Trigonometric Wavelets and the Uncertainty Principle

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Dedicated to Prof. L. Berg on the occasion of his 65th birthday

Abstract

The time-frequency localization of trigonometric wavelets is discussed. A good measure is provided by a periodic version of the Heisenberg uncertainty principle. We consider multiresolution analyses generated by de la Vallée Poussin means of the Dirichlet kernel. For the resulting interpolatory and orthonormal scaling functions and wavelets, the uncertainty product can be bounded from above by a constant.

1 Introduction

In the Hilbert space $L^2(\mathbb{R})$, the well-known Heisenberg uncertainty principle can be phrased as

$$\Delta t(f)\,\Delta\omega(f) \;\; \geq \;\; \frac{\pi}{2}\,\|f\|^4\,,$$

where, for $f \in L^2_{2\pi}$,

$$\begin{aligned} \Delta t(f) \;&:=\; \int_{-\infty}^{\infty} (t-t_0)^2\,|f(t)|^2\,dt\,, \qquad t_0 \;:=\; \int_{-\infty}^{\infty} t\,|f(t)|^2\,dt\,, \\ \Delta\omega(f) \;&:=\; \int_{-\infty}^{\infty} (\omega-\omega_0)^2\,|\hat{f}(\omega)|^2\,d\omega\,, \qquad \omega_0 \;:=\; \int_{-\infty}^{\infty} \omega\,|\hat{f}(\omega)|^2\,d\omega\,. \end{aligned}$$

As Meyer has pointed out [3, 4], basis functions with good time-frequency localization behaviour can be thought of as functions $f_h = h^{-1/2} f(\cdot/h),\; h > 0$, satisfying

$$\Delta t(f_h) \;\; \leq \;\; h^2 \|f_h\|^2\,, \qquad \Delta\omega(f_h) \;\; \leq \;\; \frac{C}{h^2}\|f_h\|^2\,,$$

with some constant C independent of h, such that the uncertainty product is uniformly bounded from above in the form

$$\Delta t(f_h)\,\Delta\omega(f_h) \;\; \leq \;\; C\,\|f_h\|^4 \qquad \text{for all } h > 0\,.$$

In [8], a class of periodic polynomial multiresolution analyses and corresponding wavelet spaces has been constructed for the time-frequency analysis in $L^2_{2\pi}$. The scaling functions and wavelets can be chosen as orthonormal or interpolatory translates of one function in each space. Naturally, the question arises how to express their time-frequency localization mathematically in order to estimate the qualitative and the quantitative approximation behaviour of these functions. In [1], Breitenberger investigated angle variances suitable for replacing Δt in the Hilbert space $L^2_{2\pi}$. The resulting uncertainty principle has been applied by Narcowich and Ward [5] to scaling functions and wavelets generated by continously differentiable periodic functions with positive Fourier coefficients. Those functions have an uncertainty product of order $O(N)$, where N is the dimension of the periodic sampling or wavelet space, respectively. As will be seen later, this is equivalent to the behaviour of the kernel of the Fourier sum operator, whereas the scaling functions and wavelets constructed from de la Vallée Poussin means of Dirichlet kernels yield a product which can be bounded from above by a constant.

2 The uncertainty principle for periodic functions

By $L^2_{2\pi}$, we denote the Hilbert space of square-integrable 2π-periodic functions with the inner product and norm

$$\langle f\,,g\rangle \;:=\; \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\overline{g(t)}\,dt\,, \qquad \|f\| \;:=\; \langle f\,,f\rangle^{1/2}\,.$$

Any function $f \in L^2_{2\pi}$ can be represented by its Fourier series

$$f(t) \;=\; \sum_{k=-\infty}^{\infty} c_k(f)\,e^{ikt}, \qquad c_k(f) \;:=\; \frac{1}{2\pi}\int_0^{2\pi} f(t)\,e^{-ikt}\,dt \quad (k\in\mathbb{Z}).$$

It holds that $(c_k(f))_{k\in\mathbb{Z}}$ is square summable and that

$$\|f\|^2 \;=\; \sum_{k=-\infty}^{\infty} |c_k(f)|^2.$$

Let us introduce the first trigonometric moment as

$$\tau(f) \;:=\; \frac{1}{2\pi}\int_0^{2\pi} e^{it}|f(t)|^2\,dt \;=\; \sum_{k=-\infty}^{\infty} c_k(f)\,\overline{c_{k+1}(f)}\,,$$

which yields $|\tau(f)| \le \|f\|^2$. For any $f \in L^2_{2\pi}$, $f \neq 0$, we define the angle variance

$$\mathrm{var}_A(f) \;:=\; \frac{\|f\|^4 - |\tau(f)|^2}{|\tau(f)|^2} \;=\; \left(\frac{\sum_{k=-\infty}^{\infty}|c_k(f)|^2}{|\sum_{k=-\infty}^{\infty} c_k(f)\,\overline{c_{k+1}(f)}|}\right)^2 - 1\,,$$

and the frequency variance

$$\mathrm{var}_F(f) \;:=\; \frac{1}{\|f\|^2}\sum_{k=-\infty}^{\infty} k^2|c_k(f)|^2 - \frac{1}{\|f\|^4}\left(\sum_{k=-\infty}^{\infty} k|c_k(f)|^2\right)^2.$$

Note that, for real functions, we have $c_k(f) = \overline{c_{-k}(f)}$, and thus, the frequency variance reduces to

$$\operatorname{var}_F(f) \;:=\; \frac{1}{\|f\|^2} \sum_{k=-\infty}^{\infty} k^2 |c_k(f)|^2 .$$

Theorem 1 *For any real-valued $f \in L^2_{2\pi}$, $f \neq const$, the uncertainty principle is given by*

$$\operatorname{var}_A(f)\ \operatorname{var}_F(f) \;\geq\; \frac{1}{4}\ . \tag{1}$$

PROOF: For the first trigonometric moment, the Cauchy-Schwarz inequality yields

$$|\tau(f)| \;=\; \langle e^{i\cdot} f\,, f\rangle \;\leq\; \|e^{i\cdot} f\|\, \|f\| \;=\; \|f\|^2\,,$$

where the equality holds iff $e^{i\cdot} f = cf$ a.e. for some constant $c \in \mathbb{C}$. Hence, for any real-valued $f \in L^2_{2\pi}$, $f \neq 0$, we conclude that $\operatorname{var}_A(f) > 0$. Moreover, if $f \neq const$, we have $\operatorname{var}_F(f) > 0$. If the product on the left-hand side is infinite, the inequality in (1) trivially holds. If both of the variances are finite, then

$$f' = i \sum_{k=-\infty}^{\infty} k\, c_k(f)\, e^{ik\cdot} \;\in\; L^2_{2\pi}\ .$$

So, we can write $\operatorname{var}_F(f) = \|f'\|^2/\|f\|^2$. Assume that $\|f\| = 1$. Then,

$$1 - |\tau(f)|^2 \;=\; \|(e^{i\cdot} - \tau(f))\, f\|^2,$$

and, by the Cauchy-Schwarz inequality and integration by parts,

$$\begin{aligned}
\|f'\|^2\ \|(e^{i\cdot} - \tau(f))\, f\|^2 \;&\geq\; \left|\frac{1}{2\pi}\int_0^{2\pi} (e^{it} - \tau(f))\, f(t)\, f'(t)\, dt\right|^2 \\
&=\; \left|\frac{1}{4\pi}\int_0^{2\pi} (e^{it} - \tau(f))\, (f^2)'(t)\, dt\right|^2 \\
&=\; \frac{1}{4}\left|\frac{1}{2\pi}\int_0^{2\pi} e^{it}\, f^2(t)\, dt\right|^2 \;=\; \frac{1}{4}\,|\tau(f)|^2,
\end{aligned} \tag{2}$$

if $f = \overline{f}$. For unnormalized f, inserting $f/\|f\|$ in (2) yields (1). ■

3 Trigonometric wavelets

Here, we briefly recall from [8] the construction of polynomial 2π-periodic scaling functions and wavelets. We start with the Dirichlet kernel

$$D_n(t) \;:=\; \sum_{k=-n}^{n} e^{ikt} \;=\; 1 + 2\sum_{k=1}^{n} \cos kt \qquad (n \in \mathbb{N}_0)\,,$$

and denote de la Vallée Poussin means, for $N, M \in \mathbb{N}\,,\ N \geq M$, by

$$\varphi_N^M(t) \;:=\; \frac{1}{2M}\sum_{k=N-M}^{N+M-1} D_k(t) \;=\; \begin{cases} \dfrac{\sin Nt \sin Mt}{2M\sin^2\frac{t}{2}} & t \in \mathbb{R}\backslash 2\pi\mathbb{Z}\,, \\ 2N & t \in 2\pi\mathbb{Z}\,. \end{cases}$$

Then, we define sampling spaces $V_j \subset L^2_{2\pi}$ as

$$V_j \;:=\; \mathrm{span}\{\phi_{j,n}:\ n = 0, \ldots, 2N_j - 1\}\,, \tag{3}$$

with the scaling functions $\phi_{j,n}(t) := \frac{1}{\sqrt{2N_j}}\,\varphi_{N_j}^{M_j}\left(t - \frac{n\pi}{N_j}\right)$ and parameters

$$N_j \;:=\; c\,2^j\,, \qquad M_j \;:=\; \begin{cases} 1 & j < \lambda\,, \\ 2^{j-\lambda} & j \geq \lambda\,, \end{cases} \qquad (c \in \mathbb{N},\ \lambda \in \mathbb{N}_0 \text{ fixed}).$$

Proposition 2 [8] *The sampling spaces in* (3) *constitute a (nonstationary) periodic multiresolution analysis* $\{V_j\}_{j\in\mathbb{N}_0}$ *in* $L^2_{2\pi}$ *, for fixed* $c \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ *, if*

$$\frac{N_j}{M_j} \;=\; c\,2^\lambda \;\geq\; 3\,.$$

For the concept of periodic multiresolution analysis see e.g. [2, 6]. By means of the parameters c and λ, we have obtained a whole class of multiresolution analyses including the two limit cases [7] $N_j = 3M_j$ or $M_j = 1$ for all $j \in \mathbb{N}_0$; i.e., $c = 3,\ \lambda = 0$ or $c = 1,\ \lambda = \infty$, respectively. The first case corresponds in some sense to cardinal Meyer wavelets, while the second case occurs when the modified Fourier sum operator,

$$S^*_{N_j} f = \sum_{k=-N_j+1}^{N_j-1} c_k(f)\,e^{ik\cdot} + \frac{1}{2}\left(c_{N_j}(f)\,e^{iN_j\cdot} + c_{-N_j}(f)\,e^{-iN_j\cdot}\right)\,,$$

the orthogonal projection onto V_j, for $M_j = 1$, is applied.

The wavelet spaces $W_j := V_{j+1} \ominus V_j$ are generated by the wavelets

$$\psi_{j,n}(t) := \sqrt{2}\,\phi_{j+1,0}\left(t - \tfrac{(2n+1)\pi}{2N_j}\right) - \phi_{j,0}\left(t - \tfrac{(2n+1)\pi}{2N_j}\right)\,,$$

such that $W_j \;:=\; \mathrm{span}\{\psi_{j,n}:\ n = 0, \ldots, 2N_j - 1\}$, and $L^2_{2\pi} = \mathrm{clos}_{L^2_{2\pi}}\left(V_0 \oplus \bigoplus_{j=0}^\infty W_j\right)$.

For further computations, we will use the Fourier coefficients which are

$$c_k(\phi_{j,n}) \;=\; \frac{1}{\sqrt{2N_j}}\,e^{-\frac{ikn\pi}{N_j}} \times \begin{cases} 1 & |k| \leq N_j - M_j\,, \\ \frac{N_j+M_j-|k|}{2M_j} & N_j - M_j < |k| < N_j + M_j\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$c_k(\psi_{j,n}) \;=\; \frac{1}{\sqrt{2N_j}}\,e^{-\frac{ik(2n+1)\pi}{2N_j}} \times \begin{cases} \frac{|k|-N_j+M_j}{2M_j} & N_j - M_j < |k| < N_j + M_j\,, \\ 1 & N_j + M_j \leq |k| \leq N_{j+1} - M_{j+1}\,, \\ \frac{N_{j+1}+M_{j+1}-|k|}{2M_{j+1}} & N_{j+1} - M_{j+1} < |k| < N_{j+1} + M_{j+1}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Orthonormal shift-invariant bases can be easily obtained from the interpolatory bases. We are dealing with the following functions given by their Fourier coefficients,

$$
c_k(\mathcal{O}\phi_{j,n}) \;=\; \frac{1}{\sqrt{2N_j}}\, e^{-\frac{ikn\pi}{N_j}} \times \begin{cases} 1 & |k| \le N_j - M_j\,, \\ \frac{N_j+M_j-|k|}{\sqrt{2}\sqrt{M_j^2+(N_j-|k|)^2}} & N_j - M_j < |k| < N_j + M_j\,, \\ 0 & \text{otherwise,} \end{cases}
$$

$$
c_k(\mathcal{O}\psi_{j,n}) \;=\; \frac{1}{\sqrt{2N_j}}\, e^{-\frac{ik(2n+1)\pi}{2N_j}} \times \begin{cases} \frac{|k|-N_j+M_j}{\sqrt{2}\sqrt{M_j^2+(N_j-|k|)^2}} & N_j - M_j < |k| < N_j + M_j\,, \\ 1 & N_j + M_j \le |k| \le N_{j+1} - M_{j+1}\,, \\ \frac{N_{j+1}+M_{j+1}-|k|}{\sqrt{2}\sqrt{M_{j+1}^2+(N_{j+1}-|k|)^2}} & N_{j+1} - M_{j+1} < |k| < N_{j+1} + M_{j+1}\,, \\ 0 & \text{otherwise,} \end{cases}
$$

satisfying, for $\ell, n = 0, \ldots, 2N_j - 1$,

$$
\left\langle \mathcal{O}\phi_{j,\ell}\,, \mathcal{O}\phi_{j,n} \right\rangle \;=\; \left\langle \mathcal{O}\psi_{j,\ell}\,, \mathcal{O}\psi_{j,n} \right\rangle \;=\; \delta_{n,\ell}\,.
$$

These functions $\mathcal{O}\phi_{j,n} \in V_j$ and $\mathcal{O}\psi_{j,n} \in W_j$ are again time-localized real-valued functions, which can be seen from the formulas

$$
\mathcal{O}\phi_{j,n}(x) \;=\; \tfrac{1}{\sqrt{2N_j}}\Big(D_{N_j-M_j}\left(x - \tfrac{n\pi}{N_j}\right) + \sqrt{2}\sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j-k}{\sqrt{M_j^2+k^2}} \cos(N_j+k)\left(x - \tfrac{n\pi}{N_j}\right)\Big)\,,
$$

$$
\begin{aligned}
\mathcal{O}\psi_{j,n}(x) \;=\; \tfrac{1}{\sqrt{2N_j}}\Bigg(& \sqrt{2}\sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j+k}{\sqrt{M_j^2+k^2}} \cos(N_j+k)(x - \tfrac{(2n+1)\pi}{2N_j}) \\
& + 2\sum_{\ell=N_j+M_j}^{N_{j+1}-M_{j+1}} \cos \ell (x - \tfrac{(2n+1)\pi}{2N_j}) \\
& + \sqrt{2}\sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}-k}{\sqrt{M_{j+1}^2+k^2}} \cos(N_{j+1}+k)(x - \tfrac{(2n+1)\pi}{2N_j})\Bigg)\,.
\end{aligned}
$$

4 The uncertainty product for trigonometric scaling functions and wavelets

We consider the uncertainty product

$$
U(f) \;:=\; \mathrm{var}_A(f)\ \mathrm{var}_F(f)\,.
$$

The scaling functions $\phi_{j,n}$ and the wavelets $\psi_{j,n}$ are real-valued. Thus, we can apply Theorem 1, which gives $U(\phi_{j,n}) \ge 1/4$ and $U(\psi_{j,n}) \ge 1/4$.

Theorem 3 *For the scaling functions $\phi_{j,n}$ and wavelets $\psi_{j,n}$ associated with a multiresolution analysis as defined in Section 3 with fixed parameters $c \in \mathbb{N}$, $\lambda \in \mathbb{N}_0$, the uncertainty product is asymptoticly, for $j \to \infty$,*

$$
U(\phi_{j,n}) \;=\; O\left(\frac{N_j}{M_j}\right)\,, \qquad U(\psi_{j,n}) \;=\; O\left(\frac{N_j}{M_j}\right)\,.
$$

PROOF: The $L^2_{2\pi}$-norms are

$$
\begin{aligned}
\|\phi_{j,n}\|^2 &= 1 - \tfrac{M_j}{3N_j} + \tfrac{1}{12N_jM_j}\,,\\
\|\psi_{j,n}\|^2 &= \begin{cases} 1 - \tfrac{1}{2N_j} & j < \lambda\,,\\ 1 - \tfrac{M_j}{N_j} + \tfrac{1}{8N_jM_j} & j \geq \lambda\,. \end{cases}
\end{aligned}
$$

The first trigonometric moments can be computed for real functions by

$$
\tau(f) = 2\sum_{k=0}^{\infty} c_k(f)\,\overline{c_{k+1}(f)}\,.
$$

Moreover, $\tau(f(\cdot - x)) = e^{ix}\tau(f)$. Hence, we obtain that

$$
\begin{aligned}
|\tau(\phi_{j,n})| &= \tfrac{1}{N_j}\left(N_j - M_j + \tfrac{1}{4M_j^2}\sum_{k=1}^{2M_j-1} k(k+1)\right)\\
&= 1 - \tfrac{M_j}{3N_j} - \tfrac{1}{6N_jM_j}\,,\\
|\tau(\psi_{j,n})| &= \tfrac{1}{N_j}\left(N_j - M_j - M_{j+1} + \tfrac{1}{4M_j^2}\sum_{k=1}^{2M_j-1} k(k+1) + \tfrac{1}{4M_{j+1}^2}\sum_{k=1}^{2M_{j+1}-1} k(k+1)\right)\\
&= \begin{cases} 1 - \tfrac{1}{N_j} & j < \lambda\,,\\ 1 - \tfrac{M_j}{N_j} - \tfrac{1}{4N_jM_j} & j \geq \lambda\,. \end{cases}
\end{aligned}
$$

Thereby we get the angle variances $\mathrm{var}_A(f) = (\|f\|^2/|\tau(f)|)^2 - 1$,

$$
\begin{aligned}
\mathrm{var}_A(\phi_{j,n}) &= \left(1 + \tfrac{3}{12N_jM_j - 4M_j^2 - 2}\right)^2 - 1 = \tfrac{3(24N_jM_j - 8M_j^2 - 1)}{4(6N_jM_j - 2M_j^2 - 1)^2)}\,,\\
\mathrm{var}_A(\psi_{j,n}) &= \begin{cases} \left(1 + \tfrac{1}{2(N_j-1)}\right)^2 - 1 = \tfrac{4N_j - 3}{4(N_j-1)^2} & j < \lambda\,,\\ \left(1 + \tfrac{3}{8N_jM_j - 8M_j^2 - 2}\right)^2 - 1 = \tfrac{3(16N_jM_j - 16M_j^2 - 1)}{4(4N_jM_j - 4M_j^2 - 1)^2} & j \geq \lambda\,. \end{cases}
\end{aligned}
$$

The frequency variances $\mathrm{var}_F(f) = 2\sum\limits_{k=1}^{\infty} k^2|c_k(f)|^2$ are

$$
\begin{aligned}
\mathrm{var}_F(\phi_{j,n}) &= \frac{1}{\|\phi_{j,n}\|^2 N_j}\left(\sum_{k=1}^{N_j-M_j} k^2 + \tfrac{1}{4M_j^2}\sum_{k=1}^{2M_j-1} k^2(N_j + M_j - k)^2\right)\\
&= \frac{4N_j(N_j - M_j) + \tfrac{N_j}{M_j} + 4M_j^2 - 2 - \tfrac{4M_j^3}{5N_j} + \tfrac{M_j}{N_j} - \tfrac{1}{5N_jM_j}}{12 - \tfrac{4M_j}{N_j} + \tfrac{1}{N_jM_j}}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{var}_F(\psi_{j,n}) &= \frac{1}{\|\psi_{j,n}\|^2 N_j}\Bigg(\tfrac{1}{4M_j^2}\sum_{k=1}^{2M_j-1} k^2(N_j-M_j+k)^2 + \sum_{k=0}^{N_j-M_j-M_{j+1}}(N_j+M_j+k)^2 \\
&\qquad + \tfrac{1}{4M_{j+1}^2}\sum_{k=1}^{2M_{j+1}-1} k^2(2N_j+M_{j+1}-k)^2\Bigg) \\
&= \begin{cases} \dfrac{7N_j^2-3N_j+\frac{1}{2}}{3-\frac{1}{2N_j}} & j<\lambda\,, \\[3ex] \dfrac{28N_j^2-36N_jM_j+\frac{4N_j}{M_j}+28M_j^2-2-\frac{36M_j^3}{5N_j}+\frac{3M_j}{N_j}-\frac{12}{40N_jM_j}}{4-\frac{4M_j}{N_j}+\frac{1}{2N_jM_j}} & j\geq\lambda. \end{cases}
\end{aligned}
$$

Now, we can estimate the uncertainty products, using $N_j \geq 3M_j$. For $M_j = 1$, i.e., $j < \lambda$, we obtain that

$$
\begin{aligned}
U(\phi_{j,n}) &< \frac{7N_j}{36}+\frac{7}{48} < \frac{N_j}{5}+\frac{1}{6}\,, \\
U(\psi_{j,n}) &< \frac{21N_j}{8}+\frac{3(15N_j-6)}{8(2N_j-3)} < 3N_j+6\,,
\end{aligned}
$$

and for $M_j \geq 2$, i.e., $j > \lambda$, that

$$
\begin{aligned}
U(\phi_{j,n}) &< \frac{7N_j}{36M_j}-\frac{7(4N_jM_j(M_j-1)-3N_j-6)}{16\cdot 27M_j^2(N_j-M_j)} < \frac{N_j}{5M_j}\,, \\
U(\psi_{j,n}) &< \frac{31N_j(7N_j^2-9N_jM_j+\frac{3N_j}{4M_j}+6)}{8\cdot 14M_j(N_j^2-2N_jM_j-\frac{N_j}{8M_j}+M_j^2)} \\
&< \frac{31N_j}{16M_j}+\frac{31\cdot 5}{8\cdot 14}+\frac{31\cdot 13}{16\cdot 14M_j^2}+\frac{31\cdot 24M_j^2+31\cdot 26+5+\frac{31\cdot 13}{M_j^2}}{16\cdot 64(M_j^2-1)} \\
&< \frac{2N_j}{M_j}+2.78\,.
\end{aligned}
$$

This gives the assertion. ■

By this Theorem, it is proved that, for $M_j = 1$ for all $j \in \mathbb{N}_j$, i.e., in the limit case $\lambda = \infty$, we have $U(\phi_{j,n}) = O(N_j)$ and $U(\psi_{j,n}) = O(N_j)$, for $j \to \infty$. For finite $\lambda \in \mathbb{N}_0$, the uncertainty product of the scaling functions and wavelets is bounded from above by a constant. The smaller the quotient $N_j/M_j = c\cdot 2^\lambda$, the better the time-frequency localization of these functions becomes. And for $N_j = 3M_j,\ \lambda = 0$, one can directly compute $U(\phi_{j,n}) < 0.49$ and $U(\psi_{j,n}) < 8.38$, uniformly for all $j > 0$. The significant difference between these two values is due to the frequency variance, which gives bigger weight to higher frequencies.

Numerical calculations show that for fixed parameters c and λ the uncertainty products $U(\phi_{j,n})$ and $U(\psi_{j,n})$ are decreasing, for $j \to \infty$. From the variances in the proof of Theorem 3, we can derive the asymptotic bounds which depend clearly on the quotient N_j/M_j, namely

$$U(\phi_{j,n}) \;\geq\; \frac{\frac{N_j}{M_j} - 1 + \frac{M_j}{N_j} - \frac{M_j^2}{5N_j^2}}{6 - 4\frac{M_j}{N_j} + \frac{2M_j^2}{3N_j^2}}\,, \qquad U(\psi_{j,n}) \;\geq\; \frac{7\frac{N_j}{M_j} - 9 + 7\frac{M_j}{N_j} - \frac{9M_j^2}{5N_j^2}}{4(1-\frac{M_j}{N_j})^2}\,.$$

Now, we examine the orthonormal scaling functions $\mathcal{O}\phi_{j,n}$ and wavelets $\mathcal{O}\psi_{j,n}$. They are also real-valued. Hence, by Theorem 1, we have $U(\mathcal{O}\phi_{j,n}) \geq 1/4$ and $U(\mathcal{O}\psi_{j,n}) \geq 1/4$.

Theorem 4 *For the orthonormal scaling functions and wavelets associated with a multiresolution analysis as defined in Section* 3 *with fixed parameters* $c \in \mathbb{N},\ \lambda \in \mathbb{N}_0$ *, the uncertainty product is asymptoticly, for* $j \to \infty$ *,*

$$U(\mathcal{O}\phi_{j,n}) \;=\; O\left(\frac{N_j}{M_j}\right), \qquad U(\mathcal{O}\psi_{j,n}) \;=\; O\left(\frac{N_j}{M_j}\right).$$

PROOF: The first trigonometric moments are given by

$$\begin{aligned}
|\tau(\mathcal{O}\phi_{j,n})| &= \tfrac{1}{N_j}\left(N_j - M_j + \tfrac{1}{2}S_{M_j}\right),\\
|\tau(\mathcal{O}\psi_{j,n})| &= \tfrac{1}{N_j}\left(N_j - M_j - M_{j+1} + \tfrac{1}{2}S_{M_j} + \tfrac{1}{2}S_{M_{j+1}}\right),
\end{aligned}$$

where

$$S_M \;:=\; \sum_{k=-M}^{M-1} \tfrac{M-k}{\sqrt{M^2+k^2}}\,\tfrac{M-(k+1)}{\sqrt{M^2+(k+1)^2}}\,.$$

Let us estimate S_M from below. With

$$(M^2+k^2)(M^2+(k+1)^2) \;\leq\; (M^2+k^2+k+1/2)^2$$

we find that

$$\begin{aligned}
S_M &\geq \sum_{k=-M}^{M-1} \tfrac{M^2+k^2-2Mk-M+k}{M^2+k^2+k+1/2} \;=\; 2M - \sum_{k=-M}^{M-1} \tfrac{2Mk+M+1/2}{M^2+(k+1/2)^2+1/4}\\
&\geq 2M - \sum_{k=-M}^{M-1} \tfrac{2Mk+M+1/2}{M^2+(k+1/2)^2}\,.
\end{aligned}$$

Since $(k+1/2)^2 = (-(k+1)+1/2)^2$, it follows that

$$\begin{aligned}
S_M &\geq 2M - \sum_{k=0}^{M-1} \tfrac{2Mk-2M(k+1)+2M+1}{M^2+(k+1/2)^2}\\
&= 2M - \tfrac{1}{M^2}\sum_{k=0}^{M-1} \tfrac{1}{1+(\frac{k}{M}+\frac{1}{2M})^2}\,.
\end{aligned}$$

The function $f(x) = (1+x^2)^{-1}$ is decreasing in $[0,1]$. We can approximate its integral from below by quadrature,

$$\begin{aligned} I_1 \;:=\; \int_0^1 \frac{dx}{1+x^2} \;&\geq\; \frac{f(\frac{1}{2M})}{2M} + \frac{1}{M}\sum_{k=1}^{M-1} f(\tfrac{k}{M}+\tfrac{1}{2M}) + \frac{f(1)}{2M} \\ &\geq\; \frac{1}{M}\sum_{k=0}^{M-1} \frac{1}{1+(\frac{k}{M}+\frac{1}{2M})^2} - \frac{1}{2M} + \frac{1}{4M}\,. \end{aligned}$$

On the other hand, $I_1 = \arctan 1 = \pi/4$, which gives us

$$S_M \;\geq\; 2M - \tfrac{\pi}{4M} - \tfrac{1}{4M^2}\,. \tag{4}$$

For small M, we rather compute S_M directly. With $S_1 = \sqrt{2}$ and $S_2 = (3\sqrt{2}+4)/\sqrt{5}$, for $j \leq \lambda$, we have

$$\begin{aligned} |\tau(\mathcal{O}\phi_{j,n})| \;&=\; 1 - \tfrac{\sqrt{2}-1}{\sqrt{2}N_j}\,, \\ |\tau(\mathcal{O}\psi_{j,n})| \;&=\; 1 - \tfrac{2-\sqrt{2}}{N_j} \qquad (j < \lambda)\,, \\ |\tau(\mathcal{O}\psi_{j,n})| \;&\geq\; 1 - \tfrac{5}{11\,N_j} \qquad (j = \lambda)\,. \end{aligned}$$

For $j > \lambda$, i.e. for $M_{j+1} = 2M_j$, with (4), we obtain

$$\begin{aligned} |\tau(\mathcal{O}\phi_{j,n})| \;&\geq\; 1 - \tfrac{M_j\pi+1}{8N_jM_j^2}\,, \\ |\tau(\mathcal{O}\psi_{j,n})| \;&\geq\; \tfrac{1}{N_j}\left(N_j - \tfrac{M_j\pi+1}{8M_j^2} - \tfrac{M_{j+1}\pi+1}{8M_{j+1}^2}\right) \;=\; 1 - \tfrac{6M_j\pi+5}{32N_jM_j^2}\,. \end{aligned}$$

Now, we compute an upper bound for the angle variances $\mathrm{var}_A(f) = (\|f\|^2/|\tau(f)|)^2 - 1$. At first, for $j \leq \lambda$, with $N_j \geq 3$, we have

$$\begin{aligned} \mathrm{var}_A(\mathcal{O}\phi_{j,n}) \;&=\; \left(1 + \frac{1-\frac{1}{\sqrt{2}}}{N_j - 1 + \frac{1}{\sqrt{2}}}\right)^2 - 1 \;=\; \frac{1-\frac{1}{\sqrt{2}}}{N_j-1+\frac{1}{\sqrt{2}}}\left(2 + \frac{1-\frac{1}{\sqrt{2}}}{N_j-1+\frac{1}{\sqrt{2}}}\right) \\ &\leq\; \frac{2}{3N_j-1}\,. \end{aligned}$$

For $j > \lambda$, the angle variance can be estimated using $N_j \geq 3M_j$ and $M_j \geq 2$, such that

$$\begin{aligned} \mathrm{var}_A(\mathcal{O}\phi_{j,n}) \;&\leq\; \left(1 + \tfrac{M_j\pi+1}{8N_jM_j^2 - M_j\pi - 1}\right)^2 - 1 \\ &=\; \tfrac{\pi+1/M_j}{8N_jM_j-\pi-1/M_j}\left(2 + \tfrac{\pi+1/M_j}{8N_jM_j-\pi-1/M_j}\right) \\ &\leq\; \tfrac{\pi+1/2}{8N_jM_j-\pi-1/2}\left(2 + \tfrac{\pi+1/2}{96-\pi-1/2}\right) \\ &\leq\; \tfrac{30}{32N_jM_j-15}\,. \end{aligned}$$

In the same way, we obtain estimates for the angle variances of the orthonormal wavelets. For $j \leq \lambda$, with $N_j \geq 3$, we have

$$\mathrm{var}_A(\mathcal{O}\psi_{j,n}) \begin{cases} = \left(1 + \frac{2-\sqrt{2}}{N_j-2+\sqrt{2}}\right)^2 - 1 \;\leq\; \frac{10}{7N_j-5}\,, & j < \lambda\,, \\ \leq \left(1 + \frac{5/11}{N_j-5/11}\right)^2 - 1 \;\leq\; \frac{1}{N_j-5/11}\,, & j = \lambda\,, \end{cases}$$

and, for $j > \lambda$, with $N_j \geq 3M_j \geq 6$,

$$\mathrm{var}_A(\mathcal{O}\psi_{j,n}) \;\leq\; \left(1 + \tfrac{6M_j\pi+5}{32N_jM_j^2-6M_j\pi-5}\right)^2 - 1 \;\leq\; \tfrac{11}{8N_jM_j-6}\,.$$

The frequency variances $\mathrm{var}_F(f) = 2\sum_{k=1}^{\infty} k^2|c_k(f)|^2$ are

$$\begin{aligned}
\mathrm{var}_F(\mathcal{O}\phi_{j,n}) &= \tfrac{1}{N_j}\left(\sum_{k=1}^{N_j-M_j} k^2 + \tfrac12 \sum_{k=-M_j+1}^{M_j-1} (N_j+k)^2 \tfrac{(M_j-k)^2}{M_j^2+k^2}\right),\\
\mathrm{var}_F(\mathcal{O}\psi_{j,n}) &= \tfrac{1}{N_j}\left(\sum_{k=N_j+M_j}^{2N_j-M_{j+1}} k^2 + \tfrac12 \sum_{k=-M_j+1}^{M_j-1} (N_j+k)^2 \tfrac{(M_j+k)^2}{M_j^2+k^2}\right.\\
&\qquad\left. + \tfrac12 \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} (2N_j+k)^2 \tfrac{(M_{j+1}-k)^2}{M_{j+1}^2+k^2}\right).
\end{aligned}$$

For $j \leq \lambda$, we compute

$$\mathrm{var}_F(\mathcal{O}\phi_{j,n}) \;=\; \tfrac{(N_j-1)(2N_j-1)}{6} + \tfrac{N_j}{2} \;=\; \tfrac{N_j^2}{3} + \tfrac16\,,$$

$$\mathrm{var}_F(\mathcal{O}\psi_{j,n}) \;=\; \begin{cases} \tfrac{(2N_j-1)(4N_j-1)}{3} - \tfrac{(N_j+1)(2N_j+1)}{6} + \tfrac{N_j}{2} + 2N_j \;=\; \tfrac{7N_j^2}{3} + \tfrac16 & j < \lambda\,,\\ \tfrac{1}{N_j}\left(\tfrac{(2N_j-1)(2N_j-2)(4N_j-3)}{6} - \tfrac{N_j(N_j+1)(2N_j+1)}{6} + \tfrac{N_j^2}{2} + \tfrac12 \sum_{k=-1}^{1} \tfrac{(2N_j+k)^2(2-k)^2}{4+k^2}\right) & \\ < \tfrac{7N_j^2}{3} + 1 & j = \lambda. \end{cases}$$

For $j > \lambda$, we have

$$\begin{aligned}
\mathrm{var}_F(\mathcal{O}\phi_{j,n}) &= \tfrac{1}{N_j}\left(\tfrac{(N_j-M_j)(N_j-M_j+1)(2N_j-2M_j+1)}{6} + \tfrac12 \sum_{k=-M_j+1}^{M_j-1}\left((N_j+k)^2 - \tfrac{2M_jk(N_j+k)^2}{M_j^2+k^2}\right)\right)\\
&= \tfrac{(N_j-M_j)(N_j-M_j+1)(2N_j-2M_j+1)}{6N_j} + \tfrac{(2M_j-1)N_j^2}{2N_j} + \tfrac{M_j(M_j-1)(2M_j-1)}{6N_j} - 4M_j \sum_{k=1}^{M_j-1} \tfrac{k^2}{M_j^2+k^2}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{var}_F(\mathcal{O}\psi_{j,n}) &= \tfrac{1}{N_j}\Bigg(\tfrac{(2N_j-2M_j)(2N_j-2M_j+1)(4N_j-4M_j+1)}{6} - \tfrac{(N_j+M_j)(N_j+M_j-1)(2N_j+2M_j-1)}{6}\\
&\quad + \tfrac12 \sum_{k=-M_j+1}^{M_j-1}\left((N_j+k)^2 + \tfrac{2M_jk(N_j+k)^2}{M_j^2+k^2}\right) + \tfrac12 \sum_{k=-2M_j+1}^{2M_j-1}\left((2N_j+k)^2 - \tfrac{4M_jk(2N_j+k)^2}{4M_j^2+k^2}\right)\Bigg)\\
&= \tfrac{1}{N_j}\Bigg(\tfrac{(2N_j-2M_j)(2N_j-2M_j+1)(4N_j-4M_j+1)}{6} - \tfrac{(N_j+M_j)(N_j+M_j-1)(2N_j+2M_j-1)}{6}\\
&\quad + \tfrac{(2M_j-1)N_j^2}{2} + \tfrac{M_j(M_j-1)(2M_j-1)}{6} + \tfrac{(4M_j-1)4N_j^2}{2} + \tfrac{M_j(2M_j-1)(4M_j-1)}{3}\\
&\quad + 4N_jM_j \sum_{k=1}^{M_j-1} \tfrac{k^2}{M_j^2+k^2} - 16N_jM_j \sum_{k=1}^{2M_j-1} \tfrac{k^2}{4M_j^2+k^2}\Bigg).
\end{aligned}$$

In order to evaluate the remaining sums in these expressions, we consider the integral

$$I_2 \;:=\; \int_0^1 \frac{x^2\,dx}{1+x^2} \;=\; 1 - \arctan 1 \;=\; 1 - \frac{\pi}{4}\,.$$

Since the function $x^2/(1+x^2)$ is decreasing in $[0,1]$, we have by means of the rectangular rule, for $M \geq 2$,

$$\frac{1}{M}\sum_{k=1}^{M-1}\frac{\left(\frac{k}{M}\right)^2}{1+\left(\frac{k}{M}\right)^2} \;\leq\; I_2 \;\leq\; \frac{1}{M}\sum_{k=1}^{M-1}\frac{\left(\frac{k}{M}\right)^2}{1+\left(\frac{k}{M}\right)^2}+\frac{1}{2M}\,.$$

Thereby, the above expressions, for $j > \lambda$, can be reduced to

$$\begin{aligned}\mathrm{var}_F(\mathcal{O}\phi_{j,n}) &< \frac{N_j^2}{3}+(\pi-3)M_j^2+M_j+\frac{1}{6}\,,\\ \mathrm{var}_F(\mathcal{O}\psi_{j,n}) &< \frac{7N_j^2}{3}+7(\pi-3)M_j^2+5M_j+\frac{1}{6}\,.\end{aligned}$$

Finally, we obtain the uncertainty products. For $j \leq \lambda$, with $N_j \geq 3$, we compute

$$U(\mathcal{O}\phi_{j,n}) \;<\; \frac{2N_j^2+1}{3(3N_j-1)} \;<\; \frac{2N_j}{9}+\frac{1}{8}\,,$$

$$U(\mathcal{O}\psi_{j,n}) \;<\; \begin{cases}\frac{5(14N_j^2+1)}{3(7N_j-5)} < \frac{10}{3}(N_j+1)\,, & j<\lambda\,,\\ \frac{2(7N_j^2+3)}{3(2N_j-1)} < \frac{7N_j+5}{3}\,, & j=\lambda\,,\end{cases}$$

and for $j > \lambda$, with $N_j \geq 3M_j$ and $M_j \geq 2$,

$$U(\mathcal{O}\phi_{j,n}) \;<\; \frac{5(2N_j^2+6(\pi-3)M_j^2+6M_j+1)}{32N_jM_j-15} \;<\; \frac{5}{16}\left(\frac{N_j}{M_j}+1\right)\,,$$

$$U(\mathcal{O}\psi_{j,n}) \;<\; \frac{11(7N_j^2+21(\pi-3)M_j^2+15M_j+1/2)}{3(8N_jM_j-6)} \;<\; \frac{77N_j}{24M_j}+\frac{5}{4}\,.$$

This gives the assertion. ■

For the orthonormal translates we cannot compute such exact bounds as for the interpolatory functions. However, numerical computations yield the very same asymptotic behaviour for $U(\mathcal{O}\phi_{j,n})$ and $U(\mathcal{O}\psi_{j,n})$ to have a lower bound depending on the quotient N_j/M_j. It also turns out that the uncertainty products of the orthonormal functions are always but once greater than the respective products of the interpolatory ones. The exception is for the wavelets in the case $N_j/M_j = 3$.

5 Conclusion

The de la Vallée Poussin averaging method is not only advantageous with respect to the shape of the fundamental basis functions and capable to supply fast numerical algorithms, but also essential for the asymptotic properties of projections. In [8], we have shown that the approximation error of the interpolation operator $L_j : C_{2\pi} \to V_j$ can be estimated by

$$\|f-L_jf\|_\infty \;\leq\; \left(4(\pi+1)+\frac{4\pi+3}{3}\ln\frac{N_j}{2M_j}\right)\,E_{N_j-M_j}(f)$$

where $E_n(f) := \inf \{\|f - t_n\|_\infty : t_n \in T_n\}$ is the best approximation by trigonometric polynomials of degree at most n. In that factor, we already see the importance of the quotient N_j/M_j to be bounded.

The uncertainty product U provides another measure of goodness emphasizing the aspect of localization properties of the basis functions in the time-frequency plane. We can differentiate qualitatively between "simple" basis functions like the Dirichlet kernel or compositions as in [5] on the one hand, where U is not uniformly bounded from above for increasing space dimension, and more sophisticated constructions like the de la Vallée Poussin means on the other hand, where it is bounded. For our polynomial, i.e., band-limited scaling functions and wavelets, we have shown it as well as the fact that our orthonormalization procedure doesn't harm this satisfying result.

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