most 4 and an isolated vertex, except for finitely many n. In this paper we scetch the computer aided proof of the nonexistence of such an ODC for $n = 11$.

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1 Introduction

Let K_n be the (undirected) complete graph on n vertices and $\mathcal{C} = \{P_0, P_1, \ldots, P_{n-1}\}$ a collection of spanning subgraphs of K_n, called *pages*. Throughout this paper we will not distinguish between a graph and its set of edges, if not noticed differently. We call $\mathcal{C}$ an *Orthogonal Double Cover (ODC)* of K_n if the two below conditions are satisfied:

(1) Every edge of K_n is contained in exactly two pages of $\mathcal{C}$.

(2) Every two pages of $\mathcal{C}$ intersect in exactly one edge.

There are several interesting results on ODC's of special structure. For instance Ganter and Gronau [3] verified a conjecture of Demetrovics, Füredi and Katona [2] by showing for all $n \equiv 1 (\mathrm{mod}\ 3)$, $n \neq 10$ the existence of an ODC of K_n whose pages consist of disjoint triangles and an isolated vertex.

Gronau, Mullin and Schellenberg [6] proved a conjecture of Chung and West [1] stating that for all n there is an ODC of K_n such that every page has maximum degree 2. A sharpening of this was given by Ganter, Gronau and Mullin [4]: For all $n \geq 4$, $n \neq 8$ there is an ODC of K_n in which each of the pages P_i $(i = 0, \ldots, n-1)$ consists of the isolated vertex $i+1$ and the union of disjoint cycles of length $3, 4$ or 5 only.

It is natural to raise the question of determining those n for which an ODC of K_n exists such that every page consists of an isolated vertex and cycles of length 3 or 4 only. Gronau, Mullin and Schellenberg [5] showed that this is the case for all but finitely many n. Their result is valid even for the directed case (which is not considered in this paper). The only values of n, for which the existence of the desired ODC is undecided, are $11, 12, 14$ and 30. We turn now to the scetch of the proof for the nonexistence of such an ODC for $n = 11$ (in the undirected and consequently also in the directed case).

2 Preliminary observations

From now on we shall call an ODC of K_{11} *proper* if all of its pages consist of pairwise vertex–disjoint cycles of length at most 4 and an isolated vertex. The definition of an ODC of K_n immediately implies that every page contains exactly $n-1$ edges, where double edges are impossible. Consequently, every page of a proper ODC of K_{11} must be isomorphic to $C_4 \dot\cup C_3 \dot\cup C_3 \dot\cup v$.

Proposition 1 *If $\mathcal{C} = \{P_0, \ldots, P_{10}\}$ is a proper ODC of K_{11}, then there exist an edge e, pages P_i, P_j $(0 \le i < j \le 10)$ and triangles T_i, T_j such that $e \in T_i \subset P_i$ and $e \in T_j \subset P_j$.*

Proof. If all triangles contained in any page of $\mathcal{C}$ were disjoint, then their union would consist of $6 \cdot 11 = 66$ edges. On the other hand, the K_{11} has just $\binom{11}{2} = 55$ edges. □

Assume that $\mathcal{C} = \{P_0, P_1, \ldots, P_{10}\}$ is a proper ODC of K_{11}. Let us agree upon writing down just the cycle structure of pages of $\mathcal{C}$ instead of the set of edges. According to Proposition 1, without loss of generality we may choose

$$P_0 = \{(1,2,3), (4,5,6), (7,8,9,10), (11)\},$$

where $\{1,2,x\} \in P_1$ and $x \in \{4,7,11\}$.

In Table 1 we listed all possibilities to choose P_1 such that there are no two isomorphic pairs (P_0, P_1). Although it is a bit tedious from the mathematical point of view, let us briefly describe our way of listing, since we want make it possible for the reader to check within a few minutes that we did not forget a choice for P_1.

For every $x \in \{4, 7, 11\}$ we first list all nonisomorphic choices for the second triangle of P_1, and then for every choice of the second triangle all choices of the remaining 4–cycle. For $i = 1, 2$ and $y \in \{1, 2, \dots, 11\}$ let $p_i(y) = 1$ if y is the isolated vertex of P_i and $p_i(y) = a$ if y is contained in an a–cycle of P_i. In order to avoid isomorphisms ϕ of the form $\phi(P_0, P_1) = (P_1', P_0)$, we demand $p_1(x) \leq p_2(3)$. If $p_1(x) = p_2(3)$, we further demand $p_1(v) \leq p_2(11)$, where v is the isolated vertex of P_2. Proceeding like this, we obtain the 32 possibilities given in Table 1 and additionally $P_1' = \{(1, 2, 4), (3, 5, 7), (6, 8, 10, 11), (9)\}$, where there is an isomorphism mapping P_1' to P_0 and P_0 to page number 2 in Table 1.

3 Algorithm

In the following we describe an algorithm that finds ODC's with all pages isomorphic to a given startpage P_0, if existing. Let n be the number of vertices of P_0 and let the edges of P_0 be numbered with 1 to $n - 1$. The algorithm consists of two parts.

In the first step we determine all graphs orthogonal to P_0. For this purpose we permute P_0 in every possible way and check whether the obtained graph is orthogonal to P_0. We sort the set of all orthogonal graphs on the edge they have in common with P_0 and obtain sets $\mathcal{M}_1$ to $\mathcal{M}_{n-1}$. Obviously, every ODC $\mathcal{C}$ consists of P_0 and exactly one element of every $\mathcal{M}_i,\ i = 1, \dots, n - 1$.

The second step is a backtracking on $\{\mathcal{M}_1, \dots, \mathcal{M}_{n-1}\}$ which checks orthogonality in every level and therefore minimizes the number of orthogonal combinations of elements from $\mathcal{M}_1, \dots, \mathcal{M}_{n-1}$. We define a procedure `find_orth` that determines all elements of every $\mathcal{M}_i$ which are orthogonal to every element of a given set $\{P_0, \dots, P_k\}$. These graphs are stored in $\mathcal{M}_1^{k+1}, \dots, \mathcal{M}_{n-1}^{k+1}$. Now we can describe the recursive search :

Table 1: Possible choices for P_1

1.	{	(1,2,4)	,	(3,5,7)	,	(6,8,11,10)	,	(9)	}
2.	{	(1,2,4)	,	(3,5,7)	,	(6,9,11,10)	,	(8)	}
3.	{	(1,2,4)	,	(3,7,9)	,	(5,8,6,10)	,	(11)	}
4.	{	(1,2,4)	,	(3,7,9)	,	(5,10,6,11)	,	(8)	}
5.	{	(1,2,4)	,	(3,7,9)	,	(6,8,10,11)	,	(5)	}
6.	{	(1,2,4)	,	(3,7,9)	,	(6,8,11,10)	,	(5)	}
7.	{	(1,2,4)	,	(5,7,9)	,	(3,6,8,10)	,	(11)	}
8.	{	(1,2,4)	,	(5,7,9)	,	(3,6,11,10)	,	(8)	}
9.	{	(1,2,4)	,	(5,7,9)	,	(3,8,6,10)	,	(11)	}
10.	{	(1,2,4)	,	(5,7,9)	,	(3,8,11,10)	,	(6)	}
11.	{	(1,2,4)	,	(5,7,9)	,	(3,10,6,11)	,	(8)	}
12.	{	(1,2,4)	,	(5,7,9)	,	(3,10,8,11)	,	(6)	}
13.	{	(1,2,4)	,	(5,7,11)	,	(3,6,8,10)	,	(9)	}
14.	{	(1,2,4)	,	(5,7,11)	,	(3,8,6,10)	,	(9)	}
15.	{	(1,2,4)	,	(5,7,11)	,	(3,9,6,11)	,	(8)	}
16.	{	(1,2,4)	,	(7,9,11)	,	(3,5,10,6)	,	(8)	}
17.	{	(1,2,4)	,	(7,9,11)	,	(3,6,8,10)	,	(5)	}
18.	{	(1,2,4)	,	(7,9,11)	,	(3,8,6,10)	,	(5)	}
19.	{	(1,2,7)	,	(4,8,10)	,	(3,5,9,6)	,	(11)	}
20.	{	(1,2,7)	,	(4,8,10)	,	(3,5,11,6)	,	(9)	}
21.	{	(1,2,7)	,	(4,8,10)	,	(3,6,11,9)	,	(5)	}
22.	{	(1,2,7)	,	(4,8,10)	,	(3,9,6,11)	,	(5)	}
23.	{	(1,2,7)	,	(4,8,11)	,	(3,9,6,10)	,	(5)	}
24.	{	(1,2,7)	,	(4,9,11)	,	(3,8,6,10)	,	(5)	}
25.	{	(1,2,7)	,	(8,10,11)	,	(3,5,9,6)	,	(4)	}
26.	{	(1,2,11)	,	(3,4,7)	,	(5,8,6,10)	,	(9)	}
27.	{	(1,2,11)	,	(3,4,7)	,	(5,9,6,10)	,	(8)	}
28.	{	(1,2,11)	,	(3,7,9)	,	(5,8,6,10)	,	(4)	}
29.	{	(1,2,11)	,	(4,7,9)	,	(3,5,10,6)	,	(9)	}
30.	{	(1,2,11)	,	(4,7,9)	,	(3,6,8,10)	,	(5)	}
31.	{	(1,2,11)	,	(4,7,9)	,	(3,8,6,10)	,	(5)	}
32.	{	(1,2,11)	,	(4,7,9)	,	(5,8,6,10)	,	(3)	}

```
procedure get_odc(P_0,...,P_k)
begin
  if k < n-1 then begin
   find_orth(P_0,...,P_k)
   for every element E of M_{k+1}^{k+1} do get_odc(P_0,...,P_k,E)
  end else SOLUTION
end
```

The search starts with get_odc(P_0). Obviously, $\mathcal{M}_i^1 = \mathcal{M}_i$, $i = 1, \ldots, n-1$. If $\mathcal{M}_1^1 \neq \emptyset$, the first element of $\mathcal{M}_1^1$ is taken as P_1 and get_odc is started recursively with the parameters (P_0, P_1) again. If $\mathcal{M}_{k+1}^{k+1} = \emptyset$ for some $k < n-1$, then P_k is exchanged with the next element of $\mathcal{M}_k^k$.

4 Implementation

We implemented the algorithm described in section 3. It works for every n, but for $n = 11$ we had already more than 2000 elements in every $\mathcal{M}_i$, and to check all combinations for fixed P_0, P_1 took about 12 hours on an HP 735/125 with not many other jobs. So our program would run about 3 years to check all combinations. Therefore we used the results of section 2 to reduce $\mathcal{M}_1$ to 32 elements and could check all combinations in about 16 days.

5 Result

There is no ODC with all pages isomorphic to $C_4 \,\dot\cup\, C_3 \,\dot\cup\, C_3 \,\dot\cup\, v$.

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