Numerical methods versus asymptotic expansion for torsion of hollow elastic beams

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Abstract

We consider the torsion problem for a hollow elastic beam. In a series of papers [4, 5, 6] a uniform method for the derivation of classical formulas for the torsional rigidity by Bredt, Prandtl and Vlasov was derived using an asymptotic expansion. We show that this expansion yields usefull approximations for torsional rigidiy if properly applied. Note that it does not converge in general. For the evaluation we use a numerical solution obtained by a Finite Difference Method. Finally, we examine the results of both methods for two example domains.

Keywords: Elliptic boundary value problems, torsion problems, asymptotic expansions
AMS subject classification: 35R30, 65N30

1 Introduction

In Saint Venant's theory of torsion there occurs a pair of 2D elliptic boundary value problems, one for the warping function u, another for the Prandtl function v. The warping function u describes the displacement along the axis of the beam, it satisfies Neumann type boundary conditions. For the Prandtl function v componentwise constant Dirichlet boundary conditions are posed. While in the case of a simply connected cross section the boundary value of the Prandtl function is meaningless, the choice of the different constants in the case of multiply connected cross sections is important. Conditions for the integrability of a first order pde for the warping function u – integrals of the tangential derivatives of u along contours surrounding each of the holes

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should vanish – fix the value of the flux of the gradient of Prandtl function. In this way it is uniquely determined up to an additive constant.
For closed hollow cross sections of annular type (cf. [4]) a transformation to arclength-thickness coordinates (s, z) proved useful. The pde for v contains a small parameter ϵ describing the thickness of the section. An expansion with respect to powers of ϵ reduces the 2D elliptic problem to a sequence of 1D boundary value problems in z-direction for each value of s. The four first approximations yield all known explicit solutions to special cases and the classical formulas for the torsional rigidity ([6]). However, till now there was no proof of the convergence of the method.
The plan of the present paper is as follows. In Section 2 we develop a recurrence scheme for the members of the asymptotic expansion. In Section 3 we present a numerical method for the solution of the original boundary value problem with special emphasis on the integrability condition. In Section 4 we give an estimate for the partial sum of the expansion. Then we choose two test domains for which the asymptotic expansion (in the present form) does not converge and compare the results of both analytical and numerical approach.

2 Asymptotic expansion

We restrict ourself to cross sections Ω of the annular type. In this case we have $\Omega = \Omega_1 \setminus \Omega_0$ with $\Omega_0 \subset \Omega_1$, both Ω_0 and Ω_1 simply connected, bounded domains, and the Prandtl function v is defined by the following bdvp

$$\begin{aligned} -\Delta v &= 2 \qquad \text{in } \Omega\,, & (1)\\ v &= 0 \qquad \text{on } \Gamma_1\,, & (2)\\ v &= c \qquad \text{on } \Gamma_0\,, & (3)\\ \oint_\Gamma \nabla v\, n ds &= -2A\,. & (4) \end{aligned}$$

Here we denote by Γ an arbitrary Jordan curve in Ω not homotopic to a point, $\Gamma_0 = \partial\Omega_0$, and $\Gamma_1 = \partial\Omega_1$. A is the area inside Γ, n is the outer normal unit vector to the domain inside Γ. We consider a family of plane domains for the Prandtl problem parametrized by ϵ. The domain Ω_ϵ will be obtained as the union of z-lifted curves from the given curve Γ_0. The latter is defined by

$$r : s \mapsto r(s) \quad s \in [0, l), \quad r(s) \in \mathbb{R}^2 \qquad (5)$$

where s is the arclength of the curve, i. e.,

$$\Omega_\epsilon = \{r(s, z) := r(s) - Qr'(s)z\epsilon\delta(s); \qquad 0 < z < 1; \qquad 0 \le s < l\} \qquad . \quad (6)$$

In our notation $\delta(s)$ is the thickness of Ω in the point of the coordinate s along Γ_0 and $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a $-\pi/2$ rotation matrix which maps the tangent vector $Qr'(s)$ onto the normal vector.

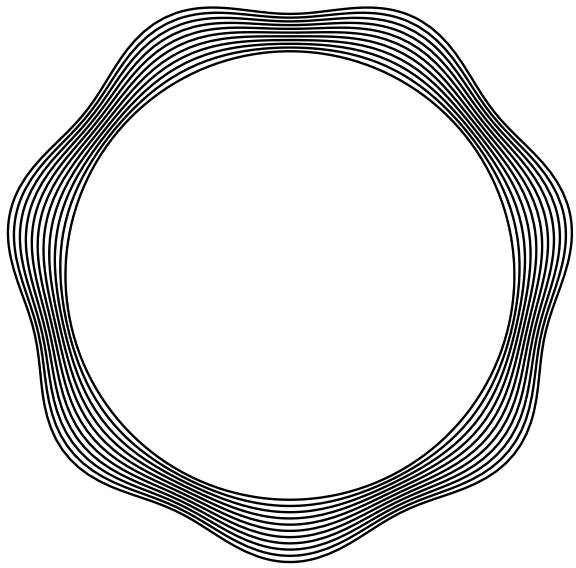

Figure 1: Test domain, cf. Sec. 4

Introducing a formal asymptotic expansion for the Prandtl function

$$v^\epsilon = \sum_{n=0}^{\infty} \epsilon^n v_n : \Omega_\epsilon \mapsto \mathbb{R} \tag{7}$$

and using the expansion of (1) computed in [5] we get the following recurrence scheme to determine the second derivative with respect to z of the function v_n.

$$\begin{aligned} v_{n,zz} = -\Big\{ & 2\varphi_n + \Big(3z\lambda v_{n-1,zz} + \lambda v_{n-1,z}\Big) + \Big(\delta^2 v_{n-2,ss} - 2z\delta\delta' v_{n-2,zs} \\ & + z^2\Big(3\lambda^2 + \delta'^2\Big) v_{n-2,zz} + \Big(2z\delta'^2 - z\delta\delta'' + 2z\lambda^2\Big) v_{n-2,z}\Big) \\ & + \Big(z\delta^2\lambda v_{n-3,ss} - 2z^2\delta\delta'\lambda v_{n-3,zs} + z^3\Big(\lambda^3 + \lambda\delta'^2\Big) v_{n-3,zz} \\ & - z\delta^3 K' v_{n-3,s} + z^2\Big(2\lambda\delta'^2 + \lambda^3 - \delta^2\left(\delta' K\right)'\Big) v_{n-3,z}\Big)\Big\}, \end{aligned} \tag{8}$$

where $K = K(s)$ is the curvature along Γ_0, $\lambda = \delta K$ and

$$\varphi_n = \begin{cases} \binom{3}{n-2} \delta^n K^{n-2} z^{n-2} & \text{for} \quad n = 2 \dots 5 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

corresponds to the right-hand side of (1). The function v_n itself is obtained by making use of the expansion of the boundary conditions

$$\begin{aligned} v_n(s,1) &= 0 \quad \forall \ s \in [0,l) \\ v_n(s,0) &= c_n \quad \forall \ s \in [0,l) \\ \int_{\Gamma_0} \frac{\partial v_n}{\partial z} \frac{d\,s}{\delta(s)} &= 0 \quad \forall \ n \neq 1 \\ \int_{\Gamma_0} \frac{\partial v_1}{\partial z} \frac{d\,s}{\delta(s)} &= 2A \qquad . \end{aligned} \tag{10}$$

Starting with $v_n = 0 \; \forall n \leq 0$ we obtain that v_n is of polynomial structure in z

$$v_n(s,z) = \sum_{i=0}^{n} a_{n,i}(s) z^i \,. \tag{11}$$

With the help of (8-11) we find the recurrence formula for the coefficients $a_{n,i}(s)$

$$\begin{aligned}
a_{n+1,j+1} \;&=\; \frac{-1}{j(j+1)} \Big\{ \boldsymbol{\delta}_{nj} b_{n+1} + (3j-2)j\lambda a_{n,j} + \delta^2 a''_{n-1,j-1} - 2(j-1)\delta\delta' a'_{n-1,j-1} \\
&\quad + \Big(j(j-1)\delta'^2 + (3j-4)(j-1)\lambda^2 - (j-1)\delta\delta'' \Big) a_{n-1,j-1} \\
&\quad + \delta^2 \lambda a''_{n-2,j-2} - \Big(\delta^3 K' + 2(j-2)\delta\delta'\lambda \Big) a'_{n-2,j-2} \\
&\quad + \Big((j-2)^2\lambda^3 + (j-2)(j-1)\delta'^2\lambda - (j-2)\delta^2(\delta' K)' \Big) a_{n-2,j-2} \Big\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, n \geq 1\,,\; 1 \leq j \leq n \\
a_{n+1,0} \;&=\; -\frac{1}{I_1} \int_{\Gamma_0} \sum_{j=2}^{n+1} a_{n+1,j}(s) \frac{d\,s}{\delta(s)} \qquad\qquad \forall\, n \geq 1 \\
a_{n+1,1} \;&=\; -\sum_{j=2}^{n+1} a_{n+1,j} \;-\; a_{n+1,0} \qquad\qquad \forall\, n \geq 1
\end{aligned} \tag{12}$$

with $\boldsymbol{\delta}_{nj}$ the Kronecker symbol, $a_{n,j} = 0$ for $n \leq 0 \vee j \leq 0$ and

$$b_n = \begin{cases} 2\binom{3}{n-2}\delta^2\lambda^{n-2} & n = 2\ldots 5 \\ 0 & \text{otherwise} \end{cases} \,.$$

We immediately get $a_{1,1} = -a_{1,0} = -\dfrac{2A}{I_1}$ with

$$I_1 = \int_{\Gamma_0} \frac{d\,s}{\delta(s)} \,.$$

To compare this approach with numerical results the recurrence formula (12) is implemented in Mathematica. The results for two test domains will be discussed in Sec. 4.

3 Numerical solution

The determination of the constant c in (3) from the integral condition (4) is the most crucial problem connected with the numerical solution of the problem (1-4). With c given, the problem (1–3) does not cause serious trouble.
Hence, we solve the problem in three steps. Since the integral on the left-hand side of (4) is an affine scalar function in c, say $\phi(c) = \phi_0 + \phi_1 c$ we determine the coefficients ϕ_0, ϕ_1 of this function by calculating two suitable auxiliary functions v_0 and v_1. Then we solve (4) for c and obtain the solution of (1–4) as a linear combination of v_0 and v_1. The auxiliary problems are

$$-\Delta v_0 = 2 \qquad \text{in } \Omega\,, \tag{13}$$

$$v_0 = 0 \qquad \text{on } \partial\Omega\,, \tag{14}$$

and

$$\begin{aligned} -\Delta v_1 &= 0 \qquad \text{in } \Omega\,, & (15)\\ v_1 &= 0 \qquad \text{on } \Gamma_1\,, & (16)\\ v_1 &= 1 \qquad \text{on } \Gamma_0\,. & (17)\end{aligned}$$

As we have

$$v = v_0 + c v_1\,, \tag{18}$$

and

$$\phi_0 = \oint_\Gamma \nabla v_0\, dn\,, \qquad \phi_1 = \oint_\Gamma \nabla v_1\, dn\,. \tag{19}$$

for c we obtain the value

$$c = \frac{-2A - \phi_0}{\phi_1}\,. \tag{20}$$

Remark 1 *For the choice $\Gamma = \Gamma_0$ we obtain as a matter of course the inequalities $\phi_0 > 0$, $\phi_1 < 0$. Further, ϕ_1 is independent of the choice of Γ, as it is the flux (through Γ) of the divergence free field ∇v_1 . Hence c is well defined and positive. It is also independent of Γ.*

For the discretization of the above bvp's we introduce a uniform rectangular grid with stepsize h, $\Omega_h = \Omega \cap \mathbb{Z}_h \times \mathbb{Z}_h$, $\mathbb{Z}_h = h\mathbb{Z} = \{hi \,|\, i \in \mathbb{Z}\}$. Given any curve γ, we define γ_h by

$$x_{ij} = h(i,j) = (ih, jh) \in \gamma_h \qquad \text{iff} \qquad \exists x \in \gamma \,:\, \text{dist}\,(x_i, x) < h/2\,.$$

Here we use the metric $\text{dist}\,(x,y) = \max\,(|x_1 - y_1|, |x_2 - y_2|)$.
That way we define Γ_h, Γ_{0_h}, and Γ_{1_h}, and prescribe the boundary values to the discrete functions v_0 and v_1 on the corresponding gridded contours. For the discretization of the Laplacian we use the standard 5-point stencil

$$\begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array}$$

while the right-hand side is simply discretized by $f_{ij} = 0$ or $f_{ij} = -2$, respectively. Since the right-hand side of (1) is constant we obtain the same system of equations by FDM and by linear finite elements on a standard triangulation.
The discrete equations are solved very effectively by iterative methods. We chose SOR with the overrelaxation parameter $\omega = 1.85$. Because of the special geometry of the considered domains and the constant boundary conditions we observed very quick convergence. The number of iterations needed to achieve a given error reduction is $O(h)$ as typical in 1D-problems, rather than $O(h^2)$. We solved problems with up to 10^5 equations with about 250 iterations, the error bound for the residuum being 10^{-10}.
Given a numerical solution v_h to (13,14) or (15-17), the integration (19) has to be carried out. It is not advisable to choose Γ as one of the components of the boundary because of the rough approximation there and the restriction to one-sided approximations for the normal derivative. Instead, we prefer a contour Γ_h composed entirely of inner points from Ω_h.

Let x_{ij} and $(x_{i'j'})$ be neighboring grid points on Γ_h. Then the four points $x_{ij} \pm y$, $x_{i'j'} \pm y$ with $y = h(j'-j, -i'+i)$ lay in $\Omega_h \cup \Gamma_i \cup \Gamma_o$. We obtain a numerical approximation of the contour integral as the sum of the differences between the value of v_h at the first two and at the second two of them, added up over all of Γ_h and divided by four. By Gauss' theorem, we calculate $2A$ by summing $h^2\left(i\left(j'-j\right)-j\left(i'-i\right)\right)$. Thus the numerical calculation of c by (20) and finally v by (18) is now straightforward.

Remark 2 *In order to obtain reasonable results the stepsize h should be at least 10 times smaller than the minimum of the section thickness. Consequently, for larger perimeter-to-thicknes ratio (> 100) the above method might result in very large systems of equations. However, in those cases the solution is almost perfectly linear in the direction normal to the contour, and the classical Bredt formulas should be applied.*
A FEM with much better approximation of the boundaries and considerably smaller systems of equations will be studied in a forthcoming paper.

4 (Non-)Convergence of ϵ-Expansion

We consider again the family Ω_ϵ of domains introduced in sec. 2. With w_n^ϵ we denote the partial sum of the formal expansion of v^ϵ, i.e.

$$w_n^\epsilon = \sum_{i=0}^{n} \epsilon^i v_i \ . \tag{21}$$

An estimate for the difference between partial sum and true solution of (1-4) gives the following

Lemma 1 *Let $\delta(s), K(s) \in C^{n+2}(\mathbb{R})$ l-periodic and Γ_0 smooth. Then it holds*

$$\|v^\epsilon - w_n^\epsilon\|_{C(\Omega_\epsilon)} \le C_n \epsilon^{n+1} \ ,$$

where $C_n = C_n(\delta, K, \Gamma_0)$.

Proof: We denote $\psi_n = v^\epsilon - w_n^\epsilon$. Using the expansion of the Laplacian from ([6]) we get

$$\begin{array}{rcll} \Delta\psi_n &=& \epsilon^n f_n & \text{in } \Omega_\epsilon \\ \psi_n &=& 0 & \text{on } \partial\Omega_\epsilon \end{array} \tag{22}$$

with

$$\psi_n = \varphi_{n+1} + 3z\lambda v_{n,zz} + \lambda v_{n,z} + o(\epsilon)$$

where $\varphi_n = 0$ for $n \ge 5$ (see (8)). As it is easily seen from (12) v_n depends only on Γ_0 and on $\delta(s), K(s)$ resp. their derivatives up to order n. Thus we have under the given smoothnes assumptions

$$\|\psi_n\|_{C(\Omega_\epsilon)} \le C_n$$

with some suitable constant. This together with (22) implies immediately the above inequality. ∎

Remark 3 *The sequence of constants C_n of Lemma 1 may grow very fast. So for practical applications the above result is not useful since the thickness of a given domain is fix. For a class of thickness functions $\delta(s)$ it is possible to show that the formal expansion will not converge for any fixed ϵ.*

We consider domains of the following type:

$$\Omega = \left\{ (x,y) : r \leq \sqrt{x^2+y^2} \leq r + \delta(s) \right\} , \tag{23}$$

with

$$\delta(s) = c_1 + c_2 * \sin(k * s/r) \, ; \, c_1 > c_2 > 0 \, ; \, k \in \mathbb{Z}^+ \, .$$

In this case Γ_0 is simply a circle and $K(s) = 1/r$ is a constant. The following picture shows the behaviour of the coefficients $a_{n,j}(s)$ in the asymptotic expansion (7,11) for the domain Ω with $r = 0.9; k = 7; c_1 = 0.2; c_2 = 0.05$.

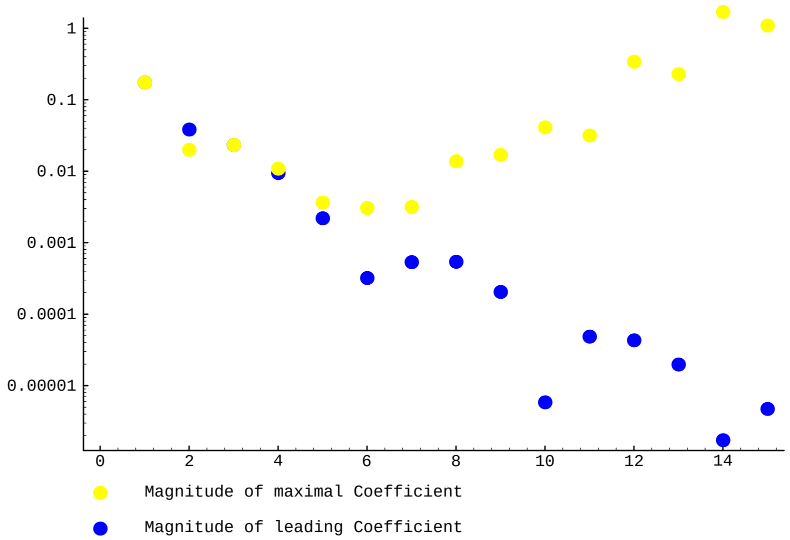

Figure 2: Logarithmic plot of the coefficients in the asymptotic for $s = 0.2$

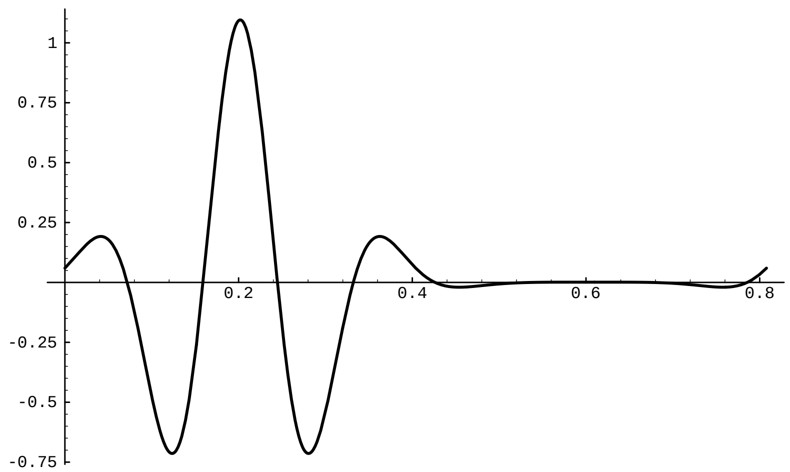

Figure 3: Coefficient $a_{15,4}(s)$ of (11)

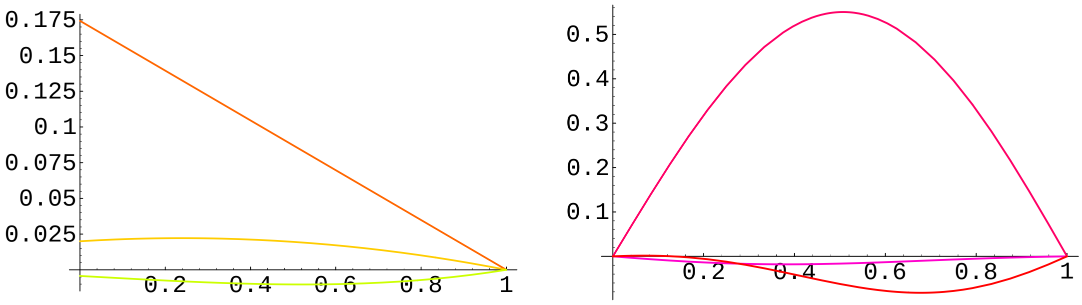

Figure 4: Functions v_n of the expansion series for $s = 0.2$

In the left part we see a cut at arclength $s = 0.2$ for the functions v_1, v_2, v_3 and in the right part for v_{13}, v_{14}, v_{15}. We remark that the functions v_n are very small for $n = 4, \cdots, 12$. For bigger n they grow rapidly.
Nevertheless, the next picture shows that we obtain from the ϵ-Expansion for this domain usefull results for torsional rigidity. The line represents the value obtained from the numerical method.

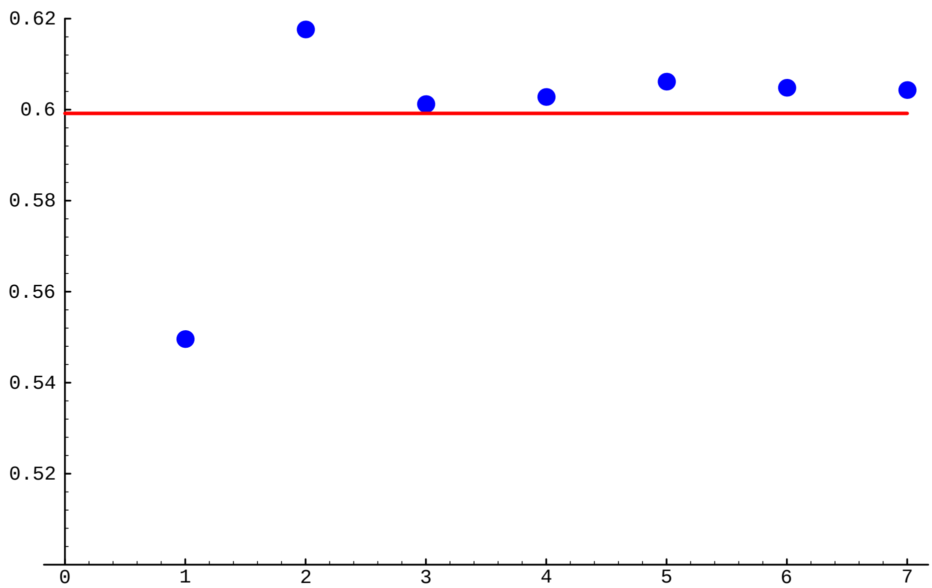

Figure 5: Torsional rigidity for $k = 7$

For a domain with 13 humps, i. e. $k = 13$, we get the following picture:

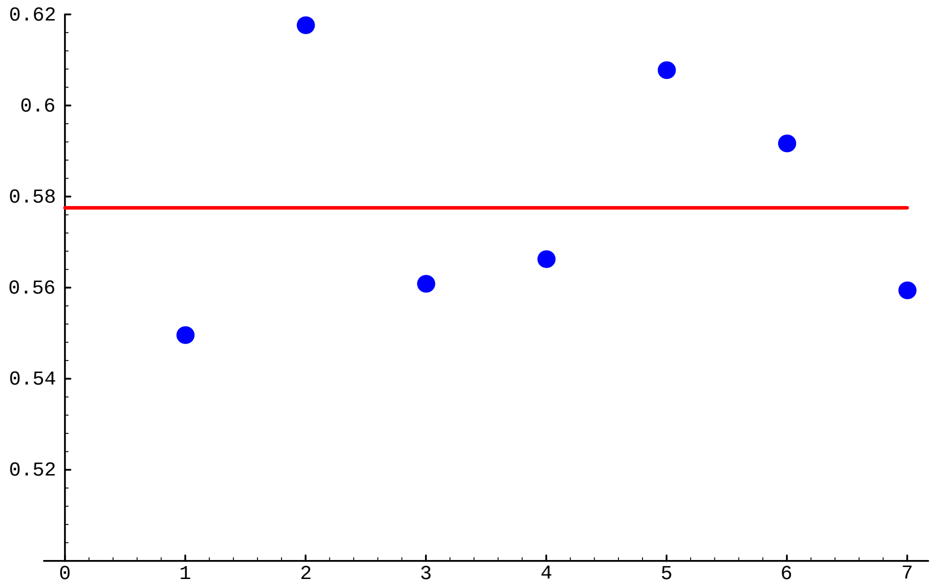

Figure 6: Torsional rigidity for $k = 13$

Finally, we present the numerical solution for the domain Ω with 7 humps.

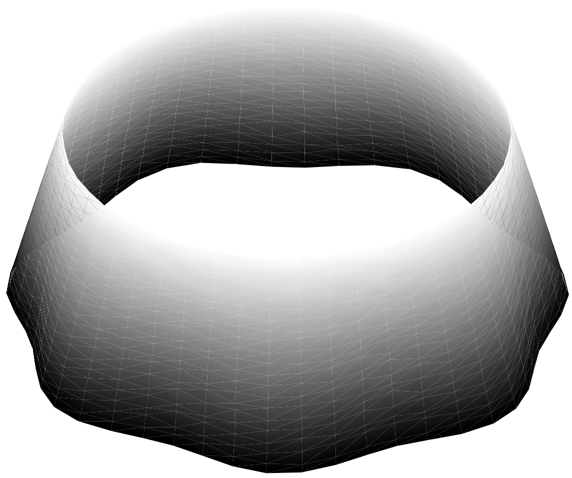

Figure 7: Numerical solution

5 Conclusions

We proved that also for not simply connected cross sections a direct numerical calculation of the Prandtl function is possible – contrary to assertions in [7]. Even a very

rough and simple approach gives accurate results at a reasonably low computational cost.
On the other hand, analytical formulas as derived by the asmyptotic method are always very appealing. However, despite the elegance, the cost for deriving higher order formulas, and for calculating numerical results from them, often exceeds the cost of the direct numerical approach.
Further, it is essential to take the right number of terms in the power series. For "good natured domains" there is a first interval of apparant convergence, we can pick any small number, say 3, 4 or 5, to obtain a good approximation. For domains with a highfrequent thickness variation, however, there is no stable behaviour at all.
Generally, the first term (Bredt formula) underestimates the torsional rigidity, the second usually overshots the exact value. Considering cost and accuracy, we recommend the third as a reasonable compromise, if no numerical procedure is desired or at hand. In such cases, e. g. for shape optimization, we strongly suggest to check for the given class of sections the behaviour of the chosen approximation with a numerical solution.
Finally, we want to stress that our aim was not to diminish the meaning of asymptotic methods, but rather to learn about their behaviour in a simple test case with easily available numerical solution before applying them to a 3-D problem. In that case we assume that an advantage of the expansion method is quite possible.

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