Factorization of refinement masks of function vectors

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Dedicated to Prof. E. W. Cheney on the occasion
of his 65th birthday

Abstract. Considering the set of closed shift–invariant subspaces V_j ($j \in \mathbb{Z}$) of $L^2(\mathbb{R})$ generated by a refinable function vector $\mathbf{\Phi}$, we give necessary and sufficient conditions for the refinement mask of $\mathbf{\Phi}$ ensuring controlled approximation order m. In particular, algebraic polynomials can be exactly reproduced in V_0 if and only if the refinement mask of $\mathbf{\Phi}$ can be factorized. The results are illustrated by B–splines with multiple knots.

§1 Introduction

The idea of considering a ladder of imbedded subspaces V_j of a Hilbert space for approximating functions has extensively been used in many applications. In the case of multiresolution analysis of $L^2(\mathbb{R})$, the subspaces V_j are usually *generated* by a single function $\phi \in L^2(\mathbb{R})$,

$$V_j := \operatorname{clos}_{L^2} \operatorname{span} \{\phi(2^j \cdot -l) : l \in \mathbb{Z}\}.$$

In order to ensure the condition $V_j \subset V_{j+1}$ ($j \in \mathbb{Z}$), we need a *refinable* scaling function, *i.e.*, ϕ has to satisfy a functional equation of the type

$$\phi = \sum_{l \in \mathbb{Z}} p_l \, \phi(2 \cdot -l) \quad (\{p_l\}_{l \in \mathbb{Z}} \in l^2). \tag{1.1}$$

A lot of papers have been delt with solutions of (1.1) and with their properties. Functions satisfying (1.1) not only arise in the context of multiresolution analysis and wavelets. They also play an important role in subdivision schemes. By convenient choice of the *refinement mask*

$$P := \sum_{l \in \mathbb{Z}} p_l \, e^{-il\cdot}$$

pp. 1–8.

one is able to influence properties of ϕ like regularity or support and, at the same time, properties of the generated set of subspaces V_j of $L^2(\mathbb{R})$. The close connection between the refinement mask of ϕ and the structure of the shift–invariant subspaces generated by ϕ is the clue for many successful applications of the theory of multiresolution and corresponding wavelets. For example, V_0 provides controlled approximation order m if and only if the refinement mask P factorizes

$$P(u) = \left(\frac{1+e^{-iu}}{2}\right)^m \; S(u) \tag{1.2}$$

with an appropriate chosen 2π-periodic function S (cf. [2–4]).
In the last time, also the generalized *multiresolution analysis of multiplicity* r ($r \in \mathbb{N}$) of $L^2(\mathbb{R})$ has been considered in more detail (cf. [5–8]). Now the set of imbedded closed subspaces V_j of $L^2(\mathbb{R})$ is *generated* by a function vector $\mathbf{\Phi} := (\phi_\nu)_{\nu=0}^{r-1}$ with $\phi_\nu \in L^2(\mathbb{R})$ ($\nu = 0, \ldots, r-1$),

$$V_j := \operatorname{clos}_{L^2} \operatorname{span} \{\phi_\nu(2^j \cdot -l) : l \in \mathbf{Z},\, \nu = 0, \ldots, r-1\}.$$

The vector of scaling functions $\mathbf{\Phi}$ has to be *refinable*, *i.e.*, $\mathbf{\Phi}$ has to satisfy a functional equation of the type

$$\mathbf{\Phi} = \sum_{l\in\mathbf{Z}} \mathbf{P}_l\, \mathbf{\Phi}(2 \cdot -l) \quad (\mathbf{P}_l \in \mathbb{R}^{r\times r}),$$

where the sequences of entries of coefficient matrices $\mathbf{P}_l$ ($l \in \mathbf{Z}$) are in l^2.
The purpose of this paper is an investigation of the structure of the *refinement mask*

$$\mathbf{P} := \sum_{l\in\mathbf{Z}} \mathbf{P}_l\, e^{-il\cdot}, \tag{1.3}$$

if the corresponding shift–invariant subspace V_0 is assumed to provide controlled approximation order m. It turns out that, under some mild conditions on $\mathbf{\Phi}$, algebraic polynomials of degree $< m$ can be exactly reproduced in V_0 if and only if $\mathbf{P}$ can be factorized in a certain manner. This result is a natural generalization of the result (1.2) for a single scaling function.
In Section 2, we will introduce some notations and present the main theorem on the factorization of the refinement mask. The example of B–splines with multiple knots is used to illustrate this result in Section 3.

§2 Factorization of the refinement mask

Let us introduce some notations. Consider the Hilbert space $L^2 = L^2(\mathbb{R})$ of all square integrable functions on $\mathbb{R}$. The Fourier transform of $f \in L^2(\mathbb{R})$

is defined by $\hat{f} := \int_{-\infty}^{\infty} f(x)e^{-ix\cdot}\,\mathrm{d}x$. Further, let $BV(\mathbb{R})$ be the set of all functions which are of bounded variation over $\mathbb{R}$ and normalized by

$$f(-\infty) = f(-\infty + 0) = 0, \qquad f(\infty) = f(\infty - 0) = 0,$$
$$f(x) = \frac{1}{2}(f(x+0) + f(x-0)) \quad (-\infty < x < \infty).$$

If $f \in L^2(\mathbb{R}) \cap BV(\mathbb{R})$, then the Poisson Summation Formula

$$\sum_{l\in\mathbf{Z}} f(l)\,e^{-iul} = \sum_{j\in\mathbf{Z}} \hat{f}(u + 2\pi j) \quad (u \in \mathbb{R})$$

is satisfied. For a measurable function f on $\mathbb{R}$ and $m \in \mathbb{N}$ let

$$\|f\|_2 := (\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x)^{1/2},$$

$$|f|_{m,2} := \|\mathrm{D}^m f\|_2, \qquad \|f\|_{m,2} := \sum_{k=0}^{m} \|\mathrm{D}^k f\|_2.$$

Here and in the following, D denotes the differential operator $\mathrm{D} := \mathrm{d}/\mathrm{d}\,\cdot$. Let $W_2^m(\mathbb{R})$ be the usual Sobolev space with the norm $\|\cdot\|_{m,2}$. The l^2–norm of a sequence $\mathbf{c} := \{c_l\}_{l\in\mathbf{Z}}$ is defined by $\|\mathbf{c}\|_{l^2} := (\sum_{l\in\mathbf{Z}} |c_l|^2)^{1/2}$.
The set $\mathcal{B}(\mathbf{\Phi}) := \{\phi_\nu(\cdot - l) : l \in \mathbf{Z},\ \nu = 0, \ldots, r-1\}$ forms a *Riesz basis* of V_0 if there exist constants $0 < A \leq B < \infty$ with

$$A \sum_{\nu=0}^{r-1} \|\mathbf{c}_\nu\|_{l^2}^2 \leq \| \sum_{\nu=0}^{r-1} \sum_{l\in\mathbf{Z}} c_{\nu,l}\,\phi_\nu(\cdot - l)\|_{L^2}^2 \leq B \sum_{\nu=0}^{r-1} \|\mathbf{c}_\nu\|_{l^2}^2$$

for any sequences $\mathbf{c}_\nu = \{c_{\nu,l}\}_{l\in\mathbf{Z}}$ $(\nu = 0, \ldots, r-1)$.
We say that V_0, generated by the function vector $\mathbf{\Phi}$, provides *controlled approximation order* m if for each $f \in W_2^m(\mathbb{R})$ there are sequences $\mathbf{c}_\nu^h = \{c_{\nu,l}^h\}_{l\in\mathbf{Z}}$ $(\nu = 0, \ldots, r-1; h > 0)$ such that for a constant c independent of h the following three conditions are satisfied:

(1) $$\|f - h^{-1/2} \sum_{\nu=0}^{r-1} \sum_{l\in\mathbf{Z}} c_{\nu,l}^h\,\phi_\nu(\cdot/h - l)\|_2 \leq c\,h^m\,|f|_{m,2}.$$

(2) We have
$$\|\mathbf{c}_\nu^h\|_{l^2} \leq c\,\|f\|_2 \quad (\nu = 0, \ldots, r-1).$$

(3) There is a constant δ independent of h such that for $l \in \mathbf{Z}$

$$\operatorname{dist}(lh, \operatorname{supp} f) > \delta \quad \Rightarrow \quad c_{\nu,l}^h = 0 \quad (\nu = 0, \ldots, r-1).$$

Now we want to generalize the known result (1.2) for the principal space V_0 and consider the connections between the refinement mask (1.3) and the structure of the finitely generated closed subspace V_0 of $L^2(\mathbb{R})$. First, let us define the $(r \times r)$–matrix $\mathbf{C} := (C_{j,k})_{j,k=0}^{r-1}$ by the vector $\mathbf{y} = (y_0, \ldots, y_{r-1})^{\mathrm{T}} \in \mathbb{R}^r$, $\mathbf{y} \neq \mathbf{o}$. Here and in the following $\mathbf{o}$ denotes the zero vector. Let $j_0 := \min\{j;\, y_j \neq 0\}$ and $j_1 := \max\{j;\, y_j \neq 0\}$. Further, let for all j with $y_j \neq 0$ be $d_j := \min\{k > j, y_k \neq 0\}$. Then for $j_0 < j_1$, the entries of $\mathbf{C}$ are defined for $j,k = 0, \ldots, r-1$ by

$$C_{j,k}(u) := \begin{cases} y_j^{-1} & y_j \neq 0 \quad \text{and} \quad j = k, \\ 1 & y_j = 0 \quad \text{and} \quad j = k, \\ -y_j^{-1} & y_j \neq 0 \quad \text{and} \quad d_j = k, \\ -e^{-iu}/y_{j_1} & j = j_1 \quad \text{and} \quad k = j_0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

For $j_0 = j_1$, $\mathbf{C}$ is a diagonal matrix of the form

$$\mathbf{C}(u) = \mathrm{diag}(\underbrace{1, \ldots, 1}_{j_0}, (1 - e^{-iu})/y_{j_0}, \underbrace{1, \ldots, 1}_{r-1-j_0}). \tag{2.2}$$

In particular, for $r = 1$ we have $\mathbf{C}(u) := (1 - e^{-iu})/y_0$. Now we can show:

Theorem 2.1 *Let $m \in \mathbb{N}$ be fixed, and let $\mathbf{\Phi} := (\phi_\nu)_{\nu=0}^{r-1}$ be a refinable vector of functions $\phi_\nu \in L^2(\mathbb{R}) \cap BV(\mathbb{R})$ satisfying the decay properties $|\phi_\nu(x)| = O(|x|^{-m-1-\epsilon})$ $(x \to \infty)$ for $\nu = 0, \ldots, r-1$ and $\epsilon > 0$. Further, let $\mathcal{B}(\mathbf{\Phi})$ form a Riesz basis of V_0. Then the following assertions are equivalent:*
(a) *The finitely generated subspace V_0 provides controlled approximation order m.*
(b) *Algebraic polynomials of degree $< m$ can be exactly reproduced in V_0.*
(c) *The refinement mask $\boldsymbol{P}$ of $\mathbf{\Phi}$ satisfies the following conditions: The elements of $\boldsymbol{P}$ are $(m-1)$–times continuously differentiable functions in $L^2_{2\pi}(\mathbb{R})$, and there are vectors $\boldsymbol{y}_k \in \mathbb{R}^r$; $\boldsymbol{y}_0 \neq \boldsymbol{o}$ $(k = 0, \ldots, m-1)$ such that for $n = 0, \ldots, m-1$ we have*

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^{\mathrm{T}} \, (2i)^{k-n} \, (\mathrm{D}^{n-k} \boldsymbol{P})(0) &= 2^{-n} \, (\boldsymbol{y}_n)^{\mathrm{T}}, \\ \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_k)^{\mathrm{T}} \, (2i)^{k-n} \, (\mathrm{D}^{n-k} \boldsymbol{P})(\pi) &= \boldsymbol{o}^{\mathrm{T}}. \end{aligned} \tag{2.3}$$

(d) *There are vectors $\mathbf{x}_0, \ldots, \mathbf{x}_{m-1} \in \mathbb{R}^r$ such that $\boldsymbol{P}$ factorizes*

$$\boldsymbol{P}(u) = \frac{1}{2^m} \boldsymbol{C}_{m-1}(2u) \, \ldots \, \boldsymbol{C}_0(2u) \, \boldsymbol{S}(u) \, \boldsymbol{C}_0(u)^{-1} \, \ldots \, \boldsymbol{C}_{m-1}(u)^{-1}, \tag{2.4}$$

where the $(r \times r)$–*matrices* $\boldsymbol{C}_k$ *are defined by* $\mathbf{x}_k$ $(k = 0, \ldots, m-1)$ *as in* (2.1)–(2.2) *and* $\boldsymbol{S}(u)$ *is an* $(r \times r)$–*matrix with* $(m-1)$–*times continuously differentiable entries in* $L^2_{2\pi}(\mathbb{R})$. *In particular, for the determinant of* $\boldsymbol{P}$ *we have*

$$\det \boldsymbol{P}(u) = \left(\frac{1 + e^{-iu}}{2^r}\right)^m \det \boldsymbol{S}(u).$$

In order to prove Theorem 2.1, we extensively use the assertion that a shift–invariant closed finitely generated subspace of $L^2(\mathbb{R})$ provides controlled approximation order m if and only if the generating function vector $\boldsymbol{\Phi}$ satisfies the Strang–Fix conditions of order m, i.e., there is a finitely supported sequence of vectors $\{\mathbf{a}_l\}_{l \in \mathbf{Z}}$ such that $f := \sum_{l \in \mathbf{Z}} \mathbf{a}_l^{\mathrm{T}} \boldsymbol{\Phi}(\cdot - l)$ satisfies

$$\hat{f}(0) \neq 0; \qquad \mathrm{D}^n \hat{f}(2\pi l) = 0 \quad (l \in \mathbf{Z} \setminus \{0\};\ n = 0, \ldots, m-1).$$

This equivalence is already shown in [9]. For a complete proof of Theorem 2.1 we refer to [11,12].

Note that the known results for the principle space V_0 (cf. [2–4]) are obtained from Theorem 2.1 in the special case $r = 1$. In particular, (2.3) simplifies to

$$\begin{aligned} \mathrm{D}^n P(\pi) &= 0 \quad (n = 0, \ldots, m-1), \\ P(0) &= 1. \end{aligned}$$

Further, for $r = 1$, it follows from (d) that the refinement mask is of the form (1.2) with $S(0) = 1$, where S is a 2π–periodic $(m-1)$–times continuously differentiable function.

It can be shown that the coefficient vectors $\mathbf{y}_k \in \mathbb{R}^r$ $(k = 0, \ldots, m-1)$ occuring in Theorem 2.1 (c) satisfy the equalities

$$\sum_{l \in \mathbf{Z}} \left(\sum_{k=0}^{n} \binom{n}{k} l^{n-k} \mathbf{y}_k^{\mathrm{T}} \right) \boldsymbol{\Phi}(x - l) = x^n \quad (x \in \mathbb{R};\ n = 0, \ldots, m-1) \tag{2.5}$$

(cf. [11], Theorem 3.2).

§3 Example: B–splines with multiple knots

We are going to illustrate the results of Theorem 2.1. Let $r \in \mathbb{N}$ and $m \in \mathbb{N}_0$ be given integers. We consider equidistant knots with multiplicity r, $x_l := \lfloor l/r \rfloor$ $(l \in \mathbf{Z})$, where $\lfloor x \rfloor$ means the integer part of $x \in \mathbb{R}$. Let $N_\nu^{m,r}$ $(\nu \in \mathbf{Z})$ denote the cardinal B–splines of order m and defect r with respect to the knots $x_\nu, \ldots, x_{\nu+m}$. For $x_{\nu+m} > x_\nu$, we have the recursion formulas

$$(x_{\nu+m} - x_\nu) N_\nu^{m,r}(x) = (x - x_\nu)\, N_\nu^{m-1,r}(x) + (x_{\nu+m} - x)\, N_{\nu+1}^{m-1,r}(x)$$

and

$$(x_{\nu+m} - x_\nu)\,(\mathrm{D}N_\nu^{m,r}) = m\,(N_\nu^{m-1,r} - N_{\nu+1}^{m-1,r}). \tag{3.1}$$

For $x_\nu = x_{\nu+m} = 0$, we define $N_\nu^{m,r}$ according to the distribution theory by

$$N_\nu^{m,r} := \frac{\mathrm{D}^{r-m-1-\nu}\delta}{r-1-\nu},$$

where δ denotes the Dirac distribution. Further, note that

$$N_{\nu+lr}^{m,r} = N_\nu^{m,r}(\cdot - l) \qquad (l \in \mathbf{Z}).$$

For $m \ge r$, we have $N_\nu^{m,r} \in C^{m-r-1}(\mathbb{R})$. We put $\mathbf{N}_m^r := (N_\nu^{m,r})_{\nu=0}^{r-1}$ and $\hat{\mathbf{N}}_m^r := (\hat{N}_\nu^{m,r})_{\nu=0}^{r-1}$. In particular, we obtain

$$\hat{\mathbf{N}}_0^r(u) := \left(\frac{(iu)^{r-1}}{r-1}, \ldots, \frac{(iu)^1}{1}, 1\right)^{\mathrm{T}} \qquad (u \in \mathbb{R}).$$

The spline functions $N_\nu^{m,r}(\cdot - l)\,(l \in \mathbf{Z};\ \nu = 0, \ldots r-1)$ form a Riesz basis of the spline space

$$V_0 := \mathrm{clos}_{L_2}\,\mathrm{span}\,\{N_\nu^{m,r}(\cdot - l) : \ l \in \mathbf{Z},\ \nu = 0, \ldots, r-1\}$$

(cf. [1]). It is well–known that V_0 provides controlled approximation order m. We want to compute the vectors $\mathbf{x}_k$ and $\mathbf{y}_k$ $(k = 0, \ldots, r-1)$ occuring in Theorem 2.1. First we observe the following recursion relation:

Lemma 3.1 *Let $r \in \mathbb{N}$ be fixed. Then we have for $m \ge 1$*

$$(iu)\,\hat{\boldsymbol{N}}_m^r(u) = m\,\boldsymbol{C}_{m-1}(u)\,\hat{\boldsymbol{N}}_{m-1}^r(u) \quad (u \in \mathbb{R}) \tag{3.2}$$

with $\boldsymbol{C}_{m-1}$ defined by the vector of spline knots

$$\mathbf{x}_{m-1} := (x_m, \ldots, x_{m+r-1})^{\mathrm{T}}$$

as in (2.1)–(2.2).

Proof: Applying Fourier transform to formula (3.1), we find for $u \in \mathbb{R}$

$$(iu)\,\hat{N}_\nu^{m,r}(u) = \frac{m}{x_{\nu+m} - x_\nu}\,(\hat{N}_\nu^{m-1,r}(u) - \hat{N}_{\nu+1}^{m-1,r}(u)).$$

Thus, by $\hat{N}_r^{m-1,r}(u) = e^{-iu}\,\hat{N}_0^{m-1,r}$ the assertion follows for $m > r$. For $m = r$, the B–splines $N_\nu^{m,m}$ $(\nu = 0, \ldots, m-1)$ coincide with the Bernstein polynomials of degree $m-1$ satisfying

$$\mathrm{D}N_\nu^{m,m} = m(N_\nu^{m-1,m} - N_{\nu+1}^{m-1,m}) \quad (\nu = 0, \ldots, m-1)$$

with $N_0^{m-1,m} := \delta/m$, $N_m^{m-1,m} := \delta(\cdot - 1)/m$ and $N_\nu^{m-1,m} := N_{\nu-1}^{m-1,m-1}$ $(\nu = 1, \ldots, m-1)$. Hence, by Fourier transform, the assertion is true for $m = r$. Finally, for $m < r$, the proof follows analogously, observing that

$$\hat{\mathbf{N}}_m^r(u) = \left(\frac{(iu)^{r-m-1}}{r-1}, \ldots, \frac{(iu)^0}{m}, \hat{\mathbf{N}}_m^m(u)\right)^{\mathrm{T}}. \quad \blacksquare$$

Now the following recursion for the refinement mask $\mathbf{P}_m$ of $\mathbf{N}_m^r$ can be shown.

Theorem 3.2 *Let $r \in \mathbb{N}$ be fixed. Then for $m \geq 1$, the refinement mask $\boldsymbol{P}_m$ of $\boldsymbol{N}_m^r$ satisfies the recursion formula*

$$\boldsymbol{P}_m(u) = \frac{1}{2}\, \boldsymbol{C}_{m-1}(2u)\, \boldsymbol{P}_{m-1}(u)\, \boldsymbol{C}_{m-1}(u)^{-1}$$

with $\boldsymbol{C}_{m-1}$ defined by $\mathbf{x}_{m-1} := (x_m, \ldots, x_{m+r-1})^{\mathrm{T}}$ as in (2.1)–(2.2) *and*

$$\boldsymbol{P}_0(u) := \operatorname{diag}(2^{r-1}, \ldots, 2^0).$$

The proof of Theorem 3.2 is similar to that of Theorem 3.1 in [10]. Repeated application of Theorem 3.2 yields

$$\mathbf{P}_m(u) = \frac{1}{2^m}\, \mathbf{C}_{m-1}(2u) \ldots \mathbf{C}_0(2u)\, \mathbf{P}_0(u)\, \mathbf{C}_0(u)^{-1} \ldots \mathbf{C}_{m-1}(u)^{-1}$$

with $\mathbf{C}_k$ defined by the vector of spline knots $\mathbf{x}_k := (x_{k+1}, \ldots, x_{k+r})^{\mathrm{T}}$ $(k = 0, \ldots, m-1)$. Hence, the refinement mask factorizes in the form (2.4) with $\mathbf{S}(u) := \mathbf{P}_0(u)$.

For the computation of the coefficient vectors $\mathbf{y}_k$ $(k = 0, \ldots, m-1)$ we use the relation (2.5). Introducing the polynomial vector

$$\mathbf{Q}(u) := \left(x_{m+\nu} \prod_{\mu=1}^{m-1} (u - x_{\mu+\nu}) \right)_{\nu=0}^{r-1},$$

we have by [13], Theorem 4.21,

$$x^n = \sum_{l \in \mathbf{Z}} (\mathbf{a}_l^n)^{\mathrm{T}}\, \mathbf{N}_m^r(x - l) \quad (n = 0, \ldots, m-1)$$

with

$$\begin{aligned} \mathbf{a}_l^n &= (-1)^n \frac{n!}{(m-1)!} (\mathrm{D}^{m-n-1}\mathbf{Q})(-l) \\ &= \sum_{k=0}^{n} \binom{n}{k} l^{n-k} (-1)^k \frac{k!}{(m-1)!} (\mathrm{D}^{m-k-1}\mathbf{Q})(0). \end{aligned}$$

Hence, for the coefficient vectors in Theorem 2.1 (c) it follows that

$$\mathbf{y}_k := \mathbf{a}_0^k = (-1)^k \frac{k!}{(m-1)!} (\mathrm{D}^{m-k-1}\mathbf{Q})(0) \quad (k = 0, \ldots, m-1).$$

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Fast Direct Solvers for PDE's in Shift-Invariant Periodic Spaces

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Abstract. Using interpolation in periodic shift-invariant spaces, we give a new approach to the construction of fast direct solvers for elliptic differential equations. Further, this method provides a numerical algorithm and error estimates in a very convenient way. Numerical results with bivariate box splines are presented.

Fast direct solvers for partial differential equations have been studied for a few time, for instance in [9, 8, 13]. The increasing importance of direct solvers with respect to domain decomposition requires a more systematic investigation of their construction and of the related error estimates.

In this paper, we give a new approach to fast direct solvers on the torus. The idea described in the first section is to apply periodic interpolation by translates in order to find an approximate solution of a differential equation. This method leads to an efficient algorithm based on FFT. It generalizes the eigenfunction expansion method in [8]. Our construction immediately provides error estimates in the L_2–norm, which are given in §2. In the third section, we present numerical results for the solution of the Helmholtz equation with bivariate box splines. Compared with [6] which considers the solution of differential equations with box splines, iteration and extrapolation procedures are not needed in our method.

§1. Construction of the fast direct solver

Let $L(D_x) := \sum_{|\alpha|\le 2\mu} a_\alpha D_x^\alpha$ be a differential operator of order 2μ. Here, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{Z}_+^d$ is multiindex with $|\alpha| := \alpha_1 + \cdots + \alpha_d$. The partial derivatives $D_x^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_d}^{\alpha_d}$ with $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ are given by $D_{x_j} := -\mathrm{i}\partial/\partial x_j$ for $j = 1, \ldots, d$. The differential operator $L(D_x)$ is said to be elliptic if $L_0(\xi) := \sum_{|\alpha|=2\mu} a_\alpha \xi^\alpha \neq 0$ for all $\xi \in \mathbb{R}^d \backslash \{0\}$. The torus $\mathbf{T}^d$ is represented by the unit cube $\mathbf{T}^d := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d;$

pp. 9–16.

$|x_j| \leq 1/2,\ j = 1, \ldots, d\}$, the opposite sides of which are identified. In the usual way, the Fourier coefficients $c_k(f)$ of $f \in L_1(\mathbf{T}^d)$ are defined by $c_k(f) := \int_{\mathbf{T}^d} f(t)\, \mathrm{e}^{-2\pi \mathrm{i} kt}\, \mathrm{d}t$ for $k \in \mathbf{Z}^d$. We use $W_2^s(\mathbf{T}^d)$ with $s > 0$ as notation for the Sobolev space of order s, which is equipped with the norm $\|f\|_{s,2}^2 := \sum_{k \in \mathbf{Z}^d} \left(1 + |2\pi k|_2^2\right)^s |c_k(f)|^2$, where $|\cdot|_2$ is the Euclidian norm in $\mathbb{R}^d$. Further, the $L_2(\mathbf{T}^d)$–norm is denoted by $\|\cdot\|_2$. The following proposition can be shown analogously to [1, Theorem 2, II, §3.4].

Lemma 1.1. *The differential operator $L(D_x)$ is isomorphism from $W_2^s(\mathbf{T}^d)$ onto $W_2^{s-2\mu}(\mathbf{T}^d)$ if and only if*

(1) *L is elliptic and*
(2) *$L(2\pi k) \neq 0$ for all $k \in \mathbf{Z}^d$.*

We suppose in our further considerations that the conditions (1) and (2) of the previous lemma are satisfied for L. Now, we want to determine $u \in W_2^s(\mathbf{T}^d)$ which suffices the differential equation

$$L(D_x)\, u = f \tag{1.1}$$

for given $f \in W_2^{s-2\mu}(\mathbf{T}^d)$. In terms of Fourier coefficients this equation leads to $L(2\pi k)\, c_k(u) = c_k(f)$ for $k \in \mathbf{Z}^d$. Thus, we obtain for the exact solution

$$c_k(u) = \frac{c_k(f)}{L(2\pi k)}, \qquad k \in \mathbf{Z}^d.$$

In general, the Fourier coefficients of the given function f are not known. They have to be replaced by approximate values. Our approach consists in constructing an interpolant for f using periodic interpolation by translates. This interpolation method is well discussed for instance in [12, 5, 7]. Here, we only describe some basic concepts.

Put $B_n := \{\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{Z}^d\,;\ -n/2 < \alpha_j \leq n/2,\ j = 1, \ldots, d\}$. Furthermore, Λ is choosen to be a 1–periodic Lagrange function, *i.e.*, $\Lambda(h\alpha) = \delta_{0,\alpha}^n$ for $\alpha \in \mathbf{Z}^d$ with step size $h := 1/n$. The Kronecker data $\delta_{0,\alpha}^n$ are defined by $\delta_{0,\alpha}^n = 1$ if $\alpha \equiv 0 \pmod{n}$, and $= 0$ otherwise. The space $\mathcal{S}_\Lambda$ is given by $\mathcal{S}_\Lambda := \operatorname{span}\{\Lambda(\cdot - h\beta)\,;\ \beta \in B_n\}$. Obviously, the interpolant $\mathcal{I}(f) \in \mathcal{S}_\Lambda$ constructed by $\mathcal{I}(f) := \sum_{\beta \in B_n} f(h\beta)\, \Lambda(\cdot - h\beta)$ interpolates f with $f(h\alpha) = \mathcal{I}(f)(h\alpha)$ for $\alpha \in B_n$. Thus, we get for the corresponding Fourier coefficients

$$c_k(\mathcal{I}(f)) = c_k^n(f)\, n^d c_k(\Lambda), \qquad k \in \mathbf{Z}^d,$$

where $c_k^n(f)$ denotes the discrete Fourier coefficients of f, which are defined by $c_k^n(f) := n^{-d} \sum_{l \in B_n} f(hl)\, \mathrm{e}^{-2\pi \mathrm{i} kl/n}$ for $k \in B_n$. A useful relation between $c_k^n(f)$ and $c_k(f)$ is the aliasing formula

$$c_k^n(f) = \sum_{l \in \mathbf{Z}^d} c_{k+nl}(f), \qquad k \in \mathbf{Z}^d, \tag{1.2}$$

which applies for $\{c_k(f)\}_{k\in\mathbf{Z}^d} \in \ell_1(\mathbf{Z}^d)$. The Fourier coefficients of the Lagrange function Λ are also known as attenuation factors [12, 7].

Now, returning to our given differential equation, we replace the original problem (1.1) by

$$L(D_x)\,\bar{u} = \mathcal{I}(f)\,, \tag{1.3}$$

where $\bar{u}$ is an approximate solution we want to determine. Thus, the Fourier coefficients of $\bar{u}$ satisfy the equation

$$c_k(\bar{u}) = \frac{c_k(\mathcal{I}(f))}{L(2\pi k)}\,, \qquad k \in \mathbf{Z}^d.$$

The corresponding Fourier series

$$\bar{u} = \sum_{k\in\mathbf{Z}^d} \frac{c_k^n(f)}{L(2\pi k)}\, n^d c_k(\Lambda)\, \mathrm{e}^{2\pi \mathrm{i} k\cdot}$$

immediately leads to a numerical algorithm based on FFT–methods. Note that a similar algorithm for periodic interpolation was given in [10, 11].

Algorithm (Fast direct solver for PDE's)

Input: $n \in \mathbb{N}$, $N = \nu n$ ($\nu \in \mathbb{N}$) and $f(l/n)$ ($l \in B_n$).

1. Compute $c_k^n(f)$ for $k \in B_n$ by d–variate FFT$(n \times \cdots \times n)$.
2. Compute $\bar{u}_{l/N} := \displaystyle\sum_{k\in B_N} \frac{c_k^n(f)}{L(2\pi k)}\, n^d c_k(\Lambda)\, \mathrm{e}^{2\pi \mathrm{i} kl/N} \qquad (l \in B_N)$ by d–variate FFT$(N \times \cdots \times N)$.

Output: $\bar{u}_{l/N}$ approximate values of $\bar{u}(l/N)$ ($l \in B_N$).

In contrary to customary direct solvers, our method yields the approximate values of $\bar{u}$ *on the finer grid* $N^{-1}B_N$. The choice $N = n$ is also possible.

Remark A simple choice for the Lagrange function is the Dirichlet kernel $\Lambda := \sum_{k\in B_n} \mathrm{e}^{2\pi \mathrm{i} k\cdot}$. In this case, our algorithm leads to a form similar to a fast Poisson solver constructed in [8].

In the next chapter, we give a detailed error analysis in the L_2–norm.

§2. Error estimates for the direct solver

We are interested in an estimate of the form $\|u - \bar{u}\|_2 \le$ *const* $h^\varrho\, \|f\|_{s-2\mu,2}$. Under the conditions (1) and (2) of Lemma 1.1 the differential operator $L(D_x)$ is bounded from below in the following sense

$$|L(2\pi k)| \ge\; c_L(1 + |2\pi k|_2^2)^\mu\,, \qquad k \in \mathbf{Z}^d \tag{2.1}$$

with some positive constant c_L.

Assume that Λ satisfies the *periodic Strang–Fix condition of order m* ($m \in \mathbb{R}_+$) introduced in [3], *i.e.*, $\{c_k(\Lambda)\}_{k\in\mathbf{Z}^d} \in \ell_1(\mathbf{Z}^d)$ and for $k \in B_n$ the inequalities

$$\begin{aligned} |1 - n^d c_k(\Lambda)| &\le c_0\,|2\pi k/n|_2^m\,, \\ |n^d c_{k+nl}(\Lambda)| &\le c_l\,|2\pi k/n|_2^m\,, \qquad l \in \mathbf{Z}^d\backslash\{0\} \end{aligned} \tag{2.2}$$

are valid with $\gamma_1^2 := \sum_{l\in\mathbf{Z}^d} c_l^2 < \infty$.

Our first estimate in Theorem 2.1 concerns the case that the given function f is a trigonometric polynomial of $\mathcal{T}_n := \operatorname{span}\{e^{2\pi i k\cdot}\,;\ k \in B_n\}$.

Theorem 2.1. *Let $f \in \mathcal{T}_n$ be given. Further, $L(D_x)$ and Λ are assumed to satisfy* (2.1) *and* (2.2), *respectively. Then*

$$\|u - \bar{u}\|_2 \le c_L^{-1}\,\gamma_1\,h^m\,\|f\|_{m-2\mu,2}\,,$$

where the constants c_L, γ_1 are given by (2.1) – (2.2).

Proof: Applying Parseval's identity and (1.2), we obtain

$$\begin{aligned} \|u-\bar{u}\|_2^2 &= \sum_{k\in\mathbf{Z}^d}\left|\frac{c_k(f)}{L(2\pi k)}(1-n^d c_k(\Lambda)) - \sum_{l\in\mathbf{Z}^d\backslash\{0\}}\frac{c_{k+nl}(f)}{L(2\pi k)}\,n^d c_k(\Lambda)\right|^2 \\ &= \sum_{k\in B_n}\frac{|c_k(f)|^2}{|L(2\pi k)|^2}|1-n^d c_k(\Lambda)|^2 + \sum_{k\in B_n}|c_k(f)|^2\sum_{l\in\mathbf{Z}^d\backslash\{0\}}\frac{|n^d c_{k+nl}(\Lambda)|^2}{|L(2\pi(k+nl))|^2}. \end{aligned}$$

The last equality is valid, since $f \in \mathcal{T}_n$ implies $c_k(f) = 0$ for $k \notin B_n$. Using $|L(2\pi(k+nl))| \ge c_L(1+|2\pi k|_2^2)^\mu$ and (2.2), we conclude

$$\begin{aligned} \|u-\bar{u}\|_2^2 &\le \frac{1}{c_L^2}\sum_{l\in\mathbf{Z}^d} c_l^2\,\frac{1}{n^{2m}}\sum_{k\in B_n}\frac{|2\pi k|_2^{2m}}{(1+|2\pi k|_2^2)^{2\mu}}|c_k(f)|^2 \\ &\le \frac{\gamma_1^2}{c_L^2}\,h^{2m}\,\|f\|_{m-2\mu,2}^2. \qquad \blacksquare \end{aligned}$$

Next, we want to extend our estimate to functions $f \in W_2^{s-2\mu}(\mathbf{T}^d)$. The n–th Fourier sum of f is defined by $S_n(f) := \sum_{k\in B_n} c_k(f)\,e^{2\pi i k\cdot}$. Furthermore, we introduce the function

$$\bar{u}_n := \sum_{k\in\mathbf{Z}^d}\frac{c_k^n(S_n(f))}{L(2\pi k)}\,n^d c_k(\Lambda)\;e^{2\pi i k\cdot}\,,$$

which is the solution of $L(D_x)\,\bar{u}_n = \mathcal{I}(S_n(f))$. Applying the triangle inequality, this yields $\|u-\bar{u}\|_2 \le \|S_n(u)-\bar{u}_n\|_2 + \|u-S_n(u)\|_2 + \|\bar{u}_n-\bar{u}\|_2$. The estimate of the first term of the right side is given in Theorem 2.1. The last two expressions will be considered in the Theorems 2.2. and 2.3.

Theorem 2.2. *Let $f \in W_2^{s-2\mu}(\mathbf{T}^d)$ and let $L(D_x)$ satisfy* (2.1)*. Then*

$$\|u - S_n(u)\|_2 \le c_L^{-1}\, \pi^{-s}\, h^s\, \|f\|_{s-2\mu,2}$$

with c_L from (2.1)*.*

Proof: We conclude with Parseval's identity and (2.1)

$$\|u - S_n(u)\|_2^2 = \sum_{k\in\mathbf{Z}^d\setminus B_n} \frac{|c_k(f)|^2}{|L(2\pi k)|^2}$$

$$\le \frac{1}{c_L^2} \sum_{k\in\mathbf{Z}^d\setminus B_n} \frac{(1+|2\pi k|_2^2)^{s-2\mu}}{(1+|2\pi k|_2^2)^s}\, |c_k(f)|^2 \le \frac{1}{c_L^2\, \pi^{2s}}\, \frac{1}{n^{2s}}\, \|f\|_{s-2\mu,2}^2. \quad \blacksquare$$

The estimate of $\|\bar{u}_n - \bar{u}\|_2$ in the next theorem points out that the smoothness of the given function f is still more restrictive with respect to the error than in the previous theorem. The reason is that $\bar{u} \notin \mathcal{S}_\Lambda$.

Theorem 2.3. *Let $f \in W_2^{s-2\mu}(\mathbf{T}^d)$ with $s - 2\mu > d/2$ be given. Further, let $L(D_x)$ satisfy* (2.1)*. Then*

$$\|\bar{u}_n - \bar{u}\|_2 \le c_L^{-1}\, \gamma_2\, \gamma_3\, h^{s-2\mu}\, \|f\|_{s-2\mu,2}$$

with $\gamma_2^2 := \max_{j\in B_n} \sum_{l\in\mathbf{Z}^d} |c_{j+nl}(\Lambda)|^2$, $\gamma_3^2 := \max_{j\in B_n} \sum_{l\in\mathbf{Z}^d\setminus\{0\}} |2\pi(j/n + l)|_2^{-2(s-2\mu)}$ *and c_L from* (2.1)*.*

Proof: Using Parseval's identity and the estimate $|L(2\pi k)| \ge c_L$, we get

$$\|\bar{u}_n - \bar{u}\|_2^2 = \sum_{k\in B_n} |c_k^n(S_n(f)) - c_k^n(f)|^2 \sum_{l\in\mathbf{Z}^d} \frac{|n^d c_{k+nl}(\Lambda)|^2}{|L(2\pi(k+nl))|^2}$$

$$\le \max_{j\in B_n} \sum_{l\in\mathbf{Z}^d} |c_{j+nl}(\Lambda)|^2\, \frac{1}{c_L^2} \sum_{k\in B_n} |c_k^n(S_n(f)) - c_k^n(f)|^2.$$

Further, the last factor is estimated by means of (1.2) and the Cauchy-Schwarz inequality by

$$\sum_{k\in B_n} |c_k^n(S_n(f)) - c_k^n(f)|^2 \le \sum_{k\in B_n} \Big[\sum_{l\in\mathbf{Z}^d\setminus\{0\}} |c_{k+nl}(f)| \Big]^2$$

$$\le \sum_{k\in B_n} \Big[\sum_{l\in\mathbf{Z}^d\setminus\{0\}} (1+|2\pi(k+nl)|_2^2)^{-(s-2\mu)} \Big]$$

$$\cdot \Big[\sum_{l\in\mathbf{Z}^d\setminus\{0\}} (1+|2\pi(k+nl)|_2^2)^{s-2\mu} |c_{k+nl}(f)|^2 \Big]$$

$$\leq \frac{1}{n^{2(s-2\mu)}} \left[\max_{j \in B_n} \sum_{l \in \mathbf{Z}^d \setminus \{0\}} |2\pi(j/n + l)|_2^{-2(s-2\mu)} \right] \|f\|^2_{s-2\mu,2} \,.$$

Both results put together complete the proof of the theorem. ■

The next theorem characterizes the complete L_2–error. The proof follows immediately from the Theorems 2.1. – 2.3.

Theorem 2.4. *Let $f \in W_2^{s-2\mu}(\mathbf{T}^d)$ with $s - 2\mu > d/2$ be given. Further, let the differential operator $L(D_x)$ and the Lagrange function Λ satisfy* (2.1) *and* (2.2), *respectively. Then*

$$\|u - \bar{u}\|_2 \leq c_{m,s-2\mu}\, h^{\varrho}\, \|f\|_{s-2\mu,2}$$

with $\varrho := \min\{m, s-2\mu\}$ and

$$c_{m,s-2\mu} := \begin{cases} c_L^{-1}(\gamma_1 + \pi^{-s} + \gamma_2\gamma_3) & \text{for } s - 2\mu \geq m \\ & \text{and } s \geq m \geq s - 2\mu, \\ c_L^{-1}(\gamma_1(\sqrt{d+1}\pi)^{m-s} + \pi^{-s} + \gamma_2\gamma_3) & \text{for } m \geq s. \end{cases}$$

The constants c_L, γ_1 appear in (2.1) – (2.2), *while γ_2 and γ_3 are the same constants as in Theorem* 2.3.

Theorem 2.4. shows that the given function $f \in W_2^{s-2\mu}(\mathbf{T}^d)$ has to be smooth enough in order to get the full power m for the factor h^m. Otherwise, we can expect only the approximation order $s - 2\mu$. The numerical tests of the next section will emphasize this statement.

§3 Numerical examples

Numerical experiments have been done on the two–dimensional torus $\mathbf{T}^2$ for the Helmholtz equation $-\triangle u + u = f$. We have used bivariate 3– and 4–directional box splines for the interpolation. The Fourier coefficients of the 3–directional box spline $M_{p,q,r}$ are given by

$$n^2 c_k(M_{p,q,r}) = \left(\operatorname{sinc} \frac{\pi}{n} k_1\right)^p \left(\operatorname{sinc} \frac{\pi}{n} k_2\right)^q \left(\operatorname{sinc} \frac{\pi}{n}(k_1 + k_2)\right)^r$$

with $k = (k_1, k_2) \in \mathbf{Z}^2$. Further, we have for the Lagrange function [7]

$$n^2 c_k(\Lambda) = \frac{c_k(M_{p,q,r})}{c_k^n(M_{p,q,r})}\,, \qquad k \in \mathbf{Z}^2.$$

The 4–directional box spline $M_{p,p,q,q}$ has the Fourier coefficients

$$c_k(M_{p,p,q,q}) = c_k(M_{p,p,q}) \left(\operatorname{sinc} \frac{\pi}{n}(k_1 - k_2)\right)^q\,, \qquad k \in \mathbf{Z}^2.$$

It is known [11] that $c_k^n(M_{p,p,q,q}) = 0$ if and only if $k \in (n/2, n/2) + n\mathbf{Z}^2$. Thus, we construct the Lagrange function in a modified way as

$$n^2 c_k(\Lambda) = \begin{cases} \dfrac{c_k(M_{p,p,q,q})}{c_k^n(M_{p,p,q,q})} & \text{if } k \in \mathbf{Z}^2 \backslash [(n/2, n/2) + n\mathbf{Z}^2], \\ 1 & \text{if } k = (n/2, n/2), \\ 0 & \text{if } k \in (n/2, n/2) + n\mathbf{Z}^2 \backslash \{0\}. \end{cases}$$

We refer to [2] for more details on box splines. The discrete Fourier coefficients of $M_{2,2,2}$ and $M_{2,2,1,1}$ (see [2, 3]) are

$$n^2 c_k^n(M_{2,2,2}) = \frac{1}{2} + \frac{1}{6}\left(\cos\frac{2\pi}{n}k_1 + \cos\frac{2\pi}{n}k_2 + \cos\frac{2\pi}{n}(k_1+k_2)\right),$$

$$n^2 c_k^n(M_{2,2,1,1}) = \frac{5}{12} + \frac{1}{4}\left(\cos\frac{2\pi}{n}k_1 + \cos\frac{2\pi}{n}k_2\right) + \frac{1}{12}\cos\frac{2\pi}{n}k_1 \cos\frac{2\pi}{n}k_2\,.$$

Due to the algorithm, we have worked with $n = 64$ and $N = 128$. The figures show the error $u_j - \bar{u}_{l/128}$ ($j = 1, 2$; $l \in B_{128}$) on the coarser grid $\frac{1}{64}B_{64}$. The exact solutions were given by $u_1(x,y) = \cos(2\pi x)\cos(2\pi y)$ and $u_2(x,y) = 16(\frac{1}{4} - x^2)(\frac{1}{4} - y^2)\cos(\pi x)\cos(\pi y)$ with $(x,y) \in [-\frac{1}{2}, \frac{1}{2}]^2$.

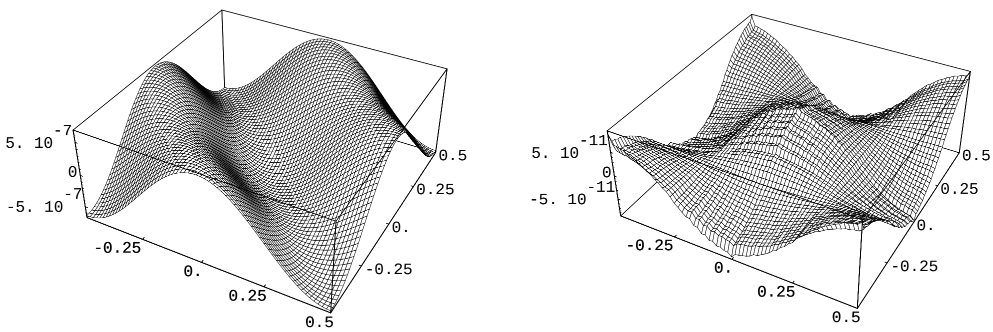

Figure 1. $u_1 - \bar{u}_{l/128}$ for $M_{2,2,2}$ (left) and $M_{2,2,1,1}$ (right).

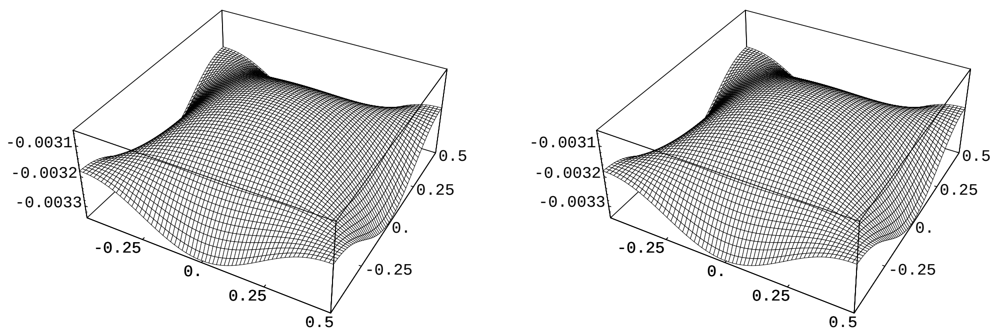

Figure 2. $u_2 - \bar{u}_{l/128}$ for $M_{2,2,2}$ (left) and $M_{2,2,1,1}$ (right).

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Interpolatory Wavelets on the Sphere

Daniel Potts and Manfred Tasche

Dedicated to Prof. E. W. Cheney on the occasion
of his 65th birthday

Abstract. In this paper, we construct an interpolatory wavelet basis on the unit sphere $S \subset \mathbb{R}^3$. Using spherical coordinates, we apply the tensor product of interpolatory trigonometric and algebraic polynomial wavelets. The described decomposition and reconstruction algorithms work in the frequency domain.

§0. Introduction

The wavelet theory is closely related to shift–invariant subspaces of $L^2(\mathbb{R}^d)$. Therefore, wavelets are naturally adapted to problems on the whole space $\mathbb{R}^d$. Very often, one has to deal with functions defined on a bounded domain such that the construction of wavelets on a domain is desirable. This problem is solved for the torus $\mathbf{T} := \mathbb{R}/2\pi\mathbf{Z}$ [5], and for compact intervals [1, 4]. In the sequel, we will apply 2π–periodic trigonometric polynomial wavelets [6] and algebraic polynomial wavelets on $I := [-1, 1]$ [3, 8] in order to construct interpolatory wavelets on the unit sphere $S \subset \mathbb{R}^3$. Other approachs to wavelets on the sphere can be found in [2, 7].

Using modified spherical coordinates $x := (x_1, x_2) \in H := \mathbf{T} \times I$, we can identify the sphere S with H by the mapping $\chi : H \to S$ with $(z_1, z_2, z_3) = \chi(x) \in S$,

$$z_1 := \cos x_1 \cos \frac{\pi x_2}{2}, \quad z_2 := \sin x_1 \cos \frac{\pi x_2}{2}, \quad z_3 := \sin \frac{\pi x_2}{2},$$

where $\mathbf{T} \times \{\pm 1\}$ corresponds to the north/south pole of S. Let $F : S \to \mathbb{R}$ be given and let $f := F \circ \chi : H \to \mathbb{R}$ be its coordinate representation.

pp. 17–24.

Then $F \in C^1(S)$ if and only if $f \in C^1(H)$ fulfils the following conditions (see [2])

$$f(x_1, \pm 1) = c_\pm, \tag{0.1}$$
$$(\partial_{x_2} f)(x_1, \pm 1) = a_\pm \cos x_1 + b_\pm \sin x_1, \tag{0.2}$$
$$(\partial_{x_1} \partial_{x_2} f)(x_1, \pm 1) = -a_\pm \sin x_1 + b_\pm \cos x_1, \tag{0.3}$$

where the mixed derivative exists for all $x_1 \in \mathbf{T}$. Here $a_\pm$, $b_\pm$, $c_\pm$ denote some constants. Note that (0.1) corresponds to the continuity of F at both poles of S. The conditions (0.2) – (0.3) assure that the tangent plane of F varies continuously at the poles of S.

The aim of this paper is to construct an almost smooth wavelet decomposition of functions defined on the unit sphere S. Roughly spoken, a function F is almost smooth on S, if $f := F \circ \chi \in C^1(H)$ with (0.1) satisfies the conditions (0.2) – (0.3) approximately.

This paper is organized as follows. In Sections 1 and 2, we briefly recall the construction of trigonometric polynomial wavelets on $\mathbf{T}$ and of algebraic polynomial wavelets on I. Using tensor product approach, we introduce a multiresolution of the weighted Hilbert space $L^2_w(H)$ in Section 3. Finally, we describe decomposition and reconstruction algorithms.

§1. Trigonometric polynomial wavelets on T

In the following, we sketch the construction of interpolatory trigonometric polynomial wavelets on $\mathbf{T}$. For $f \in L^2(\mathbf{T})$, let

$$c^1_u(f) := \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-\mathrm{i}ut}\, \mathrm{d}t \quad (u \in \mathbf{Z}).$$

With the Dirichlet kernel

$$D^1_l := \frac{1}{2} + \sum_{k=1}^{l} \cos(k\cdot) \quad (l \in \mathbb{N})$$

and $M_j := 2^{j+3}$ $(j \in \mathbb{N}_0)$, we consider the de la Vallée Poussin kernel as scaling function of level j

$$M_j\, \varphi^1_j := \frac{1}{2^j} \sum_{l=3\cdot 2^j}^{5\cdot 2^j - 1} D^1_l .$$

Hence we have $c^1_u(\varphi^1_j) = c^1_{-u}(\varphi^1_j)$ $(u \in \mathbf{Z})$ and for $u \in \mathbb{N}_0$

$$M_j\, c^1_u(\varphi^1_j) = \begin{cases} 1 & u = 0, \ldots, 3\cdot 2^j, \\ 2^{-1}(5 - 2^{-j}u) & u = 3\cdot 2^j + 1, \ldots, 5\cdot 2^j - 1, \\ 0 & u = 5\cdot 2^j, 5\cdot 2^j + 1, \ldots . \end{cases}$$

Note that the transformed two–scale relation of φ_j^1 reads as follows

$$c_u^1(\varphi_j^1) = A_{j+1,u}^1\, c_u^1(\varphi_{j+1}^1) \quad (u \in \mathbf{Z}) \tag{1.1}$$

with $A_{j+1,u}^1 = A_{j+1,-u}^1 = A_{j+1,u+M_{j+1}}$ $(u \in \mathbf{Z})$ and

$$A_{j+1,u}^1 := \begin{cases} 2 & u = 0,\dots,3\cdot 2^j, \\ 5 - 2^{-j}u & u = 3\cdot 2^j + 1,\dots,5\cdot 2^j - 1, \\ 0 & u = 5\cdot 2^j,\dots,M_j. \end{cases}$$

With $h_j^1 := 2\pi/M_j$ we consider the shift operators $\sigma_{j,k}^1 : L^2(\mathbf{T}) \to L^2(\mathbf{T})$ defined by $\sigma_{j,k}^1 f := f(\cdot - kh_j^1)$ $(k \in I_j^1)$, where I_j^1 denotes the index set $\{0,\dots,M_j - 1\}$. Introduce the shifted scaling functions $\varphi_{j,k}^1 := \sigma_{j,k}^1\, \varphi_j^1$. Then $\varphi_{j,k}^1$ are Lagrange fundamental functions with respect to the equidistant grid $G_j^1 := h_j^1\, I_j^1$, since we have $\varphi_{j,k}^1(lh_j^1) = \delta_{k,l}$ $(k,l \in I_j^1)$.

As sample space of level j we use $V_j^1 := \operatorname{span}\{\varphi_{j,k}^1 : k \in I_j^1\}$. Obviously, the operator $L_j^1 : C(\mathbf{T}) \to V_j^1$ defined by

$$L_j^1 f := \sum_{k \in I_j^1} f(kh_j^1)\, \varphi_{j,k}^1 \quad (f \in C(\mathbf{T}))$$

is the interpolation projector onto V_j^1 with respect to the grid G_j^1. Then $\{V_j^1\}_{j=0}^\infty$ forms a multiresolution of $L^2(\mathbf{T})$ (see [5, 6]).

Setting $\psi_j^1 := (2\varphi_{j+1}^1 - \varphi_j^1)(\cdot - h_{j+1}^1)$, then we obtain $c_u^1(\psi_j^1) = \overline{c_{-u}^1(\psi_j^1)}$ $(u \in \mathbf{Z})$ and for $u \in \mathbb{N}_0$

$$M_{j+1}\, c_u^1(\psi_j^1) = \begin{cases} (2^{-j}u - 3)\, \omega_{j+1}^u & u = 3\cdot 2^j + 1,\dots,5\cdot 2^j - 1, \\ 2\, \omega_{j+1}^u & u = 5\cdot 2^j,\dots,3\cdot 2^{j+1}, \\ (5 - 2^{-j-1}u)\, \omega_{j+1}^u & u = 3\cdot 2^{j+1} + 1,\dots,5\cdot 2^{j+1} - 1, \\ 0 & \text{otherwise} \end{cases}$$

with $\omega_j := \exp(-2\pi \mathrm{i}/M_j)$. Hence the transformed two-scale relation of ψ_j^1 reads as follows

$$c_u^1(\psi_j^1) = B_{j+1,u}^1\, c_u^1(\varphi_{j+1}^1) \quad (u \in \mathbf{Z}) \tag{1.2}$$

with $B_{j+1,u}^1 = \overline{B_{j+1,-u}^1} = B_{j+1,u+M_{j+1}}$ and

$$B_{j+1,u}^1 := \begin{cases} 0 & u = 0,\dots,3\cdot 2^j, \\ (2^{-j}u - 3)\, \omega_{j+1}^u & u = 3\cdot 2^j + 1,\dots,5\cdot 2^j - 1, \\ 2\, \omega_{j+1}^u & u = 5\cdot 2^j,\dots,M_j. \end{cases}$$

Let $W_j^1 := V_{j+1}^1 \ominus V_j^1$ be the wavelet space of level j. Then we obtain that $W_j^1 = \operatorname{span}\{\psi_{j,k}^1 := \psi_j^1(\cdot - kh_j^1) : k \in I_j^1\}$.

§2. Algebraic polynomial wavelets on I

Now we briefly describe interpolatory polynomial wavelets on the interval I. Let $w(x) := (1-x^2)^{-1/2}$ ($x \in (-1,\,1)$). For $f \in L^2_w(I)$, let

$$c^2_n(f) := \frac{2}{\pi} \int_I w(x)\, f(x)\, T_n(x)\,\mathrm{d}x \quad (n \in \mathbb{N}_0),$$

where T_n denotes the n-th Chebyshev polynomial. With the Chebyshev–Dirichlet kernel

$$D^2_l := \frac{1}{2} + \sum_{k=1}^{l} T_k \quad (l \in \mathbb{N})$$

and $N_j := 2^{j+2}$ ($j \in \mathbb{N}_0$), we consider the corresponding de la Vallée Poussin mean as scaling function φ^2_j of level j:

$$N_j\, \varphi^2_j := \frac{1}{2^j} \sum_{l=3\cdot 2^j}^{5\cdot 2^j - 1} D^2_l .$$

Hence we have

$$N_j\, c^2_n(\varphi^2_j) = \begin{cases} 2 & n = 0, \ldots, 3\cdot 2^j, \\ 5 - 2^{-j} n & n = 3\cdot 2^j + 1, \ldots, 5\cdot 2^j - 1, \\ 0 & n = 5\cdot 2^j, 5\cdot 2^j + 1, \ldots . \end{cases}$$

The transformed two–scale relation of φ^2_j reads as follows

$$c^2_n(\varphi^2_j) = A^2_{j+1,n}\, c^2_n(\varphi^2_{j+1}) \quad (n \in \mathbb{N}_0) \tag{2.1}$$

with $A^2_{j+1,n} = A^2_{j+1,n+N_{j+2}}$, $A^2_{j+1,N_{j+2}-l} = A^2_{j+1,l}$ ($l = 0, \ldots, N_{j+1}$) and

$$A^2_{j+1,n} := \begin{cases} 2 & n = 0, \ldots, 3\cdot 2^j, \\ 5 - 2^{-j} n & n = 3\cdot 2^j + 1, \ldots, 5\cdot 2^j - 1, \\ 0 & n = 5\cdot 2^j, \ldots, N_{j+1}. \end{cases}$$

Setting $h^2_{j,v} := \cos \frac{v\pi}{N_j}$ ($v \in \mathbf{Z}$), we introduce the Chebyshev–shift operators $\sigma^2_{j,v} : L^2_w(I) \to L^2_w(I)$ by $c^2_n(\sigma^2_{j,v} f) = h^2_{j,nv}\, c^2_n(f)$ ($f \in L^2_w(I)$). Put $\varphi^2_{j,v} := \sigma^2_{j,v}\, \varphi^2_j$ ($v \in \mathbf{Z}$). Then $\varphi^2_{j,n} \in \Pi_{5\cdot 2^j - 1}$ fulfils the interpolation properties $\varphi^2_{j,n}(h^2_{j,l}) = \varepsilon^{-1}_{j,l}\, \delta_{n,l}$ with $\varepsilon_{j,0} = \varepsilon_{j,N_j} := 1/2$, $\varepsilon_{j,l} := 1$ ($l = 1, \ldots, N_j - 1$). Hence $\varphi^2_{j,n}$ are modified Lagrange fundamental polynomials related to the Chebyshev grid $G^2_j := \{h^2_{j,k} :\, k \in I^2_j\}$ with $I^2_j := \{0, \ldots, N_j\}$.

As sample space of level j ($j \in \mathbb{N}_0$) we use $V_j^2 := \operatorname{span}\{\varphi_{j,k}^2 : k \in I_j^2\}$. Obviously, the operator $L_j^2 : C(I) \to V_j^2$ defined by

$$L_j^2 f := \sum_{k \in I_j^2} \varepsilon_{j,k}\, f(h_{j,k}^2)\, \varphi_{j,k}^2 \quad (f \in C(I))$$

is the interpolation projector onto V_j^2 with respect to the grid G_j^2. Then $\{V_j^2\}_{j=0}^\infty$ forms a multiresolution of $L_w^2(I)$ (see [3, 4, 8]).

Introducing $\psi_{j+1}^2 \in V_{j+1}^2$ by

$$N_j\, c_n^2(\psi_{j+1}^2) := \begin{cases} 2^{-j}n - 3 & n = 3 \cdot 2^j + 1, \ldots, 5 \cdot 2^j - 1, \\ 2 & n = 5 \cdot 2^j, \ldots, 3 \cdot 2^{j+1}, \\ 5 - 2^{-j-1}n & n = 3 \cdot 2^{j+1} + 1, \ldots, 5 \cdot 2^{j+1} - 1, \\ 0 & \text{otherwise}, \end{cases}$$

then we obtain the transformed two–scale relation of ψ_{j+1}^2

$$c_n^2(\psi_{j+1}^2) = B_{j+1,n}^2\, c_n^2(\varphi_{j+1}^2) \quad (n \in \mathbb{N}_0) \tag{2.2}$$

with $B_{j+1,n}^2 = B_{j+1,n+N_{j+2}}^2$ ($n \in \mathbb{N}_0$), $B_{j+1,N_{j+2}-l}^2 = B_{j+1,l}^2$ ($l \in I_{j+1}^2$) and

$$B_{j+1,n}^2 := \begin{cases} 0 & n = 0, \ldots, 3 \cdot 2^j, \\ 2^{-j}n - 3 & n = 3 \cdot 2^j + 1, \ldots, 5 \cdot 2^j - 1, \\ 2 & n = 5 \cdot 2^j, \ldots, N_{j+1}. \end{cases}$$

Let $W_j^2 := V_{j+1}^2 \ominus V_j^2$ be the wavelet space of level j. Then we have $W_j^2 = \operatorname{span}\{\psi_{j+1,2k+1}^2 := \sigma_{j+1,2k+1}^2 \psi_{j+1}^2 : k \in K_j^2\}$ with $K_j^2 := \{0, \ldots, N_j - 1\}$.

§3. Multiresolution of $L_w^2(H)$

Let $L_w^2(H)$ be the Hilbert space of all functions $f : H \to \mathbb{R}$ with

$$2\pi^2 \|f\|^2 := \int_D w(x_2)\, |f(x)|^2 \, \mathrm{d}x < \infty$$

with $D := [0, 2\pi] \times I$ and $x := (x_1, x_2)$. Applying the tensor product method, we introduce a multiresolution of $L_w^2(H)$. For $f \in L_w^2(H)$, let

$$\pi^2\, c_k(f) := \int_D w(x_2)\, f(x)\, e^{-\mathrm{i}k_1 x_1}\, T_{k_2}(x_2)\, \mathrm{d}x$$

with $k := (k_1, k_2) \in \mathbf{Z} \times \mathbb{N}_0$. As *sample space of level* j we use $V_j := V_j^1 \otimes V_j^2$. Hence we have $V_j = \operatorname{span}\{\varphi_{j,k} : k \in I_j\}$ with $I_j := I_j^1 \times I_j^2$ and

$\varphi_{j,k} := \varphi^1_{j,k_1} \otimes \varphi^2_{j,k_2}$. By (1.1) and (2.1), the transformed two–scale relation of the j-th scaling function $\varphi_j := \varphi^1_j \otimes \varphi^2_j$ reads as follows

$$c_k(\varphi_j) = A_{j+1,k}\, c_k(\varphi_{j+1}) \quad (k \in \mathbf{Z} \times \mathbb{N}_0) \tag{3.1}$$

with $A_{j+1,k} := A^1_{j+1,k_1} A^2_{j+1,k_2}$. Since we have $\epsilon_{j,l_2}\, \varphi_{j,k}(h_{j,l}) = \delta_{k_1,l_1}\, \delta_{k_2,l_2}$ $(k, l \in I_j)$ for $h_{j,l} := (l_1 h^1_j, h^2_{j,l_2})$, the shifted scaling functions $\varphi_{j,k}$ are modified Lagrange functions with respect to the grid $G_j := G^1_j \times G^2_j$. Therefore, the operator $L_j := L^1_j \otimes L^2_j : C(H) \to V_j$ given by

$$L_j f := \sum_{k \in I_j} \varepsilon_{j,k_2}\, f(h_{j,k})\, \varphi_{j,k} \quad (f \in C(H))$$

is the interpolation projector onto V_j with respect to the grid G_j. The points of $\chi(G_j)$ are lying more dense at the poles of S. If $f \in C^1(H)$ with (0.1) – (0.3) is given, then $L_j f$ fulfils (0.1) exactly and (0.2) – (0.3) approximately in the following sense that the partial derivatives in (0.2) – (0.3) must be replaced by corresponding difference quotients on G_j. From $L_j f | G_j = f | G_j$ it follows that $L_j f$ is almost smooth. Then $\{V_j\}_{j=0}^{\infty}$ forms a multiresolution of $L^2_w(H)$ in the following sense:

(M 1) $\quad V_j \subset V_{j+1}, \quad \operatorname{clos}(\bigcup_{j=0}^{\infty} V_j) = L^2_w(H)$.

(M 2) For all $j \in \mathbb{N}_0$ and for any $\alpha_{j,k} \in \mathbb{R}$ $(k \in I_j)$, we have

$$\frac{1}{4} \sum_{k \in I_j} \varepsilon_{j,k_2}\, |\alpha_{j,k}|^2 \le M_j N_j \,\|\sum_{k \in I_j} \varepsilon_{j,k_2}\, \alpha_{j,k}\, \varphi_{j,k}\|^2 \le \sum_{k \in I_j} \varepsilon_{j,k_2}\, |\alpha_{j,k}|^2 .$$

As known, the orthogonal complement W_j of V_j in V_{j+1} is generated by V^1_j, V^2_j, W^1_j and W^2_j. We obtain that $W_j = W^{\mathrm{I}}_j \oplus W^{\mathrm{II}}_j \oplus W^{\mathrm{III}}_j$ with the *wavelet spaces* $W^{\mathrm{I}}_j := V^1_j \otimes W^2_j$, $W^{\mathrm{II}}_j := W^1_j \otimes V^2_j$ and $W^{\mathrm{III}}_j := W^1_j \otimes W^2_j$. Then we have

$$W^{\mathrm{I,\,III}}_j = \operatorname{span}\{\psi^{\mathrm{I,\,III}}_{j,k} : k \in K_j\}, \quad W^{\mathrm{II}}_j = \operatorname{span}\{\psi^{\mathrm{II}}_{j,k} : k \in I_j\}$$

with $K_j := I^1_j \times K^2_j$ and with $\psi^{\mathrm{I}}_{j,k} := \varphi^1_{j,k_1} \otimes \psi^2_{j+1,2k_2+1}$, $\psi^{\mathrm{II}}_{j,k} := \psi^1_{j,k_1} \otimes \varphi^2_{j,k_2}$ and $\psi^{\mathrm{III}}_{j,k} := \psi^1_{j,k_1} \otimes \psi^2_{j+1,2k_2+1}$. By (1.1) – (1.2) and (2.1) – (2.2), the transformed two–scale relations of the wavelets $\psi^{\mathrm{I}}_j := \varphi^1_j \otimes \psi^2_{j+1}$, $\psi^{\mathrm{II}}_j := \psi^1_j \otimes \varphi^2_j$ and $\psi^{\mathrm{III}}_j := \psi^1_j \otimes \psi^2_{j+1}$ read as follows

$$c_l(\psi^{\mathrm{I-III}}_j) = B^{\mathrm{I-III}}_{j+1,l}\, c_l(\varphi_{j+1}) \quad (l = (l_1, l_2) \in \mathbf{Z} \times \mathbb{N}_0) \tag{3.2}$$

with $B^{\mathrm{I}}_{j+1,l} := A^1_{j+1,l_1}\, B^2_{j+1,l_2}$, $B^{\mathrm{II}}_{j+1,l} := B^1_{j+1,l_1}\, A^2_{j+1,l_2}$ and $B^{\mathrm{III}}_{j+1,l} := B^1_{j+1,l_1}\, B^2_{j+1,l_2}$.

§4. Decomposition and reconstruction algorithms

Now we derive efficient decomposition and reconstruction algorithms. In order to decompose a given function $f_{j+1} \in V_{j+1}$ of the form

$$f_{j+1} = \sum_{k \in I_{j+1}} \varepsilon_{j+1,k_2} \, \alpha_{j+1,k} \, \varphi_{j+1,k} \, , \tag{4.1}$$

uniquely determined functions $f_j \in V_j$ and $g_j^{\mathrm{I-III}} \in W_j^{\mathrm{I-III}}$ have to be found such that $f_{j+1} = f_j + g_j^{\mathrm{I}} + g_j^{\mathrm{II}} + g_j^{\mathrm{III}}$. Assume that the coefficients $\alpha_{j+1,k} \in \mathbb{R}$ of f_{j+1} or their transformed data

$$\hat{\alpha}_{j+1,l} := \sum_{k \in I_{j+1}} \varepsilon_{j+1,k_2} \, \alpha_{j+1,k} \, \omega_{j+1}^{k_1 l_1} \cos \frac{k_2 l_2 \pi}{N_{j+1}} \tag{4.2}$$

are known. The wanted functions $f_j \in V_j$ and $g_j^{\mathrm{I-III}} \in W_j^{\mathrm{I-III}}$ can be uniquely represented in the form (4.1) and

$$g_j^{\mathrm{I,III}} = \sum_{k \in K_j} \beta_{j,k}^{\mathrm{I,III}} \, \psi_{j+1,k}^{\mathrm{I,III}} \, , \quad g_j^{\mathrm{II}} = \sum_{k \in I_j} \varepsilon_{j,k_2} \, \beta_{j,k}^{\mathrm{II}} \, \psi_{j+1,k}^{\mathrm{II}}$$

with unknown coefficients $\alpha_{j,k}$, $\beta_{j,k}^{\mathrm{I-III}} \in \mathbb{R}$. Let $\hat{\alpha}_{j,l}$, $\hat{\beta}_{j,l}^{\mathrm{I-III}} \in \mathbb{C}$ be transformed data given in the form (4.2) or by

$$\hat{\beta}_{j,l}^{\mathrm{I,III}} := \sum_{k \in I_j} \beta_{j,k}^{\mathrm{I,III}} \, \omega_j^{k_1 l_1} \cos \frac{(2k_2+1) l_2 \pi}{N_{j+1}}, \tag{4.3}$$

$$\hat{\beta}_{j,l}^{\mathrm{II}} := \sum_{k \in K_j} \varepsilon_{j,k_2} \, \beta_{j,k}^{\mathrm{II}} \, \omega_j^{k_1 l_1} \cos \frac{k_2 l_2 \pi}{N_j}. \tag{4.4}$$

Note that the discrete transforms (4.2) – (4.4) and their inverses can be realized by the row–column method using fast Fourier transform and fast algorithms of discrete cosine transforms.

In order to reconstruct $f_{j+1} \in V_{j+1}$ ($j \in \mathbb{N}_0$), we have to compute the sum $f_{j+1} = f_j + g_j^{\mathrm{I}} + g_j^{\mathrm{II}} + g_j^{\mathrm{III}}$ with given functions $f_j \in V_j$ and $g_j^{\mathrm{I-III}} \in W_j^{\mathrm{I-III}}$. Assume that $\alpha_{j,k}$, $\beta_{j,k}^{\mathrm{I-III}} \in \mathbb{R}$ or their transformed data are known. Then $f_{j+1} \in V_{j+1}$ can be uniquely represented in the form (4.1).

The decomposition and reconstruction algorithms are based on the following connection between $\hat{\alpha}_{j+1,l}$ and $\hat{\alpha}_{j,l}$, $\hat{\beta}_{j,l}^{\mathrm{I-III}}$. Introducing for $l = (l_1, l_2) \in I_j$ the two-scale symbol matrices

$$S_{j+1,l_1}^1 := \begin{pmatrix} A_{j+1,l_1}^1 & B_{j+1,l_1}^1 \\ A_{j+1,M_j+l_1}^1 & B_{j+1,M_j+l_1}^1 \end{pmatrix},$$

$$S_{j+1,l_2}^2 := \begin{pmatrix} A_{j+1,l_2}^2 & B_{j+1,l_2}^2 \\ A_{j+1,N_{j+1}-l_2}^2 & -B_{j+1,N_{j+1}-l_2}^2 \end{pmatrix}$$

and the Kronecker product $S_{j+1,l} := S^1_{j+1,l_1} \otimes S^2_{j+1,l_2}$, then we obtain by (3.1) – (3.2) that for $l \in K_j$

$$\begin{pmatrix} \hat{\alpha}_{j+1,l_1,l_2} \\ \hat{\alpha}_{j+1,l_1,N_{j+1}-l_2} \\ \hat{\alpha}_{j+1,M_j+l_1,l_2} \\ \hat{\alpha}_{j+1,M_j+l_1,N_{j+1}-l_2} \end{pmatrix} = S_{j+1,l} \begin{pmatrix} \hat{\alpha}_{j,l} \\ \hat{\beta}^{\mathrm{I}}_{j,l} \\ \hat{\beta}^{\mathrm{II}}_{j,l} \\ \hat{\beta}^{\mathrm{III}}_{j,l} \end{pmatrix}$$

and for $l_1 \in I^1_j$

$$\begin{pmatrix} \hat{\alpha}_{j+1,l_1,N_j} \\ \hat{\alpha}_{j+1,M_j+l_1,N_j} \end{pmatrix} = S^1_{j+1,l_1} \begin{pmatrix} \hat{\alpha}_{j,l_1,N_j} \\ \hat{\beta}^{\mathrm{II}}_{j,l_1,N_j} \end{pmatrix}.$$

Note that $S_{j+1,l}$ $(l \in K_j)$ and S^1_{j+1,l_1} $(l_1 \in I^1_j)$ are regular matrices.

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Time Frequency Localization of Trigonometric Hermite Operators

Jürgen Prestin and Ewald Quak

Abstract. In this paper, localization properties of trigonometric polynomial Hermite operators are discussed. In particular, time frequency uncertainty and operator norms are compared for the different types of fundamental interpolants which serve as scaling functions for a trigonometric multiresolution analysis.

§1. Introduction

Recently, several different approaches to periodic multiresolution analyses have been presented. For example, periodic scaling functions and wavelets are discussed by Narcowich and Ward [3] who investigate their time frequency behaviour in terms of an uncertainty principle for periodic functions due to Breitenberger [1]. The periodic basis functions in [3] possess an uncertainty product of $\mathcal{O}(\sqrt{n})$ for increasing dimension n of the corresponding spaces. On the other hand, uniformly bounded uncertainty products are computed by Selig [6] for trigonometric fundamental Lagrange interpolants based on de la Vallée Poussin means. A multiresolution analysis generated by two different types of scaling functions and wavelets, namely fundamental trigonometric Hermite interpolants, is investigated in [5]. One of these scaling functions is a Fejér kernel, while the other is a conjugate Dirichlet kernel. In this paper, we explain the different behaviour of these two types of functions with respect to their localization properties. This includes the computation of the time frequency uncertainty as well as the operator norms. Let us remark here that similar differences also occur for the de la Vallée Poussin and Fourier type Lagrange interpolants which are described in [4].

pp. 25–32.

Let $L^p_{2\pi}$ denote the space of real-valued p-integrable 2π-periodic functions for $1 \le p < \infty$, and $C_{2\pi}$ the space of real-valued continuous functions, with their respective norms $\|f\|_p = \frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p\,dx^{1/p}$ and $\|f\|_C = \max_x |f(x)|$. In particular, the $L^2_{2\pi}$-norm is induced by the inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx$ for $f, g \in L^2_{2\pi}$.

The following functions are well-known and have been studied in detail, *e.g.*, in [8]. For any $n \in \mathbb{N}$, consider two different kinds of kernels

$$\phi^0_{n,0}(x) := \frac{1}{2n} + \sum_{k=1}^{2n-1} \frac{2n-k}{2n^2}\cos kx = \begin{cases} \frac{1}{4n^2}\frac{\sin^2(nx)}{\sin^2(\frac{x}{2})} & \text{for } x \notin 2\pi\mathbf{Z}, \\ 1 & \text{for } x \in 2\pi\mathbf{Z}, \end{cases}$$

and

$$\phi^1_{n,0}(x) := \frac{1}{2n^2}\left(\sum_{k=1}^{2n-1}\sin kx + \frac{1}{2}\sin 2nx\right) = \phi^0_{n,0}(x)\sin x.$$

For $k = 0, \ldots, 2n-1$, define equally spaced translates by $\phi^0_{n,k}(x) := \phi^0_{n,0}(x - k\pi/n)$ and, analogously, $\phi^1_{n,k}(x) := \phi^1_{n,0}(x - k\pi/n)$.

These functions are fundamental Hermite interpolants, *i.e.*, for each $k, s = 0, \ldots, 2n-1$, it holds that

$$\phi^0_{n,k}(\frac{s\pi}{n}) = \delta_{k,s} \quad \text{and} \quad {\phi^0_{n,k}}'(\frac{s\pi}{n}) = 0$$

and

$$\phi^1_{n,k}(\frac{s\pi}{n}) = 0 \quad \text{and} \quad {\phi^1_{n,k}}'(\frac{s\pi}{n}) = \delta_{k,s}.$$

In particular, $\phi^0_{n,k}$ and $\phi^1_{n,k}$ were used as scaling functions in the Hermite multiresolution analysis studied in [5].

For $n \in \mathbb{N}$, the spaces V^r_n are defined for $r = 0, 1$ as

$$V^r_n := \operatorname{span}\{\phi^r_{n,k} : k = 0, \ldots, 2n-1\}.$$

It is well known (see [5,8]) that

$$V_n = V^0_n + V^1_n = \operatorname{span}\{\cos rx, \sin(r+1)x : r = 0, \ldots, 2n-1\}.$$

Thus $V_n \subset V_{n+1}$ and the spaces V_n form a polynomial multiresolution analysis of $L^2_{2\pi}$.

As shown in [5], the corresponding wavelet spaces can be described as

$$W_n = W^0_n + W^1_n,$$

where for $r = 0, 1$

$$W_n^r := \operatorname{span}\{\psi_{n,k}^r : k = 0, \dots, 2n-1\},$$

$$\psi_{n,0}^0(x) = \frac{1}{2n}\cos 2nx + \frac{1}{6n^2}\sum_{\ell=2n+1}^{4n-1}(6n-\ell)\cos \ell x$$

and

$$\psi_{n,0}^1(x) = \frac{1}{6n^2}\sum_{\ell=2n+1}^{4n-1}\sin \ell x + \frac{1}{8n^2}\sin 4nx.$$

As for the scaling functions, let $\psi_{n,k}^0(x) := \psi_{n,0}^0(x - k\pi/n)$ and $\psi_{n,k}^1(x) := \psi_{n,0}^1(x - k\pi/n)$ for $k = 0, \dots, 2n-1$.

These wavelets show the same interpolatory properties as the scaling functions, namely for each $k, s = 0, \dots, 2n-1$:

$$\psi_{n,k}^0(\frac{s\pi}{n}) = \delta_{k,s} \quad \text{and} \quad {\psi_{n,k}^0}'(\frac{s\pi}{n}) = 0$$

and

$$\psi_{n,k}^1(\frac{s\pi}{n}) = 0 \quad \text{and} \quad {\psi_{n,k}^1}'(\frac{s\pi}{n}) = \delta_{k,s}.$$

The following computations are restricted to the scaling functions, but similar investigations yield that both types of wavelet functions behave like $\phi_{n,k}^1$. Let us only briefly mention at this point that by giving up the interpolatory properties, one can find another splitting of the space W_n into two wavelet spaces such that one of the two wavelets possesses the same localization properties as $\phi_{n,k}^0$.

§2. Localization by interpolation

For any $n \in \mathbb{N}$, a Hermite type interpolation operator can be defined, mapping any real-valued differentiable 2π-periodic function f into the space V_n, namely $L_n := L_n^0 + L_n^1$, where for $r = 0, 1$,

$$L_n^r f(x) = \sum_{k=0}^{2n-1} f^{(r)}(\frac{k\pi}{n})\phi_{n,k}^r(x).$$

Hence, each L_n^r is a projection onto V_n^r due to the interpolatory properties.

Note that the decomposition $V_n = V_n^0 + V_n^1$ is direct, but not orthogonal, as can be seen from the following result.

Lemma 2.1. *For $n \in \mathbb{N}$, it holds that*

$$\begin{aligned} V_n^0 = \operatorname{span}\{&(n+s)\cos(n-s)x + (n-s)\cos(n+s)x : \quad s = 0,\dots,n, \\ &(n+s)\sin(n-s)x - (n-s)\sin(n+s)x : \quad s = 1,\dots,n-1\} \end{aligned}$$

and

$$\begin{aligned} V_n^1 = \operatorname{span}\{&\cos(n-s)x - \cos(n+s)x : \quad s = 1,\dots,n-1, \\ &\sin(n-s)x + \sin(n+s)x : \quad s = 0,\dots,n\}. \end{aligned}$$

Proof: To show the first statement, it is sufficient to establish that V_n^0 is in the kernel of L_n^1. Consider

$$\begin{aligned} &L_n^1((n+s)\cos(n-s)\cdot\ + (n-s)\cos(n+s)\cdot\ ) \\ &= \sum_{k=0}^{2n-1} (s^2 - n^2)\left(\sin(n-s)\frac{k\pi}{n} + \sin(n+s)\frac{k\pi}{n}\right)\phi_{n,k}^1 \\ &= \sum_{k=0}^{2n-1} (2s^2 - 2n^2)\sin k\pi\ \cos\frac{sk\pi}{n}\ \phi_{n,k}^1 = 0. \end{aligned}$$

The rest follows analogously. ■

A further possibility to describe localization of polynomials is related to a Nikolskii type inequality which can be stated as

$$1 \le \frac{\|p_n\|_p}{\|p_n\|_q} \le C n^{\frac{1}{q}-\frac{1}{p}}$$

for all trigonometric polynomials of degree at most n and $1 \le q \le p \le \infty$ (see [7, Ch.4.9.2]). To obtain equality on the right-hand side can also be regarded as a measure of time localization.

Lemma 2.2. *For $n \in \mathbb{N}$, the norms of the fundamental interpolants $\phi_{n,k}^r$, $r = 0,1$, $k = 0,\dots,2n-1$, $1 < p \le \infty$, are as follows:*

$$\|\phi_{n,k}^r\|_p \sim n^{-\frac{1}{p}}.$$

Moreover,

$$\|\phi_{n,k}^0\|_1 = \frac{1}{2n}, \quad \|\phi_{n,k}^0\|_2 = \frac{1}{2n}\sqrt{\frac{8n^2+1}{6n}}, \quad \|\phi_{n,k}^0\|_C = 1,$$

whereas $$\|\phi_{n,k}^1\|_1 \sim \frac{\log n}{n^2} \quad \textit{and} \quad \|\phi_{n,k}^1\|_2 = \frac{1}{4n^2}\sqrt{4n - \frac{3}{2}}.$$

Proof: The asymptotic behaviour of these kernels is well-known (see *e.g.*, [8, Ch.10]).

For later use, the complex Fourier coefficients of $\phi^0_{n,0}$ can be given as $c^0_s = (2n - |s|)/(4n^2)$ for $|s| \leq 2n - 1$ and $c^0_s = 0$ for $|s| \geq 2n$. The ones for $\phi^1_{n,0}$ are $c^1_s = \operatorname{sgn} s\ i/(4n^2)$ for $0 < |s| < 2n$, $c^1_s = \operatorname{sgn} s\ i/(8n^2)$ for $|s| = 2n$ and $c^1_s = 0$, otherwise. Hence,

$$\|\phi^0_{n,k}\|_2^2 = \sum_{s=-\infty}^{\infty} |c^0_s|^2 = \frac{1}{4n^2} + \frac{1}{8n^4} \sum_{s=1}^{2n-1} s^2 = \frac{1}{3n} + \frac{1}{24n^3}$$

and

$$\|\phi^1_{n,k}\|_2^2 = \frac{1}{32n^4} + \sum_{s=1}^{2n-1} \frac{1}{8n^4} = \frac{1}{4n^3} - \frac{3}{32n^4}.$$

For the computation of the L^1-norm and C-norm, the positivity of $\phi^0_{n,k}$ can be applied. ■

§3. Time frequency uncertainty

According to Breitenberger [1] and Narcowich & Ward [3], it is possible to formulate an uncertainty principle for functions in $L^2_{2\pi}$. For a function $f \in L^2_{2\pi}$, represented by its Fourier series $f(x) = \sum_{s=-\infty}^{\infty} c_s e^{isx}$, the *first trigonometric moment* is defined as

$$\tau(f) := \frac{1}{2\pi} \int_0^{2\pi} e^{ix} |f(x)|^2 \, dx = \sum_{s=-\infty}^{\infty} c_s \overline{c_{s+1}}.$$

Then, the *angle variance* for $f \in L^2_{2\pi}$, $f \not\equiv 0$, is defined by

$$\operatorname{var}_A(f) \;:=\; \frac{\|f\|_2^4 - |\tau(f)|^2}{|\tau(f)|^2} = \left| \frac{\sum_{s=-\infty}^{\infty} |c_s|^2}{\sum_{s=-\infty}^{\infty} c_s \overline{c_{s+1}}} \right|^2 - 1,$$

and the *frequency variance* for real-valued functions by

$$\operatorname{var}_F(f) \;:=\; \sum_{s=-\infty}^{\infty} s^2 |c_s|^2.$$

The *Uncertainty Principle for real-valued functions* $f \in L^2_{2\pi}$, $f \not\equiv 0$, (see [1,3]) states that

$$\frac{\sqrt{\operatorname{var}_A(f) \operatorname{var}_F(f)}}{\|f\|_2} \geq \frac{1}{2}.$$

We now proceed to compute the uncertainty products of the two trigonometric Hermite interpolants $\phi^0_{n,k}$ and $\phi^1_{n,k}$.

Theorem 3.1. *For $n \in \mathbb{N}$, the angle variances of $\phi^0_{n,k}$ and $\phi^1_{n,k}$ are*

$$\mathrm{var}_A(\phi^0_{n,k}) = \frac{3}{4}\frac{16n^2-1}{(4n^2-1)^2} \quad \textit{and} \quad \mathrm{var}_A(\phi^1_{n,k}) = \frac{1}{4}\frac{48n-27}{(4n-3)^2},$$

while

$$\mathrm{var}_F(\phi^0_{n,k}) = \frac{16n^4-1}{120n^3} \quad \textit{and} \quad \mathrm{var}_F(\phi^1_{n,k}) = \frac{8n^2-3n+1}{24n^3}.$$

Consequently, the uncertainty products are

$$\frac{\sqrt{\mathrm{var}_A(\phi^0_{n,k})\,\mathrm{var}_F(\phi^0_{n,k})}}{\|\phi^0_{n,k}\|_2} = \sqrt{\frac{3}{10}\left(1+\frac{20n^2+1}{2(8n^2+1)(4n^2-1)}\right)}$$

and

$$\frac{\sqrt{\mathrm{var}_A(\phi^1_{n,k})\,\mathrm{var}_F(\phi^1_{n,k})}}{\|\phi^1_{n,k}\|_2} = \sqrt{n\left(1+\frac{120n^2-101n+18}{(8n-3)(4n-3)^2}\right)}.$$

Proof: As the variances are translation invariant, it suffices to deal with the case $k=0$.

One computes

$$\begin{aligned}\sum_{s=-\infty}^{\infty} c^0_s\overline{c^0_{s+1}} &= 2\sum_{s=0}^{2n-2} c^0_s c^0_{s+1} = \frac{1}{8n^4}\sum_{s=0}^{2n-2}(2n-s)(2n-s-1)\\ &= \frac{1}{8n^4}\left(\sum_{s=1}^{2n} s(s-1)\right) = \frac{4n^2-1}{12n^3},\end{aligned}$$

and similarly all the other necessary terms. ■

Discussion: According to the above theorem, $\phi^1_{n,k}$ displays an $\mathcal{O}(\sqrt{n})$ product, while in the limit, one obtains for $\phi^0_{n,k}$ a value of $\sqrt{0.3} = 0.5477..$, relatively close to the optimal value of $1/2$. This is due to the fact that the function $\phi^0_{n,0}$ can be interpreted as an approximation of the theta functions

$$\sum_{s=-\infty}^{\infty} e^{-cs^2+isx},$$

for which the time frequency localization tends to the optimal value of $1/2$ if the positive parameter c tends to 0. Details will be discussed in a forthcoming paper.

§4. Approximation Properties

Our goal in this section is to describe time-frequency localization in terms of operator norms. Let X be one of the spaces $L^p_{2\pi}$ or $C_{2\pi}$. We consider the orthoprojections P^r_n from X onto V^r_n.

Theorem 4.1. *Let $n \in \mathbb{N}$. For the operator norms of the orthoprojections, it holds that*

$$\|P^0_n\|_{X\to X} < 3$$

and

$$\|P^1_n\|_{X\to X} \sim \begin{cases} \log n, & \text{if } X = C_{2\pi} \text{ or } X = L^1_{2\pi}, \\ C_p, & \text{if } X = L^p_{2\pi}, 1 < p < \infty. \end{cases}$$

Proof: Let $f \in C_{2\pi}$ be given such that $P^0_n f = \sum_{k=0}^{2n-1} \gamma_k \phi^0_{n,k} \not\equiv 0$. By the positivity of $\phi^0_{n,k}$, we obtain

$$\begin{aligned} \|P^0_n f\|_C &= \| \sum_{k=0}^{2n-1} \gamma_k \phi^0_{n,k}\|_C \leq \left(\max_{0\leq k<2n} |\gamma_k| \right) \| \sum_{k=0}^{2n-1} \phi^0_{n,k}\|_C \\ &= \max_{0\leq k<2n} |\gamma_k| =: |\gamma_l|. \end{aligned}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} \|f\|_C \|\phi^0_{n,l}\|_1 &\geq |\langle f, \phi^0_{n,l}\rangle| = |\langle P^0_n f, \phi^0_{n,l}\rangle| = |\sum_{k=0}^{2n-1} \gamma_k \langle \phi^0_{n,k}, \phi^0_{n,l}\rangle| \\ &\geq |\gamma_l| \left(2\langle \phi^0_{n,l}, \phi^0_{n,l}\rangle - \sum_{k=0}^{2n-1} \frac{|\gamma_k|}{|\gamma_l|} \langle \phi^0_{n,k}, \phi^0_{n,l}\rangle \right) \\ &\geq |\gamma_l| \left(2\|\phi^0_{n,l}\|_2^2 - \|\phi^0_{n,l}\|_1 \right). \end{aligned}$$

Using the norm equalities from Lemma 2.2, we summarize

$$\|P^0_n\|_{C_{2\pi}\to C_{2\pi}} \leq \frac{|\gamma_l|}{\|f\|_C} \leq \left(\frac{2\|\phi^0_{n,l}\|_2^2}{\|\phi^0_{n,l}\|_1} - 1 \right)^{-1} = \frac{6n^2}{2n^2+1} < 3.$$

The uniform boundedness of $\|P^0_n\|_{C_{2\pi}\to C_{2\pi}}$ implies the one for $X = L^p_{2\pi}$, $1 \leq p < \infty$, (see [2, Ch.1, §4, Th.9]) .

To estimate $\|P^1_n\|_{X\to X}$, we use the similarity of P^1_n to the Fourier sum projection.

Namely, from Lemma 2.1, we conclude that V^1_n has the orthogonal basis $\{\sin nx \cdot \cos kx \, : \, k = 0, \ldots, n; \; \sin nx \cdot \sin kx \, : \, k = 1, \ldots, n-1\}$.

Hence, in terms of the classical Fourier sum operator S_n with respect to span$\{\cos kx : k = 0, \ldots, n;\ \sin kx : k = 1, \ldots, n-1\}$, we write

$$P_n^1 f(x) \sim S_n(f \sin n\cdot)(x)\ \sin nx,$$

which leads to an asymptotic behaviour as stated in the theorem. ■

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Algebraic Polynomial Wavelets on the Square

Frauke Sprengel

Abstract. Nested spaces of bivariate algebraic polynomials on the square are investigated. The scaling functions of these spaces are chosen as fundamental polynomials of Lagrange interpolation on a sparse Gauß–Chebyshev grid. The interpolatory properties are crucial for the approach based on Boolean sums.

§1 Introduction

We are interested here in wavelet methods on a bounded domain, in our case on the square $[-1,1]^2$, which are based on the construction of univariate polynomial wavelets on an interval. Therefore, we consider special bivariate sample and wavelet spaces of algebraic polynomials. As our starting point we take the univariate interpolating polynomial scaling functions and wavelets introduced by Kilgore and Prestin [2], which were recently further developed and generalized by Plonka, Selig, and Tasche [3,5]. We want to introduce here new bivariate interpolating polynomial scaling functions and wavelets by investigating Boolean sums of interpolation operators. In this way, we overcome the problem of a very fast growth of the dimensions that we have in the ordinary tensor product case. However, because of the simple structure of the spaces one can take use of fast algorithms provided in [3] for the univariate case. That means, we obtain sample spaces with dimension $\mathcal{O}(j\,2^j)$ while tensor products yield dimension $\mathcal{O}(2^{2j})$ for the level j. The concept of Boolean sums is often applied in interpolation and approximation (see e. g. [1]) as well as in finite element methods (see e. g. [6]). It uses essentially the interpolatory properties of the underlying univariate scaling functions and wavelets. The general multivariate case can be treated analogously.

Applying the Boolean sum method, we are able to construct multivariate scaling functions and wavelets from univariate interpolating scaling functions and wavelets. We can establish stability relations for the multivariate wavelets similar to the univariate case.

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§2 Polynomial wavelets on the interval

In the sequel we denote our reference interval by $I := [-1, 1]$ and the Chebyshev weight by $w(x) := (1-x^2)^{-1/2}$ $(x \in (-1,1))$. Let $L^2_w(I)$ be the weighted Hilbert space of all measurable functions $f : I \longrightarrow \mathbb{R}$ with

$$\int_I w(x) f(x)^2 \, dx < \infty.$$

For $f, g \in L^2_w(I)$, the corresponding inner product is given by

$$\langle f, g\rangle_I := \frac{2}{\pi} \int_I w(x) f(x) g(x) \, dx.$$

By Π_n we denote the set of all real polynomials of degree at most n restricted on I. Furthermore, let $T_n \in \Pi_n$ be the Chebyshev polynomials

$$T_n(x) := \cos(n \arccos x)$$

and $\mathcal{G}_j$ the grid of the Gauß–Chebyshev nodes

$$\mathcal{G}_j := \{g_{j,k} := \cos k\frac{\pi}{2^j}\,,\ k = 0, \ldots, 2^j\}.$$

The *polynomial scaling functions* introduced by Kilgore and Prestin [2] are of the form

$$\phi_{j,k}(x) := \frac{\varepsilon_{j,k}}{2^{j-1}} \sum_{r=0}^{2^j} \varepsilon_{j,r}\, T_r(g_{j,k})\, T_r(x)\,,$$

where here and in the following

$$\varepsilon_{j,k} := \begin{cases} \frac{1}{2} & \text{if } k = 0 \text{ or } k = 2^j, \\ 1 & \text{if } k = 1, 2, \ldots, 2^j - 1. \end{cases}$$

Further, we observe the interpolatory property

$$\phi_{j,k}(g_{j,l}) = \delta_{k,l}, \quad k, l = 0, 1, \ldots, 2^j.$$

and the equality $V_j = \Pi_{2^j}$, where V_j denotes our polynomial sample space

$$V_j := \operatorname{span}\{\phi_{j,k},\ k = 0, 1, \ldots, 2^j\}.$$

From that definition, we obtain immediately $\dim V_j = 2^j + 1$. The spaces V_j form a non–stationary multiresolution of the weighted Hilbert space $L^2_w(I)$, *i.e.*,

$$V_j \subset V_{j+1},\ j \in \mathbb{N}_0 \quad \text{and} \quad \operatorname{clos}_{L^2_w(I)} \bigcup_{j \in \mathbb{N}_0} V_j = L^2_w(I).$$

The interpolation operator L_j on the Gauß–Chebyshev grid projecting onto the space V_j can be written as

$$L_j f := \sum_{k=0}^{2^j} f(g_{j,k})\, \phi_{j,k}\,,$$

with the range $\operatorname{Im} L_j = V_j$ and the precision set $\operatorname{prec} L_j = \mathcal{G}_j$. Now let us consider the univariate wavelet spaces W_j defined by

$$V_{j+1} = V_j \oplus^{\perp} W_j,$$

where $\oplus^{\perp}$ denotes the orthogonal sum with respect to the inner product in $L^2_w(I)$. Following Kilgore and Prestin [2], they are given by

$$W_j := \operatorname{span}\{\psi_{j,k},\ k = 0, 1, \ldots, 2^j - 1\}\,,$$

with the *polynomial wavelet functions*

$$\psi_{j,k}(x) := \frac{1}{2^{j-1}} \sum_{r=2^j+1}^{2^{j+1}} \varepsilon_{j+1,r}\, T_r(g_{j+1,2k+1})\, T_r(x)\,,$$

satisfying also interpolatory conditions

$$\psi_{j,m}\left(g_{j+1,2\ell+1}\right) = \delta_{\ell,m}, \qquad \ell = 0, 1, \ldots, 2^j - 1.$$

Note that we use the original approach from [2] for simplicity. For applying Chebyshev and fast cosine transform, one has to use the ideas outlined in [3,5]. There the scaling functions and wavelets are slightly modified which does not affect the approach described below.

§3 Boolean sums of interpolation projectors

Now we want to investigate bivariate interpolation on a sparse Gauß–Chebyshev grid, the so-called blending interpolation. By the definition of the interpolation projectors L_j, it holds that

$$\operatorname{Im} L_j \subset \operatorname{Im} L_{j+1}\,, \qquad \operatorname{prec} L_j \subset \operatorname{prec} L_{j+1}.$$

Hence,

$$L_0 \ \le\ L_1 \ \le\ L_2 \ \le\ \cdots \ \le\ L_j \ \le\ L_{j+1} \ \le\ \cdots \tag{3.1}$$

is a chain of projectors, where the order relation $A \le B$ holds if and only if $AB = BA = A$.

For bivariate functions $f : I^2 \longrightarrow \mathbb{R}$, we define parametric extensions of the interpolation projectors by

$$L_j^x f(x,y) := \sum_{\ell=0}^{2^j} f(g_{j,\ell}, y)\, \phi_{j,\ell}(x),$$

$$L_k^y f(x,y) := \sum_{\ell=0}^{2^k} f(x, g_{k,\ell})\, \phi_{k,\ell}(y).$$

The investigation of their composition shows that L_j^x and L_k^y commute and yield the tensor product interpolation. From (3.1), we have for the parametric extensions that

$$L_0^x \le L_1^x \le L_2^x \le \cdots \le L_j^x \le L_{j+1}^x \le \cdots,$$
$$L_0^y \le L_1^y \le L_2^y \le \cdots \le L_k^y \le L_{k+1}^y \le \cdots$$

are chains of projectors again. Their *Boolean sum* is defined by

$$S_j := \bigoplus_{r=0}^{j} L_r^x\, L_{j-r}^y,$$

where $A \oplus B := A + B - AB$ and the representation of B_j in terms of ordinary sums is known (see [1]) to be

$$S_j = \sum_{r=0}^{j} L_r^x\, L_{j-r}^y - \sum_{r=0}^{j-1} L_r^x\, L_{j-r-1}^y. \tag{3.2}$$

The Boolean sums also form a chain (see [1])

$$S_0 \le S_1 \le S_2 \le \cdots \le S_j \le S_{j+1} \le \cdots.$$

They have the range

$$\operatorname{Im} S_j = \sum_{r=0}^{j} \operatorname{Im} L_r^x \otimes \operatorname{Im} L_{j-r}^y = \sum_{r=0}^{j} V_r \otimes V_{j-r}$$

and interpolate on the grid

$$\begin{aligned} \mathcal{G}_j^S &:= \operatorname{prec} S_j = \bigcup_{r=0}^{j} \mathcal{G}_r \times \mathcal{G}_{j-r} \\ &= \bigcup_{r=0}^{j} \{(g_{r,k}, g_{j-r,\ell}),\ k = 0,1,\ldots,2^r;\ \ell = 0,1,\ldots,2^{j-r}\}. \end{aligned}$$

In this way, the interpolation projector on the sparse Gauß–Chebyshev grid turned out to be the Boolean sum operator S_j.

Moreover, it follows that

$$\operatorname{Im} S_j \subset \operatorname{Im} S_{j+1}, \qquad \operatorname{prec} S_j \subset \operatorname{prec} S_{j+1}. \tag{3.3}$$

§4 Sample and wavelet spaces

Taking into consideration the properties of the blending interpolation operator S_j, we are now able to define new bivariate sample spaces by

$$V_j^S := \operatorname{Im} S_j = \sum_{r=0}^{j} V_r \otimes V_{j-r}. \tag{4.1}$$

The space V_j^S defined in such a way is now a subspace of the space of bivariate algebraic polynomials of total degree at most 2^j. From that definition and (3.3), we obtain that

$$V_j^S \subset V_{j+1}^S$$

and

$$V_j^S = \operatorname{span} \bigcup_{r=0}^{j} \{\phi_{r,k}(x) \cdot \phi_{j-r,\ell}(y);\ k = 0, 1, \ldots, 2^r, \\ \ell = 0, 1, \ldots, 2^{j-r}\}. \tag{4.2}$$

Hence, we can immediately characterize the functions in the sample space. Note that the system in (4.2) is only a generating system, which does not form a basis. The goal is to characterize the bivariate scaling functions again by certain interpolatory properties. For that reason, we define an index set $\mathcal{J}_j$ by

$$\mathcal{J}_j := \{(k, \ell) \in \mathbb{N}_0^2\ ;\ (g_{j,k}, g_{j,\ell}) \in \mathcal{G}_j^S\}$$

and look for fundamental interpolants $\Phi_{j,(k,\ell)} \in V_j^S$ on the sparse grid $\mathcal{G}_j^S$, *i.e.*,

$$\Phi_{j,k,\ell}(g_{j,m}, g_{j,n}) = \delta_{k,m}\delta_{\ell,n}\,, \qquad (k, \ell), (m, n) \in \mathcal{J}_j.$$

These functions can be found using tensor products of the univariate interpolating polynomial scaling functions on the level j

$$\Lambda_{j,k,\ell}\,(x, y) = \phi_{j,k}(x)\ \phi_{j,\ell}(y).$$

The application of the blending interpolation projector S_j yield

$$\begin{aligned}\Phi_{j,k,\ell}(x, y) &= S_j \Lambda_{j,k,\ell}\,(x, y)\\ &= \sum_{s=0}^{j} \phi_{s,2^{s-j}k}(x)\ \phi_{j-s,2^{-s}\ell}(y)\\ &\quad - \sum_{s=0}^{j-1} \phi_{s,2^{s-j}k}(x)\ \phi_{j-s-1,2^{-s-1}\ell}(y),\end{aligned}$$

where $(k,\ell) \in \mathcal{J}_j$ and $\phi_{s,t} \equiv 0$ for $t \notin \mathbb{N}_0$. The set

$$\mathcal{B}(V_j^S) = \{\Phi_{j,k,\ell},\ (k,\ell) \in \mathcal{J}_j\}$$

is a basis of V_j^S, what can be deduced from the relations (3.2) and (4.1).

As usual, we define the wavelet space W_j^S as the orthogonal complement of the sample space on the lower level j relative to the sample space on the higher level $j+1$, *i.e.*,

$$V_{j+1}^S = V_j^S \oplus^\perp W_j^S, \tag{4.3}$$

with respect to the inner product in $L_w^2(I^2)$

$$\langle f, g\rangle := \frac{4}{\pi^2} \iint\limits_{I^2} w(x)w(y)f(x,y)g(x,y)\,dxdy.$$

These wavelet spaces can be characterized in terms of tensor products of various univariate wavelet spaces.

Theorem 4.1 *For $j \in \mathbb{N}_0$, we have*

$$W_j^S = \bigoplus_{r=0}^{j+1}{}^{\perp}\, W_{r-1} \otimes W_{j-r} \qquad (4.4),$$

with the notation $W_{-1} := V_0$.

Proof: Denoting the right-hand side of (4.4) by U_j, we check the orthogonality $U_j \perp V_j^S$ first. For that purpose, we take arbitrary functions $u_j \in U_j$ and $v_j \in V_j^S$. By the definition of U_j, there exist uniquely determined coefficients $\gamma_{s,m,n}$ and by (4.2) some (not uniquely determined) coefficients $\alpha_{r,k,\ell}$ such that

$$u_j(x,y) = \sum_{s=0}^{j+1} \sum_{m=0}^{2^{s-1}-1} \sum_{n=0}^{2^{j-s}-1} \gamma_{s,m,n}\, \psi_{s-1,m}(x)\, \psi_{j-s,n}(y),$$

$$v_j(x,y) = \sum_{r=0}^{j} \sum_{k=0}^{2^r} \sum_{\ell=0}^{2^{j-r}} \alpha_{r,k,\ell}\, \phi_{r,k}(x)\, \phi_{j-r,\ell}(y),$$

where we denote $\psi_{-1,m} := \phi_{0,m}$ and we have to replace the index 2^{-1} by 2 whenever this occurs. Their inner product can be computed as

$$\langle v_j, u_j\rangle =$$

$$= \sum_{r=0}^{j} \sum_{s=0}^{j+1} \sum_{k=0}^{2^r} \sum_{\ell=0}^{2^{j-r}} \sum_{m=0}^{2^{s-1}-1} \sum_{n=0}^{2^{j-s}-1} \alpha_{r,k,\ell}\gamma_{s,m,n} \langle \phi_{r,k}, \psi_{s-1,m}\rangle_I \langle \phi_{j-r,\ell}, \psi_{j-s,n}\rangle_I.$$

Taking into account the orthogonality of the univariate scaling functions and wavelets, we can distinguish two cases.
(i) For $0 \le r < s \le j+1$, it holds $\langle \phi_{r,k}, \psi_{s-1,m}\rangle_I = 0$,
(ii) for $0 \le s \le r \le j$, it holds $\langle \phi_{j-r,\ell}, \psi_{j-s,n}\rangle_I = 0$.

Hence, all terms occuring in this sum vanish and we have

$$\langle v_j, u_j\rangle = 0 \qquad \text{for all } v_j \in V_j^S,\ u_j \in U_j.$$

Using the relations between the univariate polynomial sample and wavelet spaces, it is easy to prove that

$$V_j^S + U_j = \sum_{r=0}^{j} V_r \otimes V_{j-r} + \sum_{r=0}^{j+1} W_{r-1} \otimes W_{j-r} = \sum_{r=0}^{j+1} V_r \otimes V_{j+1-r} = V_{j+1}^S.$$

In this manner, we obtain that U_j satisfies the definition (4.3) of the wavelet space. Thus, $U_j = W_j^S$, which proves the theorem. ■

The representation (4.4) of the wavelet space leads to a basis in a natural way

$$\mathcal{B}(W_j^S) = \bigcup_{r=0}^{j+1} \{\psi_{r-1,k}(x) \cdot \psi_{j-r,\ell}(y);\ k = 0,1,\ldots,2^{r-1}-1, \\ \ell = 0,1,\ldots,2^{j-r}-1\}$$

with the notations of this section. Thus, the dimension is

$$\dim W_j^S = j2^{j-1} + 2^{j+2}.$$

Denoting $W_{-1}^S := V_0^S$, we can write the sample space in terms of orthogonal sums of wavelet spaces

$$V_j^S = \bigoplus_{r=-1}^{j-1}{}^{\perp}\ W_r^s = \bigoplus_{r=0}^{j}{}^{\perp} \bigoplus_{k=0}^{r}{}^{\perp}\ (W_{k-1} \otimes W_{r-1-k}) .$$

Hence, we obtain immediately $\dim V_0^S = 4$, $\dim V_1^S = 8$ and for $j \ge 2$

$$\dim V_j^S = (j-2)2^{j-1} + 2^{j+2} + 1.$$

That means, we have the dimension of the constructed bivariate polynomial sample spaces growing as $\mathcal{O}(j2^j)$ while the dimension of ordinary tensor products of univariate sample spaces grows as $\mathcal{O}(2^{2j})$.

Starting from the stability relations for the univariate polynomial sample and wavelet spaces proved in [3], we are able to formulate the stability relation for our bivariate wavelet spaces. For sequences $\underline{\alpha} \in \mathbb{R}^{\dim W_j^S}$, we use the notation

$$\|\underline{\alpha}\|^2 := \sum_{r=0}^{j+1} \sum_{k=0}^{2^{r-1}-1} \sum_{\ell=0}^{2^{j-r}-1} |\alpha_{r,k,\ell}|^2.$$

Theorem 4.2 *For $j \in \mathbb{N}_0$ and arbitrary $\underline{\alpha} \in \mathbb{R}^{\dim W_j^S}$, it holds that*

$$\frac{1}{2}\,\|\underline{\alpha}\| \le \left\| \sum_{r=0}^{j+1} \sum_{k=0}^{2^{r-1}-1} \sum_{\ell=0}^{2^{j-r}-1} \alpha_{r,k,\ell}\, 2^{j/2}\, \psi_{r-1,k}(x)\, \psi_{j-r,\ell}(y) \right\|_{L^2_w(I^2)} \le \sqrt{2}\,\|\underline{\alpha}\|.$$

For an analogous proof in the periodic case, we refer the reader to [4].

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