# Approximation Properties of Multi–Scaling Functions: A Fourier Approach


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Dedicated to Prof. L. Berg on the occasion of his 65th birthday


**Abstract**

In this paper, we consider approximation properties of a finite set of functions $\phi_\nu$ ($\nu = 0, \ldots, r-1$) which are not necessarily compactly supported, but have a suitable decay rate. Assuming that the function vector $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ is refinable, we sketch a new way, how to derive necessary and sufficient conditions for the refinement mask in Fourier domain.


## 1. Introduction

For applications of multi–wavelets in finite element methods, the problem occurs, how to construct refinable vectors $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$ ($r \in \mathbb{N}$) of functions with short support, such that algebraic polynomials of degree $< m$ ($m \in \mathbb{N}$) can be exactly reproduced by a linear combination of integer translates of $\phi_\nu$ ($\nu = 0, \ldots, r-1$).

In Heil, Strang and Strela [9] and in Plonka [13], the approximation properties of refinable function vectors $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$ were studied in some detail. In particular, new necessary and sufficient conditions for the refinement mask of $\boldsymbol{\phi}$ could be derived. In [13], it could even be shown that the function vector $\boldsymbol{\phi}$ can only provide approximation order $m$ if its refinement mask factorizes in a certain manner. For finding these results, [9] as well as [13] strongly used properties of doubly infinite matrices determined by the matrix coefficients occuring in the refinement equation (in time domain).

Now we want to sketch a way, how the necessary and sufficient conditions for the refinement mask of $\boldsymbol{\phi}$ can completely be derived in the Fourier domain.

As in [13], the functions $\phi_\nu$ are allowed to have a noncompact support if they have a suitable decay rate. The main tool of our new approach is the so called superfunction, which is contained in the span of the integer translates of $\phi_\nu$ ($\nu = 0, \ldots, r-1$) and already provides the same approximation order as $\boldsymbol{\phi}$. The results are applied to some multi–scaling functions $\phi_0$, $\phi_1$ first considered by Donovan, Geronimo, Hardin and Massopust [6, 7].

## 2. Notations

Let us introduce some notations. Consider the Hilbert space $L^2 = L^2(\mathbb{R})$ of all square integrable functions on $\mathbb{R}$. The Fourier transform of $f \in L^2(\mathbb{R})$ is defined by $\hat{f} := \int_{-\infty}^{\infty} f(x) e^{-ix\cdot}\ \mathrm{d}x$.
The function vector $\boldsymbol{\phi}$ with elements in $L^2(\mathbb{R})$ is *refinable*, if $\boldsymbol{\phi}$ satisfies a refinement equation of the form

$$\boldsymbol{\phi} = \sum_{l\in\mathbb{Z}} \boldsymbol{P}_l\, \boldsymbol{\phi}(2\cdot - l) \qquad (\boldsymbol{P}_l \in \mathbb{R}^{r\times r}),$$

or equivalently, if $\boldsymbol{\phi}$ satisfies the Fourier transformed refinement equation

$$\hat{\boldsymbol{\phi}} = \boldsymbol{P}(\cdot/2)\, \hat{\boldsymbol{\phi}}(\cdot/2) \tag{1}$$

with $\hat{\boldsymbol{\phi}} := (\hat{\phi}_\nu)_{\nu=0}^{r-1}$ and with the *refinement mask* (*two–scale symbol*)

$$\boldsymbol{P} = \boldsymbol{P}_{\boldsymbol{\phi}} := \frac{1}{2} \sum_{l\in\mathbb{Z}} \boldsymbol{P}_l\, e^{-il\cdot}. \tag{2}$$

Note that $\boldsymbol{P}$ is an $(r \times r)$–matrix of $2\pi$–periodic functions. The components $\phi_\nu$ of a refinable function vector $\boldsymbol{\phi}$ are called *multi–scaling functions.*
Let $BV(\mathbb{R})$ be the set of all functions which are of bounded variation over $\mathbb{R}$ and normalized by

$$\lim_{|x|\to\infty} f(x) = 0, \quad f(x) = \frac{1}{2} \lim_{h\to 0} (f(x+h) + f(x-h)) \quad (-\infty < x < \infty).$$

If $f \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then the Poisson summation formula

$$\sum_{l\in\mathbb{Z}} f(l)\, e^{-iul} = \sum_{j\in\mathbb{Z}} \hat{f}(u + 2\pi j)$$

is satisfied (cf. Butzer and Nessel [3]). By $C(\mathbb{R})$, we denote the set of continuous functions on $\mathbb{R}$. For a measurable function $f$ on $\mathbb{R}$ and $m \in \mathbb{N}$ let

$$\begin{aligned} \|f\|_p &:= \Big(\int_{-\infty}^{\infty} |f(x)|^p\ \mathrm{d}x\Big)^{1/p}, \\ |f|_{m,p} &:= \|\mathrm{D}^m f\|_p, \qquad \|f\|_{m,p} := \sum_{k=0}^{m} \|\mathrm{D}^k f\|_p. \end{aligned}$$

Here and in the following, D denotes the differential operator with respect to $x$ $\mathrm{D} := \mathrm{d}/\,\mathrm{d}x$. Let $W_p^m(\mathbb{R})$ be the usual Sobolev space with the norm $\|\cdot\|_{m,p}$. The $l^p$–norm of a sequence $\boldsymbol{c} := \{c_l\}_{l\in\mathbb{Z}}$ is defined by $\|\boldsymbol{c}\|_{l^p} := (\sum_{l\in\mathbb{Z}} |c_l|^p)^{1/p}$.

For $m \in \mathbb{N}$, let $E_m(\mathbb{R})$ be the space of all functions $f \in C(\mathbb{R})$ with the decay property

$$\sup_{x\in\mathbb{R}}\{|f(x)|\,(1+|x|)^{1+m+\epsilon}\} < \infty \qquad (\epsilon > 0).$$

Let $l^2_{-m} := \{\boldsymbol{c} := (c_k) : \sum_{k=-\infty}^{\infty}(1+|k|^2)^{-m}\,|c_k|^2 < \infty\}$ be a weighted sequence with the corresponding norm

$$\|\boldsymbol{c}\|_{l^2_{-m}} := \left(\sum_{l=-\infty}^{\infty}(1+|l|^2)^{-m}\,|c_l|^2\right)^{1/2}.$$

Considering the functions $\phi_\nu \in E_m(\mathbb{R})$ $(\nu = 0,\ldots,r-1)$, we call the set $\mathcal{B}(\boldsymbol{\phi}) := \{\phi_\nu(\cdot - l) : l \in \mathbb{Z},\, \nu = 0,\ldots,r-1\}$ *$L^2_{-m}$–stable* if there exist constants $0 < A \le B < \infty$ with

$$A\sum_{\nu=0}^{r-1}\|\boldsymbol{c}_\nu\|^2_{l^2_{-m}} \le \|\sum_{\nu=0}^{r-1}\sum_{l\in\mathbb{Z}} c_{\nu,l}\,\phi_\nu(\cdot - l)\|^2_{L^2_{-m}} \le B\sum_{\nu=0}^{r-1}\|\boldsymbol{c}_\nu\|^2_{l^2_{-m}}$$

for any sequences $\boldsymbol{c}_\nu = \{\boldsymbol{c}_{\nu,l}\}_{l\in\mathbb{Z}} \in l^2_{-m}$ $(\nu = 0,\ldots,r-1)$. Here $L^2_{-m}$ denotes the weighted Hilbert space $L^2_{-m} = \{f : \|f\|_{L^2_{-m}} := \|(1+|\cdot|^2)^{-m/2}\,f\|_2 < \infty\}$. Note that, if the functions $\phi_\nu$ are compactly supported, then the (algebraic) linear independence of the integer translates of $\phi_\nu$ $(\nu = 0,\ldots,r-1)$ yields the $L^2_{-m}$–stability of $\mathcal{B}(\boldsymbol{\phi})$. For $m = 0$, we obtain the well–known *$L^2$–stability* (*Riesz stability*).
For $\phi_\nu \in E_m(\mathbb{R})$ $(\nu = 0,\ldots,r-1)$, we say that $\boldsymbol{\phi}$ provides *controlled $L^p$–approximation order $m$* $(1 \le p \le \infty)$, if the following three conditions are satisfied:
For each $f \in W_p^m(\mathbb{R})$ there are sequences $\boldsymbol{c}_\nu^h = \{c_{\nu,l}^h\}_{l\in\mathbb{Z}}$ $(\nu = 0,\ldots,r-1; h > 0)$ such that for a constant $c$ independent of $h$ we have:

(1) $$\|f - h^{-1/p}\sum_{\nu=0}^{r-1}\sum_{l\in\mathbb{Z}} c_{\nu,l}^h\,\phi_\nu(\cdot/h - l)\|_p \le c\,h^m\,|f|_{m,p}.$$

(2) Furthermore,

$$\|\boldsymbol{c}_\nu^h\|_{l^p} \le c\,\|f\|_p \quad (\nu = 0,\ldots,r-1).$$

(3) There is a constant $\delta$ independent of $h$ such that for $l \in \mathbb{Z}$

$$\mathrm{dist}\,(lh, \mathrm{supp}\,f) > \delta \quad\Rightarrow\quad c_{\nu,l}^h = 0 \quad (\nu = 0,\ldots,r-1).$$

This notation of controlled $L^p$–approximation order, first introduced in Jia and Lei [11], is a generalization of the well–known definition of approximation order for compactly supported functions. In [11], the strong connection of controlled approximation order provided by $\boldsymbol{\phi}$ and the Strang–Fix conditions for $\boldsymbol{\phi}$ was shown. Note

that, instead of using the definition of Jia and Lei [11], we also could take the definition of local approximation order by Halton and Light [8]. For our considerations the equivalence to the Strang–Fix conditions is important.
The theory of closed shift–invariant subspaces of $L^2(\mathbb{R})$, spanned by integer translates of a finite set of functions has been extensively studied (cf. e.g. de Boor, DeVore and Ron [1, 2]; Jia [10]). In particular, it has been shown that the approximation order provided by a vector $\boldsymbol{\phi}$ can already be realized by a finite linear combination

$$f = \sum_{\nu=0}^{r-1} \sum_{l\in\mathbb{Z}} a_{\nu l}\, \phi_\nu(\cdot - l). \qquad (a_{\nu l} \in \mathbb{R}).$$

We call $f$ *superfunction* of $\boldsymbol{\phi}$.

## 3. Approximation by refinable function vectors

In this section we shall give a new approach to necessary and sufficient conditions for the refinement mask of a refinable vector $\boldsymbol{\phi}$ ensuring controlled $L^p$–approximation order $m$. In particular, we show, how a superfunction $f$ of $\boldsymbol{\phi}$ (providing the same approximation order as $\boldsymbol{\phi}$) can be constructed by the coefficients which occur in the linear combinations of $\phi_\nu$ reproducing the monomials.
In the following, let $r \in \mathbb{N}$ and $m \in \mathbb{N}$ be fixed. First we want to recall the result in [13] dealing with the connection between controlled $L^p$–approximation order, reproduction of polynomials and Strang–Fix conditions.

**Theorem 1** (cf. [13]) *Let $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ be a vector of functions $\phi_\nu \in E_m(\mathbb{R}) \cap BV(\mathbb{R})$. Further, let $\mathcal{B}(\boldsymbol{\phi})$ be $L^2_{-m}$–stable. Then the following conditions are equivalent:*
(a) *The function vector $\boldsymbol{\phi}$ provides controlled approximation order $m$ ($m \in \mathbb{N}$).*
(b) *Algebraic polynomials of degree $< m$ can be exactly reproduced by integer translates of $\phi_\nu$,i.e., there are vectors $\boldsymbol{y}_l^n \in \mathbb{R}^r$ ($l \in \mathbb{Z}$; $n = 0, \ldots, m-1$) such that the series $\sum_{l\in\mathbb{Z}} (\boldsymbol{y}_l^n)^{\mathrm{T}}\, \boldsymbol{\phi}(\cdot - l)$ are absolutely and uniformly convergent on any compact interval of $\mathbb{R}$ and*

$$\sum_{l\in\mathbb{Z}} (\boldsymbol{y}_l^n)^T\, \boldsymbol{\phi}(x-l) = x^n \qquad (x \in \mathbb{R};\ n = 0, \ldots, m-1).$$

(c) *The function vector $\boldsymbol{\phi}$ satisfies the Strang–Fix conditions of order $m$, i.e., there is a finitely supported sequence of vectors $\{\boldsymbol{a}_l\}_{l\in\mathbb{Z}}$, such that*

$$f := \sum_{l\in\mathbb{Z}} \boldsymbol{a}_l^{\mathrm{T}}\, \boldsymbol{\phi}(\cdot - l)$$

*satisfies*

$$\hat{f}(0) \neq 0; \quad \mathrm{D}^n \hat{f}(2\pi l) = 0 \quad (l \in \mathbb{Z} \setminus \{0\}; n = 0, \ldots, m-1).$$

The equivalence of (a) and (c) is already shown in Jia and Lei [11], Theorem 1.1. Further, (b) follows from (c) by [11], Corollary 2.3. For showing that (b) yields (c), in [13] the function

$$f := \sum_{k=0}^{m-1} \boldsymbol{a}_k^{\mathrm{T}}\, \boldsymbol{\phi}(\cdot + k), \tag{3}$$

is introduced. Here, the coefficient vectors $\boldsymbol{a}_k$ are determined by

$$(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{m-1}) := (\boldsymbol{y}_0^0, \ldots, \boldsymbol{y}_0^{m-1})\, \boldsymbol{V}^{-1}$$

with the Vandermonde matrix $\boldsymbol{V} := (k^n)_{k,n=0}^{m-1}$. Hence we have

$$\boldsymbol{y}_0^n = \sum_{k=0}^{m-1} k^n\, \boldsymbol{a}_k \qquad (n = 0, \ldots, m-1). \tag{4}$$

By Fourier transform of (3) we obtain

$$\hat{f}(u) = \boldsymbol{A}(u)^{\mathrm{T}}\, \hat{\boldsymbol{\phi}}(u)$$

with

$$\boldsymbol{A}(u) := \sum_{k=0}^{m-1} \boldsymbol{a}_k\, e^{iuk}. \tag{5}$$

That means, $\boldsymbol{A}(u)$ is an $(r \times r)$–matrix of trigonometric polynomials. Observe that by (4)

$$(\mathrm{D}^n \boldsymbol{A})(0) = \sum_{k=0}^{m-1} (ik)^n\, \boldsymbol{a}_k = i^n\, \boldsymbol{y}_0^n \quad (n = 0, \ldots, m-1).$$

Using the Poisson summation formula it can be shown that $f$ satisfies the conditions

$$(\mathrm{D}^\mu \hat{f})(2\pi l) = \delta_{0,l}\, \delta_{0,\mu} \quad (l \in \mathbb{Z};\ \mu = 0, \ldots, m-1)$$

and hence the Strang–Fix conditions of order $m$ (cf. [13]). Observe that $f$ in (3) is a superfunction of $\boldsymbol{\phi}$.

In the new proof for the following theorem, this superfunction will be the main tool.

**Theorem 2** *Let $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ be a refinable vector of functions $\phi_\nu \in E_m(\mathbb{R}) \cap BV(\mathbb{R})$. Further, let $\mathcal{B}(\boldsymbol{\phi})$ be $L^2_{-m}$–stable. Then the function vector $\boldsymbol{\phi}$ provides $L^p$–controlled approximation order $m$ if and only if the refinement mask $\boldsymbol{P}$ of $\boldsymbol{\phi}$ in (2) satisfies the following conditions:*
*There are vectors $\boldsymbol{y}_0^k \in \mathbb{R}^r$; $\boldsymbol{y}_0^0 \neq \boldsymbol{0}$ $(k = 0, \ldots, m-1)$ such that for $n = 0, \ldots, m-1$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}}\, (2i)^{k-n}\, (\mathrm{D}^{n-k} \boldsymbol{P})(0) \;=\; 2^{-n}\, (\boldsymbol{y}_0^n)^{\mathrm{T}}, \tag{6}$$

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}}\, (2i)^{k-n}\, (\mathrm{D}^{n-k} \boldsymbol{P})(\pi) \;=\; \boldsymbol{0}^{\mathrm{T}}, \tag{7}$$

*where $\boldsymbol{0}$ denotes the zero vector.*

**Proof:** Note that the conditions (6)–(7) can also be written in the form

$$\begin{aligned} \mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0} &= (\mathrm{D}^n\boldsymbol{A})^T(0) \quad (n=0,\ldots,m-1), && (8)\\ \mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi} &= \mathbf{0}^T \quad (n=0,\ldots,m-1), && (9) \end{aligned}$$

where $\boldsymbol{A}$, defined in (5), is the symbol of a superfunction $f$ of $\boldsymbol{\phi}$ in (3). From Theorem 1 we know that $\boldsymbol{\phi}$ provides controlled approximation order $m$ if and only if $f$ satisfies the Strang–Fix conditions of order $m$. Hence we only have to prove: The relations (8)–(9) are satisfied if and only if $f$ satisfies the Strang–Fix conditions of order $m$, i.e.,

$$(\mathrm{D}^n\hat{f})(2\pi l) = c_n\,\delta_{0,l} \quad (n=0,\ldots,m-1) \tag{10}$$

with constants $c_n \in \mathbb{R}$ and $c_0 \neq 0$.

1. We show that the relations (8)–(9) are satisfied if we have (10).
Note that by (1)

$$\hat{f}(2u) = \boldsymbol{A}^T(2u)\,\hat{\boldsymbol{\phi}}(2u) = \boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)\hat{\boldsymbol{\phi}}(u).$$

Taking the derivatives, it follows on the one hand

$$\begin{aligned} (\mathrm{D}^n\hat{f})(u) &= \mathrm{D}^n[\boldsymbol{A}^T(u)\,\hat{\boldsymbol{\phi}}(u)]\\ &= \sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^k\boldsymbol{A}^T)(u)\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(u) \end{aligned}$$

and on the other hand

$$\begin{aligned} 2^n\,(\mathrm{D}^n\hat{f})(2u) &= \mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)\,\hat{\boldsymbol{\phi}}(u)]\\ &= \sum_{k=0}^{n}\binom{n}{k}\mathrm{D}^k[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(u). \end{aligned}$$

2. Let us first show that the conditions (9) are satisfied. For all $l \in \mathbb{Z}$ we find by (10) that

$$0 = \hat{f}(4\pi l + 2\pi) = \boldsymbol{A}^T(4\pi l+2\pi)\,\boldsymbol{P}(2\pi l+\pi)\,\hat{\boldsymbol{\phi}}(2\pi l+\pi) = \boldsymbol{A}^T(0)\,\boldsymbol{P}(\pi)\,\hat{\boldsymbol{\phi}}(2\pi l+\pi).$$

Hence, linear independence of the sequences $\{\hat{\boldsymbol{\phi}}_\nu(\pi+2\pi l)\}_{l\in\mathbb{Z}}$ for $\nu = 0,\ldots,r-1$ gives

$$\boldsymbol{A}^T(0)\,\boldsymbol{P}(\pi) = \mathbf{0}^T.$$

Note that the linear independence of the sequences $\{\hat{\boldsymbol{\phi}}_\nu(u+2\pi l)\}_{l\in\mathbb{Z}}$ for all $u \in \mathbb{R}$ and for $\nu = 0,\ldots,r-1$ is satisfied if and only if the integer translates of $\phi_\nu$ form a $L^2$–stable basis of their closed span (cf. Jia and Micchelli [12]). This was the first step of the induction proof.

Let now $\mathrm{D}^\mu[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi} = \boldsymbol{0}^T$ be satisfied for $\mu = 0, \ldots, n-1$ $(n < m)$, and observe that by assumption (10) $(\mathrm{D}^n \hat{f})(4\pi l + 2\pi) = 0$ for all $l \in \mathbb{Z}$. Then, by the linear independence of $\{\hat{\boldsymbol{\phi}}_\nu(\pi + 2\pi l)\}_{l\in\mathbb{Z}}$ for $\nu = 0, \ldots, r-1$ and by

$$\begin{aligned} 0 &= 2^n\,(\mathrm{D}^n \hat{f})(4\pi l + 2\pi) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{D}^k[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi}\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(\pi + 2\pi l) \\ &= \mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi}\,\hat{\boldsymbol{\phi}}(\pi + 2\pi l) \end{aligned}$$

it follows that

$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi} = \boldsymbol{0}^T.$$

Thus, the relations (9) are satisfied.
3. Now we show that (10) yields (8). Let $u = 2\pi l$ $(l \in \mathbb{Z})$. Then we have on the one hand by the Strang–Fix conditions

$$\hat{f}(4\pi l) = \boldsymbol{A}^T(0)\,\boldsymbol{P}(0)\,\hat{\boldsymbol{\phi}}(2\pi l) = c_0\,\delta_{0,l}$$

and on the other hand

$$\hat{f}(2\pi l) = \boldsymbol{A}^T(0)\,\hat{\boldsymbol{\phi}}(2\pi l) = c_0\,\delta_{0,l}.$$

By linear independence of $\{\hat{\boldsymbol{\phi}}_\nu(2\pi l)\}_{l\in\mathbb{Z}}$ for $\nu = 0, \ldots, r-1$ we obtain

$$\boldsymbol{A}^T(0)\,\boldsymbol{P}(0) = \boldsymbol{A}^T(0).$$

Again, we proceed by induction. Let now $\mathrm{D}^\mu[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0} = (\mathrm{D}^\mu \boldsymbol{A}^T)(0)$ be satisfied for $\mu = 0, \ldots, n-1$ $(n < m)$ and observe that by assumption $(\mathrm{D}^n \hat{f})(2\pi l) = c_n\,\delta_{0,l}$ $(l \in \mathbb{Z},\ c_0 \neq 0)$. Then we find for all $l \in \mathbb{Z}$

$$\begin{aligned} 2^n(\mathrm{D}^n \hat{f})(4\pi l) &= \sum_{k=0}^{n} \binom{n}{k} \mathrm{D}^k[\boldsymbol{A}(2u)^T\,\boldsymbol{P}(u)]|_{u=0}\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(2\pi l) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (\mathrm{D}^k \boldsymbol{A}^T)(0)\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(2\pi l) \\ &\quad + \mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0}\,\hat{\boldsymbol{\phi}}(2\pi l) = c_n\,\delta_{0,l} \end{aligned}$$

On the other hand, for $l \in \mathbb{Z}$

$$(\mathrm{D}^n \hat{f})(2\pi l) = \sum_{k=0}^{n} \binom{n}{k} (\mathrm{D}^k \boldsymbol{A}^T)(0)\,(\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(2\pi l) = c_n\,\delta_{0,l}.$$

Hence, a comparison yields

$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0}\,\hat{\boldsymbol{\phi}}(2\pi l) = (\mathrm{D}^n \boldsymbol{A}^T)(0)\,\hat{\boldsymbol{\phi}}(2\pi l)$$

By linear independence of $\{\hat{\boldsymbol{\phi}}_\nu(2\pi l)\}_{l\in\mathbb{Z}}$ for $\nu = 0, \ldots, r-1$ we obtain

$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0} = (\mathrm{D}^n\boldsymbol{A}^T)(0).$$

Now the proof by induction is complete.
4. We are going to prove the reverse direction. Assume that the relations (8)–(9) are satisfied. We show that then the conditions $(\mathrm{D}^n\hat{f})(2\pi l) = c_n\,\delta_{0,l}$ $(n = 0, \ldots, m-1)$ hold, where $c_0 \neq 0$.
For the $\mu$-th derivative of $\hat{f}$ we find

$$\begin{aligned} 2^\mu\,(\mathrm{D}^\mu\hat{f})(4\pi l) &= \sum_{k=0}^{\mu}\binom{\mu}{k}\,\mathrm{D}^\mu[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=0}\,(\mathrm{D}^{\mu-k}\hat{\boldsymbol{\phi}})(2\pi l) \\ &= \sum_{k=0}^{\mu}\binom{\mu}{k}\,(\mathrm{D}^\mu\boldsymbol{A})(0)\,(\mathrm{D}^{\mu-k}\hat{\boldsymbol{\phi}})(2\pi l) \\ &= (\mathrm{D}^\mu\hat{f})(2\pi l) \end{aligned}$$

and

$$\begin{aligned} 2^\mu\,(\mathrm{D}^\mu\hat{f})(4\pi l + 2\pi) &= \sum_{k=0}^{\mu}\binom{\mu}{k}\,\mathrm{D}^\mu[\boldsymbol{A}^T(2u)\,\boldsymbol{P}(u)]|_{u=\pi}\,(\mathrm{D}^{\mu-k}\hat{\boldsymbol{\phi}})(2\pi l + \pi) \\ &= 0. \end{aligned}$$

Thus, we indeed obtain $(\mathrm{D}^n\hat{f})(2\pi l) = c_n\,\delta_{0,l}$. It only remains to show that $c_0 \neq 0$. By Poisson summation formula and using the $L^2$–stability of $\boldsymbol{\phi}$ we have

$$\hat{\phi}(0) = \boldsymbol{A}^T(0)\,\hat{\boldsymbol{\phi}}(0) = (\boldsymbol{y}_0^0)^T\,\hat{\boldsymbol{\phi}}(0) = (\boldsymbol{y}_0^0)^T\sum_{l\in\mathbb{Z}}\boldsymbol{\phi}(\cdot - l) \neq 0.$$

Hence $f$ satisfies the Strang–Fix conditions of order $m$. ■

**Remark 3** *For proving the second direction in Theorem 2 we do not need any stability condition if we assume that $(\boldsymbol{y}_0^0)^T\,\hat{\boldsymbol{\phi}}(0) \neq 0$. Since $\boldsymbol{y}_0^0$ and $\hat{\boldsymbol{\phi}}(0)$ are a left and a right eigenvector of $\boldsymbol{P}(0)$, respectively, this assumption is satisfied if the eigenvalue 1 of $\boldsymbol{P}(0)$ is simple.*

## 4. The GHM–multi–scaling functions

We consider the example of a vector of two multi–scaling functions $\boldsymbol{\phi} := (\phi_0, , \phi_1)^T$ treated in Donovan, Geronimo, Hardin and Massopust ([6, 7]). In the special case $s = s_0 = s_1$ (with $s \in [-1, 1]$) of their construction, the refinement equation of $\boldsymbol{\phi}$ is given by

$$\boldsymbol{\phi}(x) = \boldsymbol{P}_0\,\boldsymbol{\phi}(2x) + \boldsymbol{P}_1\,\boldsymbol{\phi}(2x-1) + \boldsymbol{P}_2\,\boldsymbol{\phi}(2x-2) + \boldsymbol{P}_3\,\boldsymbol{\phi}(2x-3), \qquad (11)$$

where

$$\boldsymbol{P}_0 := \begin{pmatrix} -\frac{s^2-4s-3}{2\,(s+2)} & 1 \\ -\frac{3(s-1)(s+1)(s^2-3s-1)}{4\,(s+2)^2} & \frac{3s^2+s-1}{2\,(s+2)} \end{pmatrix}, \quad \boldsymbol{P}_1 := \begin{pmatrix} -\frac{s^2-4s-3}{2\,(s+2)} & 0 \\ -\frac{3(s-1)(s+1)(s^2-s+3)}{4\,(s+2)^2} & 1 \end{pmatrix},$$

$$\boldsymbol{P}_2 := \begin{pmatrix} 0 & 0 \\ -\frac{3(s-1)(s+1)(s^2-s+3)}{4\,(s+2)^2} & \frac{3s^2+s-1}{2\,(s+2)} \end{pmatrix}, \quad \boldsymbol{P}_3 := \begin{pmatrix} 0 & 0 \\ -\frac{3(s-1)(s+1)(s^2-3s-1)}{4\,(s+2)^2} & 0 \end{pmatrix}.$$

For the refinement mask $\boldsymbol{P}$ we have

$$\boldsymbol{P}(u) := \frac{1}{2}\,(\boldsymbol{P}_0 + \boldsymbol{P}_1\,e^{-iu} + \boldsymbol{P}_2\,e^{-2iu} + \boldsymbol{P}_3\,e^{-3iu}).$$

Applying the result of Theorem 2 we can show that $\boldsymbol{\phi}$ provides the controlled $L^p$–approximation order $m = 2$:
Observing that

$$\boldsymbol{P}(0) = \begin{pmatrix} \frac{-s^2+4s+3}{2(s+2)} & \frac{1}{2} \\ \frac{-3(s-1)^3(s+1)}{2(s+2)^2} & \frac{3s^2+2s+1}{2(s+2)} \end{pmatrix}, \quad (\mathrm{D}\boldsymbol{P})(0) = i \begin{pmatrix} \frac{s^2-4s-3}{4(s+2)} & 0 \\ \frac{9(s-1)^3(s+1)}{4(s+2)^2} & \frac{-(3s^2+2s+1)}{2(s+2)} \end{pmatrix},$$

and

$$\boldsymbol{P}(\pi) = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{3(s^2-1)}{2(s+2)} \end{pmatrix}, \quad (\mathrm{D}\boldsymbol{P})(\pi) = i \begin{pmatrix} \frac{-s^2+4s+3}{4(s+2)} & 0 \\ \frac{3(s^2-1)(-s^2+4s+3)}{4(s+2)^2} & \frac{-3(s^2-1)}{2(s+2)} \end{pmatrix},$$

we find with

$$\boldsymbol{y}_0^0 = \left(\frac{-3(s^2-1)}{s+2}, 1\right), \qquad \boldsymbol{y}_0^1 = \left(\frac{-3(s^2-1)}{2(s+2)}, 1\right)$$

the relations

$$(\boldsymbol{y}_0^0)^T\,\boldsymbol{P}(0) = (\boldsymbol{y}_0^0)^T, \quad (\boldsymbol{y}_0^0)^T\,\boldsymbol{P}(\pi) = \boldsymbol{0}^T$$

and

$$\begin{aligned} (2i)^{-1}\,(\boldsymbol{y}_0^0)^T\,(\mathrm{D}\boldsymbol{P})(0) + (\boldsymbol{y}_0^1)^T\,\boldsymbol{P}(0) &= 2^{-1}\,(\boldsymbol{y}_0^1)^T, \\ (2i)^{-1}\,(\boldsymbol{y}_0^0)^T\,(\mathrm{D}\boldsymbol{P})(\pi) + (\boldsymbol{y}_0^1)^T\,\boldsymbol{P}(\pi) &= \boldsymbol{0}^T. \end{aligned}$$

Hence, (8)–(9) are satisfied for $m = 2$. Knowing $\boldsymbol{y}_0^0$ and $\boldsymbol{y}_0^1$, we can construct a superfunction $f$ of $\boldsymbol{\phi}$ (as defined in (3)) by

$$f(x) = (\boldsymbol{y}_0^0 - \boldsymbol{y}_0^1)^T\,\boldsymbol{\phi}(x) + (\boldsymbol{y}_0^1)^T\,\boldsymbol{\phi}(x+1)$$

obtaining

$$f(x) = \frac{3(1-s^2)}{2(s+2)}(\phi_0(x) + \phi_0(x+1)) + \phi_1(x+1).$$

Application of the refinement equation (11) on the right hand side yields

$$
\begin{aligned}
f(x) &= \frac{9(1-s^2)}{4(s+2)}(\phi_0(2x)+\phi_0(2x+1)) + \frac{3(1-s^2)}{4(s+2)}(\phi_0(2x-1)+\phi_0(2x+2)) \\
&\quad + \frac{1}{2}(\phi_1(2x+2)+\phi_1(2x)) + \phi_1(2x+1) \\
&= \frac{1}{2}f(2x-1) + f(2x) + \frac{1}{2}f(2x+1).
\end{aligned}
$$

That means, $f$ itself satisfies the refinement equation of the hat–function $h(x) := \max\{(1-|x|), 0\}$. Hence, taking a proper normalization constant, the superfunction $f$ coincides with the hat function $h$. Indeed, in [6] the approximation order 2 provided by $\boldsymbol{\phi}$ was derived by showing that the hat–function $h$ lies in the span of the integer translates of $\phi_0$, $\phi_1$.

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