Optimal Functions for a Periodic Uncertainty Principle and Multiresolution Analysis

Dedicated to Professor Lothar Berg
on the occasion of his 65th birthday

Jürgen Prestin and Ewald Quak†

Abstract. In this paper, it is shown that certain Theta functions are asymptotically optimal for the periodic time frequency uncertainty principle described by Breitenberger in [2]. These extremal functions give rise to a periodic multiresolution analysis where the corresponding wavelets also show similar localization properties.

§1. Introduction

A fundamental result on time and frequency localization of square-integrable functions on the real line is given by the Heisenberg uncertainty principle, and it is well known that the Gaussian functions serve as extremal functions for this inequality. On the other hand, uncertainty relations for periodic functions have not been studied as thoroughly as the original inequality on the real axis. In this paper, we want to focus on a concept discussed by Breitenberger in [2], where uncertainty of a periodic square-integrable function is described in terms of the product of frequency and angular variances.

A unified approach to the theory of general periodic multiresolution analyses is investigated by Plonka and Tasche [8], Koh, Lee and Tan [4], Narcowich and Ward [5]. The time frequency localization of periodic scaling functions and wavelets according to the aforementioned uncertainty relation is studied intensively by Narcowich and Ward. The periodic basis functions in [5] possess an uncertainty product of $\mathcal{O}(\sqrt{n})$ for increasing dimension n of the corresponding spaces. Uniformly bounded uncertainty products are computed by Selig [10] for trigonometric fundamental Lagrange interpolants based on de la Vallée Poussin means and for trigonometric Hermite interpolants based on Fejér kernels in [9].

† The second author was supported in part by the Department of the Air Force, Wright-Patterson AFB, OH, contract F33600-94-M-2603

The question of extremal functions to attain equality in the periodic uncertainty principle is not addressed in these papers. As it turns out that there is no function satisfying the equality, our goal is to construct asymptotically optimal functions. Not surprisingly, this leads to periodized Gaussian functions, *i.e.*, special cases of Theta functions (see Section 2). Additionally, the wavelet functions in the corresponding multiresolution analysis also show a similar asymptotic behaviour (see Section 3). In Section 4, the localization properties of the orthogonalized functions are considered.

Let us finally remark that recently, Narcowich and Ward [6,7] also studied wavelets and localization properties on higher dimensional spheres, while asymptotically optimal results for compactly supported scaling functions and wavelets on the real line can be found in the paper by Chui and Wang [3].

§2. Time Frequency Uncertainty

Let $L^2_{2\pi}$ denote the space of complex-valued square-integrable 2π-periodic functions with inner product $\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx$ and norm $\|f\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2dx\right)^{1/2}$. To state an uncertainty principle for functions in $L^2_{2\pi}$, the necessary notions of variance for periodic functions need to be introduced. We refer to Breitenberger [2] for a discussion of the background in physics and Narcowich & Ward [5] for applications to periodic basis functions in a multiresolution analysis.

Definition 2.1 For a function $f \in L^2_{2\pi}$, represented by its Fourier series, *i.e.*, $f(x) = \sum_{s=-\infty}^{\infty} c_s e^{isx}$, the *first trigonometric moment* is defined as

$$\tau(f) := \frac{1}{2\pi}\int_0^{2\pi} e^{ix}|f(x)|^2\,dx = \sum_{s=-\infty}^{\infty} c_s\overline{c_{s+1}}.$$

Then, the *angle variance* for $f \in L^2_{2\pi}$ is defined by

$$\mathrm{var}_A(f) \ := \ \frac{\|f\|_2^4 - |\tau(f)|^2}{|\tau(f)|^2} = \left|\frac{\sum_{s=-\infty}^{\infty}|c_s|^2}{\sum_{s=-\infty}^{\infty} c_s\overline{c_{s+1}}}\right|^2 - 1,$$

and the *frequency variance* for $f \in L^2_{2\pi}$ as

$$\mathrm{var}_F(f) \ := \|f'\|_2^2 + \frac{\langle f',f\rangle^2}{\|f\|_2^2} = \sum_{s=-\infty}^{\infty} s^2|c_s|^2 - \frac{\left(\sum_{s=-\infty}^{\infty} s|c_s|^2\right)^2}{\sum_{s=-\infty}^{\infty}|c_s|^2}.$$

Note that the variances attain the value ∞ iff $\tau(f) = 0$ or f' does not exist in $L^2_{2\pi}$, respectively.

In the following theorem, an uncertainty relation for $L^2_{2\pi}$ is formulated, which only excludes single frequency functions of the form ce^{ikx}, $c \in \mathbb{C}$, $k \in \mathbf{Z}$. In this case, $\mathrm{var}_F = 0$ and $\mathrm{var}_A = \infty$. Nevertheless, inequality (2.1) below actually holds for all functions in $L^2_{2\pi}$. In order to investigate the existence of optimal or asymptotically optimal functions for this uncertainty inequality, and for the sake of completeness, we give a detailed proof here.

Theorem 2.1 *Uncertainty Principle for periodic functions. ([2,5]).*
For functions $f \in L^2_{2\pi}$ which are not of the form ce^{ikx}, $c \in \mathbb{C}$, $k \in \mathbb{Z}$, it holds that

$$\frac{\sqrt{\mathrm{var}_A(f)\mathrm{var}_F(f)}}{\|f\|_2} \geq \frac{1}{2}\,.$$

Proof: The goal is to establish the inequality

$$|\tau(f)|\,\|f\|_2 \leq 2\sqrt{\|f\|_2^4 - |\tau(f)|^2}\sqrt{\|f'\|_2^2\,\|f\|_2^2 + \langle f', f\rangle^2}\,. \tag{2.1}$$

Without loss of generality one can assume $\|f\|_2 = 1$, which reduces (2.1) to

$$|\tau(f)| \leq 2\sqrt{1 - |\tau(f)|^2}\sqrt{\|f'\|_2^2 + \langle f', f\rangle^2}\,. \tag{2.2}$$

Setting for notational convenience $\tau = \tau(f) \in \mathbb{C}$, and $\mu = -i\langle f', f\rangle \in \mathbb{R}$, it holds that

$$1 - |\tau|^2 = \|(e^{i\cdot} - \tau)f\|_2^2 \tag{2.3}$$

and

$$\|f'\|_2^2 - \mu^2 = \| - if' - \mu f\|_2^2\,. \tag{2.4}$$

One directly verifies the identity

$$e^{ix}f(x) = -i[(e^{i\cdot} - \tau)f]'(x) - (e^{ix} - \tau)[-if]'(x)$$

and, consequently,

$$\begin{aligned} e^{ix}f(x) = &\left\{-i[(e^{i\cdot} - \tau)f]'(x) - \mu(e^{ix} - \tau)f(x)\right\} \\ &- \left\{(e^{ix} - \tau)(-\mu)f(x) + (e^{ix} - \tau)(-i)f'(x)\right\}\,. \end{aligned}$$

Taking inner products with f yields

$$\begin{aligned} \tau &= \langle -i[(e^{i\cdot} - \tau)f]' - \mu(e^{i\cdot} - \tau)f, f\rangle - \langle (e^{i\cdot} - \tau)(-\mu)f + (e^{i\cdot} - \tau)(-i)f', f\rangle \\ &=: A_1 + A_2\,. \end{aligned}$$

Using integration by parts in A_1 produces

$$A_1 = \langle (e^{i\cdot} - \tau)f, -if'\rangle - \langle \mu(e^{i\cdot} - \tau)f, f\rangle = \langle (e^{i\cdot} - \tau)f, -if' - \mu f\rangle\,,$$

and thus by the Cauchy-Schwarz inequality

$$|A_1| \leq \|(e^{i\cdot} - \tau)f\|_2\,\| - if' - \mu f\|_2\,.$$

For A_2, one obtains

$$A_2 = -\langle -if' - \mu f, (e^{-i\cdot} - \bar{\tau}) f \rangle$$

and, once again by the Cauchy-Schwarz inequality,

$$|A_2| \leq \| -if' - \mu f\|_2 \; \|(e^{-i\cdot} - \bar{\tau})f\|_2 \,.$$

Taking into account that $\|(e^{-i\cdot} - \bar{\tau})f\|_2 = \|(e^{i\cdot} - \tau)f\|_2$, inequality (2.2) follows from equations (2.3) and (2.4). ■

Now, we investigate the problem of optimal periodic time-frequency localization, *i.e.*, whether it is possible to achieve equality in the uncertainty principle, which in turn implies equality in (2.1). Examining the proof, in the Cauchy-Schwarz inequality actual equality only occurs iff for A_1

$$c_1(e^{ix} - \tau)f(x) = -if'(x) - \mu f(x) \quad \text{a.e.}\,,$$

and for A_2

$$c_2(e^{-ix} - \bar{\tau})f(x) = -if'(x) - \mu f(x) \quad \text{a.e.}\ ,$$

with complex constants c_1, c_2. Apart from the functions ce^{ikx} mentioned above, where $-if' - \mu f \equiv 0$, this would only be possible if on a set of positive measure and for a constant $c \in \mathbb{C}$, we have

$$e^{ix} - \tau = c(e^{-ix} - \bar{\tau})\,,$$

i.e.,

$$e^{ix} - ce^{-ix} = \tau - c\bar{\tau}\,,$$

which cannot happen as the left hand side is not constant.

Hence, there is no function f of optimal time-frequency localization. However, one can try to obtain the best possible constant 0.5 in the limit which will be the goal for the rest of this section.

Let us consider the Theta functions

$$\theta(z,q) = \sum_{s=-\infty}^{\infty} q^{s^2} z^s \qquad (z \in \mathbb{C}\backslash\{0\}, |q| < 1)\,.$$

With the transformation $z = e^{ix}$, $q = e^{-h}$, we write

$$\varphi_h(x) = \theta(e^{ix}, e^{-h}) = \sum_{s=-\infty}^{\infty} e^{-hs^2} e^{isx}\,.$$

Note that φ_h is an even positive 2π-periodic and smooth function.

Before we prove the essential theorems, we need some auxiliary results and estimates which are summarized in the following lemma.

Lemma 2.1 *For all $h > 0$, and shifts $\alpha \in \mathbb{R}$, the following identity holds*

$$(i) \qquad \sum_{s=-\infty}^{\infty} e^{-2h(s-\alpha)^2} = \sqrt{\frac{\pi}{2h}} \sum_{s=-\infty}^{\infty} \cos 2\pi s\alpha \; e^{-\frac{\pi^2 s^2}{2h}} .$$

Consequently, the following estimates hold for the function

$$u(h,\alpha) := \sum_{s=-\infty}^{\infty} e^{-2h(s-\alpha)^2} ,$$

independently of α,

$$(ii) \qquad \sqrt{\frac{\pi}{2h}} \, \frac{1-3e^{-\frac{\pi^2}{2h}}}{1-e^{-\frac{\pi^2}{2h}}} < u(h,\alpha) < \sqrt{\frac{\pi}{2h}} \, \frac{1+e^{-\frac{\pi^2}{2h}}}{1-e^{-\frac{\pi^2}{2h}}} ,$$

and more precisely for $u(h) := u(h,0)$,

$$(iii) \qquad \sqrt{\frac{\pi}{2h}} < u(h) < \sqrt{\frac{\pi}{2h}} \, \frac{1+e^{-\frac{\pi^2}{2h}}}{1-e^{-\frac{\pi^2}{2h}}} .$$

Proof: The Poisson summation formula (see also [1, Chap. 3]) can be used to establish (i)

$$\begin{aligned}
\sum_{s=-\infty}^{\infty} e^{-2h(s-\alpha)^2} &= \sum_{s=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2h(x-\alpha)^2} e^{-2\pi i s x} \, dx \\
&= \sum_{s=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2hx^2} e^{-2\pi i s(x+\alpha)} \, dx \\
&= \sqrt{\frac{\pi}{2h}} \sum_{s=-\infty}^{\infty} e^{-\frac{\pi^2 s^2}{2h}} e^{-2\pi i s\alpha} .
\end{aligned}$$

The sum $\sum_{s=-\infty}^{\infty} e^{-\frac{\pi^2 s^2}{2h}}$ can be easily estimated as a geometric series

$$1 < \sum_{s=-\infty}^{\infty} e^{-\frac{\pi^2 s^2}{2h}} < 1 + 2\sum_{s=1}^{\infty} e^{-\frac{\pi^2 s}{2h}} = \frac{1+e^{-\frac{\pi^2}{2h}}}{1-e^{-\frac{\pi^2}{2h}}},$$

yielding (iii).

Estimating $\sum_{s=-\infty}^{\infty} \cos 2\pi s\alpha \; e^{-\frac{\pi^2 s^2}{2h}}$ from below by $2 - \sum_{s=-\infty}^{\infty} e^{-\frac{\pi^2 s^2}{2h}}$, one finally obtains the lower bound in (ii). ■

It is now possible to give an estimate for the uncertainty product of the Theta functions φ_h.

Theorem 2.2 *For all* $0 < h \le 1$, *it holds that*

$$\frac{\sqrt{\mathrm{var}_A(\varphi_h)\mathrm{var}_F(\varphi_h)}}{\|\varphi_h\|_2} < \frac{1+\sqrt{h}}{2}.$$

Proof: Using the Fourier coefficients $c_{s,h} = e^{-hs^2}$ of φ_h, we write

$$\sum_{s=-\infty}^{\infty} c_{s,h}^2 = \sum_{s=-\infty}^{\infty} e^{-2hs^2} = u(h)$$

and

$$\begin{aligned}\sum_{s=-\infty}^{\infty} c_{s,h}c_{s+1,h} &= \sum_{s=-\infty}^{\infty} e^{-h(s^2+(s+1)^2)} = e^{-\frac{h}{2}} \sum_{s=-\infty}^{\infty} e^{-\frac{h}{2}(2s+1)^2} \\ &= e^{-\frac{h}{2}} \left(\sum_{s=-\infty}^{\infty} e^{-\frac{h}{2}s^2} - \sum_{s=-\infty}^{\infty} e^{-\frac{h}{2}(2s)^2} \right) = e^{-\frac{h}{2}}(u(\frac{h}{4}) - u(h)) \ .\end{aligned}$$

Hence we obtain

$$\mathrm{var}_A(\varphi_h) \ = \frac{u^2(h)}{e^{-h}(u(\frac{h}{4}) - u(h))^2} - 1 \ .$$

Now it follows by Lemma 2.1

$$\begin{aligned} u(\frac{h}{4}) - u(h) &> \sqrt{\frac{2\pi}{h}} - \sqrt{\frac{\pi}{2h}} \ \frac{1+e^{-\frac{\pi^2}{2h}}}{1-e^{-\frac{\pi^2}{2h}}} \\ &= \sqrt{\frac{\pi}{2h}} \left(\frac{1-3e^{-\frac{\pi^2}{2h}}}{1-e^{-\frac{\pi^2}{2h}}} \right) \ .\end{aligned}$$

Then

$$\begin{aligned}\mathrm{var}_A(\varphi_h) &< e^h \left(\frac{1+e^{-\frac{\pi^2}{2h}}}{1-3e^{-\frac{\pi^2}{2h}}} \right)^2 - 1 \\ &= e^h - 1 + 8e^h \left(\frac{1}{e^{\frac{\pi^2}{2h}} - 3} + \frac{2}{(e^{\frac{\pi^2}{2h}} - 3)^2} \right) \ .\end{aligned}$$

Taylor's formula and the monotonicity of $h^2(e^{\frac{\pi^2}{2h}} - 3)$ for $0 < h \le 1$ gives the estimate

$$\mathrm{var}_A(\varphi_h) < h + \frac{eh^2}{2} + 8e(\frac{h^2}{e^{\frac{\pi^2}{2}} - 3} + \frac{2h^2}{(e^{\frac{\pi^2}{2}} - 3)^2}) < h + 1.53h^2 \ .$$

Now we have to deal with $\mathrm{var}_F(\varphi_h)$. Differentiation of the Poisson summation formula with respect to h yields

$$\begin{aligned}
\mathrm{var}_F(\varphi_h) &= \sum_{s=-\infty}^{\infty} s^2 e^{-2hs^2} = -\frac{1}{2}\frac{d}{dh}\left(\sum_{s=-\infty}^{\infty} e^{-2hs^2}\right) \\
&= -\frac{1}{2}\frac{d}{dh}\left(\sqrt{\frac{\pi}{2h}}\sum_{s=-\infty}^{\infty} e^{-\frac{\pi^2 s^2}{2h}}\right) \\
&= \frac{1}{4h}\sqrt{\frac{\pi}{2h}}\sum_{s=-\infty}^{\infty} e^{-\frac{\pi^2 s^2}{2h}} - \sqrt{\frac{\pi}{2h}}\frac{\pi^2}{4h^2}\sum_{s=-\infty}^{\infty} s^2 e^{-\frac{\pi^2 s^2}{2h}} \\
&< \frac{1}{4h}\sqrt{\frac{\pi}{2h}}\sum_{s=-\infty}^{\infty} e^{-\frac{\pi^2 s^2}{2h}} = \frac{u(h)}{4h}\,,
\end{aligned}$$

where the last equality follows again from Lemma 2.1.

Putting together the corresponding estimates we obtain

$$\frac{\mathrm{var}_A(\varphi_h)\mathrm{var}_F(\varphi_h)}{\|\varphi_h\|_2^2} < \frac{(h+1.53h^2)u(h)}{u(h)4h} < \frac{1}{4} + 0.39h\,,$$

from which the assertion of Theorem 2.1 follows. ■

Note that $h=0$ yields a distribution, but not an $L^2_{2\pi}$-function. As an alternative, one can investigate the limit case $h=0$ for the partial sum $\sum_{s=-m}^{m} e^{-hs^2}e^{isx}$ of φ_h, *i.e.*,

$$\phi_m(x) = \sum_{s=-m}^{m} e^{isx}\,.$$

One obtains

$$\begin{aligned}
\frac{\sqrt{\mathrm{var}_A(\phi_m)\mathrm{var}_F(\phi_m)}}{\|\phi_m\|_2} &= \sqrt{\frac{\frac{(2m+1)^2-(2m)^2}{(2m)^2}\sum_{s=-m}^{m} s^2}{2m+1}} \\
&= \sqrt{\frac{(4m+1)m(m+1)(2m+1)}{4m^2(2m+1)3}} \\
&= \sqrt{\frac{(4m+1)(m+1)}{12m}}\,.
\end{aligned}$$

Hence,

$$\sqrt{\frac{m}{3}+\frac{5}{12}} < \frac{\sqrt{\mathrm{var}_A(\phi_m)\mathrm{var}_F(\phi_m)}}{\|\phi_m\|_2} < \sqrt{\frac{m}{3}+\frac{1}{2}}\,.$$

For similar results, see [5], [9], [10]. However, in [9], it is proved that a certain smoothing procedure, namely a Fejér mean of the partial sum polynomial ϕ_m, *i.e.*,

$$\sigma_m = \frac{1}{m+1} \sum_{k=0}^{m} \phi_k \,,$$

yields

$$\frac{\sqrt{\mathrm{var}_A(\sigma_m)\mathrm{var}_F(\sigma_m)}}{\|\sigma_m\|_2} = \sqrt{\frac{3}{10}\left(1 + \frac{20m^2+1}{2(8m^2+1)(4m^2-1)}\right)}\,,$$

which in the limit $m \to \infty$ tends to $\sqrt{0.3} = 0.5477...$

To illustrate these results, see Figure 1 for Theta functions with two different step sizes, and Figure 2 for a Dirichlet and a Fejér kernel.

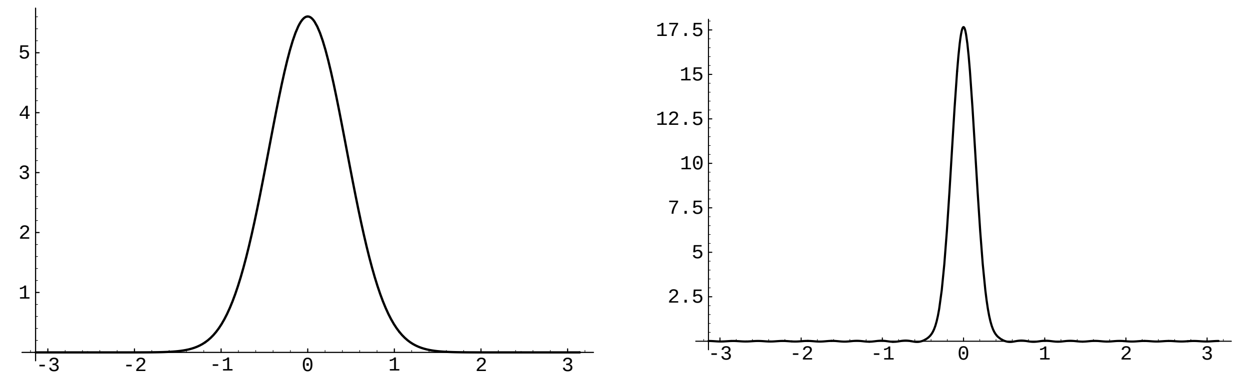


Figure 1. $\varphi_{0.1}$ (left) and $\varphi_{0.01}$ (right).

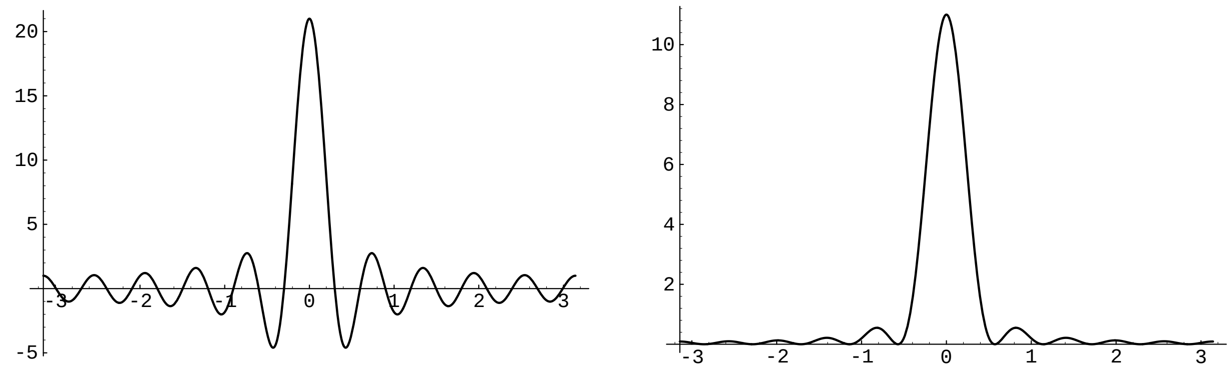


Figure 2. ϕ_{10} (left) and σ_{10} (right).

§3. Multiresolution Analysis

Now, the goal is to construct nested spaces spanned by translates of the Theta functions investigated in Section 2. For fixed h, we set

$$V_n = \operatorname{span}\{\varphi_h(\cdot - \frac{2k\pi}{n}),\ k = 0, \ldots, n-1\}\ .$$

Trivially, it holds that

$$V_n \subset V_{2n}\ .$$

Furthermore, as all the Fourier coefficients of the function φ_h are positive, we have

$$\dim V_n = n \quad \text{and} \quad L^2_{2\pi} = \operatorname{clos}_{L^2}\left(\bigcup_{j=0}^{\infty} V_{2^j}\right)\ .$$

See, for example, Lemma 2.4 (i) and Theorem 3.1 in [8].

Hence, it makes sense to consider wavelet spaces W_n, namely the relative orthogonal complements of V_n with respect to V_{2n} such that

$$V_{2n} = V_n \oplus W_n\ .$$

While the scaling functions are just translates of one single function, independent of the level n, the corresponding wavelets turn out to be dependent on n. As the major concern here, however, is the dependence on the term h, we will suppress the dependence on n in our notation by writing

$$\psi_h(x) := \sum_{s=-\infty}^{\infty} e^{-hs^2}\left(1 - \frac{\sum_{p=-\infty}^{\infty}(-1)^p e^{-2h(np-s)^2}}{\sum_{q=-\infty}^{\infty} e^{-2h(nq-s)^2}}\right) e^{is(x-\frac{\pi}{n})}\ .$$

One now obtains

Theorem 3.1 *It holds for $n \in \mathbb{N}$ that*

$$W_n = \operatorname{span}\{\psi_h(\cdot - \frac{2k\pi}{n}),\ k = 0, \ldots, n-1\}\ .$$

Proof: The fact that ψ_h is indeed an element of V_{2n} follows immediately from the $2n$-periodicity in s of the quotient

$$\frac{d_{s,h}}{c_{s,h}} = \left(1 - \frac{\sum_{p=-\infty}^{\infty}(-1)^p e^{-2h(np-s)^2}}{\sum_{q=-\infty}^{\infty} e^{-2h(nq-s)^2}}\right) e^{-is\frac{\pi}{n}}\ ,$$

where $d_{s,h}$ are the Fourier coefficients of ψ_h, see Lemma 2.1 of [8].

The orthogonality assertion is equivalent to

$$\sum_{s=-\infty}^{\infty} c_{s,h} d_{s,h} e^{2\pi i \frac{rs}{n}} = 0, \quad r = 0, \ldots, n-1\,.$$

Hence we have to prove

$$\sum_{s=-\infty}^{\infty} e^{-2hs^2} e^{\pi i s \frac{2r-1}{n}} = \sum_{s=-\infty}^{\infty} e^{-2hs^2} e^{\pi i s \frac{2r-1}{n}} \frac{\sum_{p=-\infty}^{\infty} (-1)^p e^{-2h(np-s)^2}}{\sum_{q=-\infty}^{\infty} e^{-2h(nq-s)^2}}.$$

Setting $s = n\ell + t$ yields

$$\sum_{s=-\infty}^{\infty} = \sum_{t=0}^{n-1} \sum_{\ell=-\infty}^{\infty}$$

and we obtain for the right hand side

$$\begin{aligned}
&\sum_{t=0}^{n-1} \sum_{\ell=-\infty}^{\infty} e^{-2h(n\ell+t)^2} (-1)^\ell e^{\pi i t \frac{2r-1}{n}} \frac{\sum_{p=-\infty}^{\infty} (-1)^p e^{-2h(n(p-\ell)-t)^2}}{\sum_{q=-\infty}^{\infty} e^{-2h(n(q-\ell)-t)^2}} \\
&\quad = \sum_{t=0}^{n-1} e^{\pi i t \frac{2r-1}{n}} \sum_{p=-\infty}^{\infty} (-1)^p e^{-2h(np-t)^2}
\end{aligned}$$

which equals the left hand side. ■

Additionally, one can also state the two-scale relations for the basis functions of V_n and W_n with respect to V_{2n}. We have

Corollary 3.1 *For fixed $n \in \mathbb{N}$, and $k = 0, \ldots, n-1$, the two-scale relations are*

$$\begin{aligned}
\varphi_h(x - \frac{2k\pi}{n}) &= \sum_{\ell=0}^{2n-1} \alpha_{k,\ell} \varphi_h(x - \frac{\ell\pi}{n}) \\
\psi_h(x - \frac{2k\pi}{n}) &= \sum_{\ell=0}^{2n-1} \beta_{k,\ell} \varphi_h(x - \frac{\ell\pi}{n})\,,
\end{aligned}$$

where for $m = 0, \ldots, n-1$,

$$\begin{aligned}
\beta_{k,2m+1} &= \delta_{k,m}\,, \\
\beta_{k,2m} &= \frac{1}{n} \sum_{t=0}^{n-1} e^{-\pi i t \frac{2m-2k-1}{n}} \frac{\sum_{p=-\infty}^{\infty} (-1)^p e^{-2h(np-t)^2}}{\sum_{q=-\infty}^{\infty} e^{-2h(nq-t)^2}}\,,
\end{aligned}$$

and, trivially, $\alpha_{k,\ell} = \delta_{2k,\ell}$.

Proof: The values of the $\beta_{k,\ell}$'s are obtained by applying discrete Fourier transforms to the corresponding two-scale equation, see also [8]. ■

Now, the question of time-frequency localization of the wavelet will be addressed.

Theorem 3.2 *Let $n \in \mathbb{N}$ be fixed. Then, for all $0 < h \leq \frac{1}{4n^2}$, we have*

$$\frac{\sqrt{\mathrm{var}_A(\psi_h)\mathrm{var}_F(\psi_h)}}{\|\psi_h\|_2} < \frac{1}{2} + 1.1n^2\sqrt{h}\,.$$

Proof: As the variance measures of Definition 2.1 are invariant under translation of the function, we consider $\tilde{\psi}_h = \psi_h(\cdot + \frac{\pi}{n})$ with Fourier coefficients

$$\tilde{d}_{s,h} = c_{s,h}\left(1 - \frac{\sum_{p=-\infty}^{\infty}(-1)^p e^{-2h(np-s)^2}}{\sum_{q=-\infty}^{\infty} e^{-2h(nq-s)^2}}\right).$$

First, we obtain

$$\frac{\sum_{p=-\infty}^{\infty}(-1)^p e^{-2h(np-s)^2}}{\sum_{q=-\infty}^{\infty} e^{-2h(nq-s)^2}}$$

$$= \frac{\sum_{p=-\infty}^{\infty} e^{-2h(2pn-s)^2} - \sum_{p=-\infty}^{\infty} e^{-2h(2pn+n-s)^2}}{u(n^2h, \frac{s}{n})}$$

$$= \frac{u(4n^2h, \frac{s}{2n}) - u(4n^2h, \frac{s}{2n} - \frac{1}{2})}{u(n^2h, \frac{s}{n})}\,.$$

Hence, by Lemma 2.1, with $\gamma = \frac{\pi^2}{8hn^2}$,

$$\left|\frac{\sum_{p=-\infty}^{\infty}(-1)^p e^{-2h(np-s)^2}}{\sum_{q=-\infty}^{\infty} e^{-2h(nq-s)^2}}\right| < \frac{1}{2}\left(\frac{1+e^{-\gamma}}{1-e^{-\gamma}} - \frac{1-3e^{-\gamma}}{1-e^{-\gamma}}\right)\frac{1-e^{-4\gamma}}{1-3e^{-4\gamma}}$$

$$= \frac{1}{2}\,\frac{4e^{-\gamma}}{1-e^{-\gamma}}\,\frac{1-e^{-4\gamma}}{1-3e^{-4\gamma}}\,.$$

Consequently,

$$\left|\frac{\sum_{p=-\infty}^{\infty}(-1)^p e^{-2h(np-s)^2}}{\sum_{q=-\infty}^{\infty} e^{-2h(nq-s)^2}}\right| < 2.02e^{-\frac{\pi^2}{8hn^2}}$$

and

$$(1 - 2.02e^{-\gamma})c_{s,h} \leq \tilde{d}_{s,h} \leq (1 + 2.02e^{-\gamma})c_{s,h}\,. \tag{3.1}$$

In the following, we will use that

$$\frac{1+2.02e^{-\gamma}}{1-2.02e^{-\gamma}} < 1+4.1e^{-\gamma} < 1+0.06\sqrt{h}n$$

and

$$\left(\frac{1+2.02e^{-\gamma}}{1-2.02e^{-\gamma}}\right)^4 < 1+17.2e^{-\gamma} < 1+1.98h^2n^4\,.$$

According to Section 2, we consider

$$\frac{\sqrt{\mathrm{var}_A(\psi_h)\mathrm{var}_F(\psi_h)}}{\|\psi_h\|_2} = \sqrt{-1+\left(\frac{\sum_{s=-\infty}^{\infty}\tilde{d}_{s,h}^2}{\sum_{s=-\infty}^{\infty}\tilde{d}_{s,h}\tilde{d}_{s+1,h}}\right)^2}\sqrt{\frac{\sum_{s=-\infty}^{\infty}s^2\tilde{d}_{s,h}^2}{\sum_{s=-\infty}^{\infty}\tilde{d}_{s,h}^2}}$$

and obtain as an upper bound for the first squareroot term

$$R_1 = \sqrt{-1+(1+17.2e^{-\gamma})\left(\frac{\sum_{s=-\infty}^{\infty}c_{s,h}^2}{\sum_{s=-\infty}^{\infty}c_{s,h}c_{s+1,h}}\right)^2}$$

and for the second one

$$R_2 = \sqrt{(1+4.1e^{-\gamma})^2\frac{\sum_{s=-\infty}^{\infty}s^2c_{s,h}^2}{\sum_{s=-\infty}^{\infty}c_{s,h}^2}}\,.$$

Using the corresponding estimates in the proof of Theorem 2.2, we obtain

$$R_2 < \frac{1+4.1e^{-\gamma}}{2\sqrt{h}} < \frac{1+0.06n\sqrt{h}}{2\sqrt{h}}$$

and

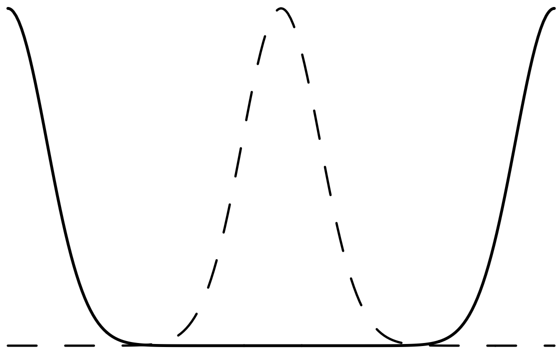
$$R_1 < \sqrt{-1+(1+1.98h^2n^4)(1+h+1.53h^2)} < \sqrt{h+4.2h^2n^4}\,.$$

Finally,

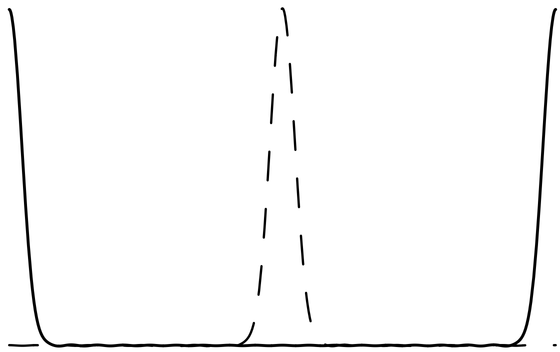
$$R_1\cdot R_2 < \frac{1}{2}(1+0.06n\sqrt{h})(1+2.05n^2\sqrt{h}) < \frac{1}{2}+1.1n^2\sqrt{h}\,. \qquad \blacksquare$$

It remains an open question whether there is a translation invariant basis of W_n which is asymptotically optimal independent of the level n.

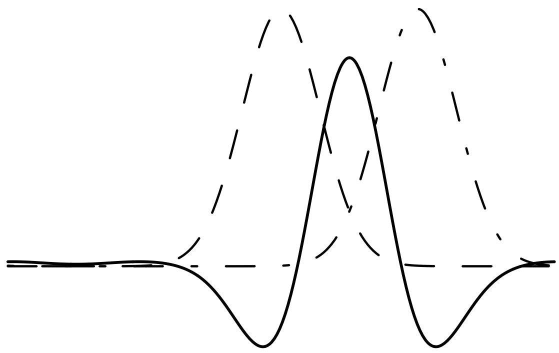
Figure 3 illustrates the influence of different values for the parameter h and the level n. For $n=1$ there is only one scaling function φ_h (dashed line) and one wavelet ψ_h (thick line). For $n=4$ and $n=8$, we show two scaling functions φ_h, $\varphi_h(\cdot-\frac{2\pi}{n})$ and the wavelet ψ_h.



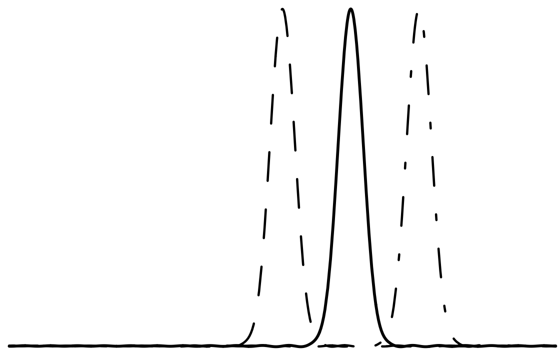
$h = 0.1$ and $n = 1$



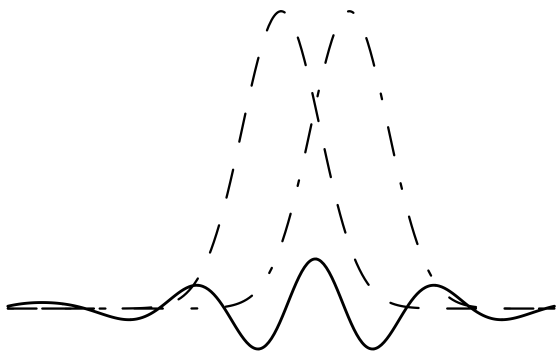
$h = 0.01$ and $n = 1$



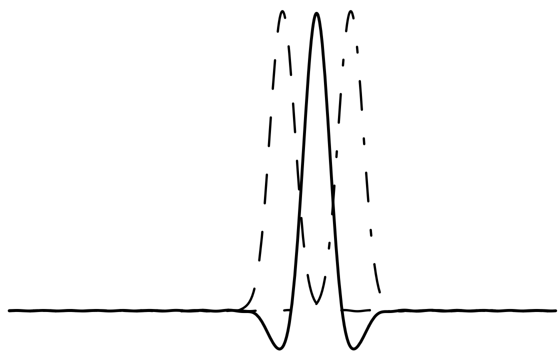
$h = 0.1$ and $n = 4$



$h = 0.01$ and $n = 4$



$h = 0.1$ and $n = 8$



$h = 0.01$ and $n = 8$

Figure 3.

§4. Orthogonalization

Finally, we will consider time-frequency localization for orthogonal scaling functions and wavelets.

Lemma 4.1 *For fixed h, let*

$$\varphi_h^{\perp}(x) := \sum_{s=-\infty}^{\infty} c_{s,h}^{\perp} e^{isx},$$

with

$$c_{s,h}^{\perp} := e^{-hs^2} \left(n \sum_{q=-\infty}^{\infty} e^{-2h(s+qn)^2} \right)^{-\frac{1}{2}},$$

and

$$\psi_h^{\perp}(x) := \sum_{s=-\infty}^{\infty} d_{s,h}^{\perp} e^{is(x-\frac{\pi}{n})},$$

with

$$d_{s,h}^{\perp} := e^{-hs^2} \left(\left(n \sum_{p=-\infty}^{\infty} e^{-2h(s+2pn)^2} \right)^{-1} - \left(n \sum_{q=-\infty}^{\infty} e^{-2h(s+qn)^2} \right)^{-1} \right)^{\frac{1}{2}}.$$

Then $\varphi_h^{\perp}(\cdot - \frac{2k\pi}{n})$, $k = 0, \ldots, n-1$, and $\psi_h^{\perp}(\cdot - \frac{2k\pi}{n})$, $k = 0, \ldots, n-1$, constitute orthonormal bases for V_n and W_n, respectively.

Proof: We have to prove $\varphi_h^{\perp} \in V_n$, $\psi_h^{\perp} \in W_n$ and the orthonormality of corresponding translates. This can be done in exactly the same manner as in the proof of Theorem 3.1 (see also [8]). ■

Now we can show that for fixed n, these orthonormalized scaling functions and wavelets are again optimally time-frequency localized.

Theorem 4.1 *Let n be fixed. Then, for all $0 < h \le \frac{1}{4n^2}$, we have*

$$\sqrt{\mathrm{var}_A(\varphi_h^{\perp})\mathrm{var}_F(\varphi_h^{\perp})} < \frac{1}{2} + n^2\sqrt{h}$$

and

$$\sqrt{\mathrm{var}_A(\psi_h^{\perp})\mathrm{var}_F(\psi_h^{\perp})} < \frac{1}{2} + 1.1n^2\sqrt{h}.$$

Proof: Because of the similarity to the proof of Theorem 3.2, we can be very brief here. Note that

$$c_{s,h}^{\perp} = c_{s,h} \left(\frac{2h}{\pi}\right)^{\frac{1}{4}} \left(u(hn^2, \frac{s}{n})\right)^{-\frac{1}{2}} \left(\frac{2hn^2}{\pi}\right)^{-\frac{1}{4}}.$$

With Lemma 2.1 and $\beta = \frac{\pi^2}{2hn^2} > 2\pi^2$, we estimate

$$\left| \left(u(hn^2, \frac{s}{n}) \right)^{-\frac{1}{2}} \left(\frac{2hn^2}{\pi} \right)^{-\frac{1}{4}} - 1 \right|$$

$$< \max \left\{ 1 - \left(\frac{1 - e^{-\beta}}{1 + e^{-\beta}} \right)^{\frac{1}{2}} ; \left(\frac{1 - e^{-\beta}}{1 - 3e^{-\beta}} \right)^{\frac{1}{2}} - 1 \right\} < 1.01 e^{-\beta} , \tag{4.1}$$

which is similar to (3.1). Analogously,

$$d_{s,h}^{\perp} = c_{s,h} \left(\frac{2h}{\pi} \right)^{\frac{1}{4}} \left(\frac{1}{u(4hn^2, \frac{s}{2n})} - \frac{1}{u(hn^2, \frac{s}{n})} \right)^{\frac{1}{2}} \left(\frac{2hn^2}{\pi} \right)^{-\frac{1}{4}} .$$

Here we estimate with $\gamma = \frac{\pi^2}{8hn^2} > \frac{\pi^2}{2}$

$$\left| \left(\frac{1}{u(4hn^2, \frac{s}{2n})} - \frac{1}{u(hn^2, \frac{s}{n})} \right)^{\frac{1}{2}} \left(\frac{2hn^2}{\pi} \right)^{-\frac{1}{4}} - 1 \right|$$

$$\begin{aligned} &< \max \left\{ 1 - \left(2\frac{1 - e^{-\gamma}}{1 + e^{-\gamma}} - \frac{1 - e^{-4\gamma}}{1 - 3e^{-4\gamma}} \right)^{\frac{1}{2}} ; \left(2\frac{1 - e^{-\gamma}}{1 - 3e^{-\gamma}} - \frac{1 - e^{-4\gamma}}{1 + e^{-4\gamma}} \right)^{\frac{1}{2}} - 1 \right\} \\ &< 2.03 e^{-\gamma} . \end{aligned} \tag{4.2}$$

With (4.1) and (4.2), we complete the estimates as in the proof of Theorem 3.2. ■

§5. Conclusion

For the periodic uncertainty principle of Theorem 2.1, we have shown that the lower bound 0.5 cannot be attained. On the other hand, the bound is optimal as we have constructed functions φ_h, whose uncertainty products, according to Theorem 2.2, are arbitrarily close to 0.5. Using equally spaced translates, these functions φ_h give rise to a multiresolution analysis. In Theorem 3.1 we construct corresponding wavelet functions which are again asymptotically optimal with respect to h. However, the localization of the wavelet is estimated only in dependence on the level n. This means that for fixed h the localization is lost for large values of n. Nevertheless for fixed n and small h, the shape of the wavelets approaches the shape of the Theta functions. Analogous results are stated in Section 4 for orthogonalized bases. It remains an open question whether there is an orthogonal basis of $L^2_{2\pi}$, which is asymptotically optimal independent from the level n.

Acknowledgments. The authors thank Dr. F. J. Narcowich for several stimulating discussions concerning the uncertainty principle for periodic functions and especially for pointing out the proof of Theorem 2.1.

References

1. L. Berg, "Asymptotische Darstellungen und Entwicklungen," Deutscher Verlag der Wissenschaften, Berlin 1968.
2. E. Breitenberger, Uncertainty measures and uncertainty relations for angle observables, *Found. Phys.* **15** (1983), 353–364.
3. C. K. Chui and J. Z. Wang, A study of compactly supported scaling functions and wavelets, in "Wavelets, Images, and Surface Fitting," P. J. Laurent, A. Le Méhauté and L. L. Schumaker (eds.), AKPeters, Boston, 1994, 121–140.
4. Y. W. Koh, S. L. Lee and H. H. Tan, Periodic orthogonal splines and wavelets, *Appl. Comput. Harmonic Anal.*, 1994, to appear.
5. F. J. Narcowich and J. D. Ward, Wavelets associated with periodic functions, CAT Report # 340, Texas A&M University, 1994, submitted for publication.
6. F. J. Narcowich and J. D. Ward, Non-stationary spherical wavelets for scattered data, in "Proceedings of the Eighth Texas International Conference on Approximation Theory," to appear.
7. F. J. Narcowich and J. D. Ward, Non-stationary wavelets on the m-sphere for scattered data, CAT Report # 347, Texas A&M University, 1995, submitted for publication.
8. G. Plonka and M. Tasche, A unified approach to periodic wavelets, in "Wavelets: theory, algorithms, and applications," C. K. Chui, L. Montefusco, and L. Puccio (eds.), Academic Press, New York, 1994, 137–151.
9. J. Prestin and E. Quak, Time frequency localization of trigonometric Hermite operators, CAT Report # 353, Texas A&M University, 1995, submitted for publication.
10. K. Selig, Trigonometric wavelets and the uncertainty principle, submitted for publication.

Jürgen Prestin
FB Mathematik
Universität Rostock
Universitätsplatz 1
D-18051 Rostock
Germany
prestin@mathematik.uni-rostock.d400.de

Ewald Quak
Center for Approximation Theory
Department of Mathematics
Texas A&M University
College Station, TX 77843-3368
USA
quak@euler.math.tamu.edu