On the Dirichlet problem for the Ekman equation

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February 6, 1995

Abstract

The Ekman partial differential equation for the streamfunction of turbulent mass flow in shallow and small sized surface waters is discussed. The Dirichlet problem for the Ekman equation is shown to be well posed in a weighted Sobolev space. Conditions for the existence of classical solutions are given. The dependence of regularity and asymptotics of the solution on the properties of the depth profile are studied.

1 Introduction

In the present paper we are concerned with the Dirichlet problem for the PDE

$$\Delta u - \frac{2}{h}(\nabla h, \nabla u) = k_A(\nabla h, QW) \qquad \text{in } \Omega \tag{1}$$

which models the wind-induced hydrodynamic flow in small and shallow surface waters. Here the bounded domain $\Omega \subset R^2$ describes the projection of the considered surface water on the plane R^2, $(\cdot,\cdot)$ denotes the Euclidean scalar product in R^2, $u : \Omega \longrightarrow R$ is the *streamfunction*, $h : \Omega \longrightarrow [0,\infty)$ is the *depth profile*, $W \in R^2$ is the *wind vector*,

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a clockwise rotation by an angle of $\pi/2$, and, finally, $k_A \geq 0$ is an empirical constant.
In [17] equation (1) was derived in the given form and discussed under the Dirichlet condition

$$u = u_0 \qquad \text{on } \partial\Omega\,, \tag{2}$$

and a FDM approximation for the numerical solution was presented.
However, typical depth profiles h vanish at least on parts of $\partial\Omega$, and up to now there has been no proof that the problem (1)–(2) is well-posed for that case. The aim of the present paper is to prove the existence and uniqueness of solutions to (1)–(2) and, furthermore, to discuss the regularity of the solution under possibly weak assumptions.
To this end we transform equation (1) into divergence form and consider an associated variational problem

$$a(u,v) = b(v) \qquad \forall v \tag{3}$$

in a weighted Sobolev space $H(\Omega)$. Using the Lax-Milgram lemma we obtain unique solvability of the Dirichlet problem.

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If, in addition, ∇h is bounded this allows us to infer the existence of an element $f \in L(\Omega)$ such that the solution u of (3) satisfies

$$\Delta u = f\,. \tag{4}$$

Hence, under standard assumptions on Ω, we conclude that $u \in H^2(\Omega)$.
We are able to strengthen this regularity result, if Ω is a polygonal domain and the depth profile fulfils $h(x) = g(x)\mathrm{dist}\,(x, \partial\Omega)^\beta$, with $g(x) \geq c > 0$. In this case we obtain $u \in H^{1+\beta}(\Omega)$.
Further, the asymptotic behaviour of the solution in a neighbourhood of smooth parts of the boundary will be derived.
The paper is completed by some numerical examples of streamfunctions u for the Greifswalder Bodden of the Baltic Sea. For the discretization we used a conforming FE method based on (3). It is worthwhile to mention that our present approach allows us to work with positive definite symmetric matrices, while the FDM applied in [17] yields nonsymmetric systems of equations. Hence, our calculations require only half the memory and CPU time compared to [17].

2 Weak formulations, existence and uniqueness

If we derived a weak formulation of (1)–(2) by multiplying (1) by a test function v and integrating by parts we would obtain a nonsymmetric bilinear form. Moreover, without assuming rather undesirable conditions on h it proves to be difficult to obtain existence and uniqueness of a weak solution that way. Weak coercivity (cf. [1]) can be obtained for $h \geq h_0 > 0$ by using a maximum principle (cf. [4]). The assumptions of the Lax-Milgram Lemma can be fitted under certain conditions for Ω and h expressed in terms of embedding constants (cf. [13]). Both results hold in $H^1(\Omega)$, and both require rather unrealistic restrictions on h.
A proper variational formulation is obtained after multiplying (1) by h^{-2}. In fact, observing that

$$h^{-2}\Delta u - 2h^{-3}(\nabla h, \nabla u) = \mathrm{div}\,(h^{-2}\nabla u)$$

we can rewrite (1) in the form

$$\mathrm{div}\,(h^{-2}\nabla u) = k_A h^{-2}(\nabla h, QW)\,. \tag{5}$$

We introduce the weighted Sobolev space

$$H = \overset{o}{H}{}^1_{h^{-2}}(\Omega) = \overline{C_0^\infty(\Omega)} \tag{6}$$

with the completion taken with respect to the norm

$$\|v\| = \|v\|_{h^{-2}} = \left(\int\limits_\Omega h^{-2}|\nabla v|^2\,dx \right)^{1/2}. \tag{7}$$

Observe that $\|\cdot\|$ makes H a Hilbert space with scalar product

$$< u, v >= \int\limits_\Omega (\nabla u, \nabla v) h^{-2} dx\,. \tag{8}$$

For the boundary conditions $\hat{u}_0$ we assume that there exists an element $u \in H^1(\Omega)$ such that

$$\mathrm{tr}\,(\hat{u}_0, \partial\Omega) = u_0 \qquad \text{and} \qquad \int\limits_\Omega h^{-2}|\nabla \hat{u}_0|^2\,dx < \infty\,. \tag{9}$$

Now we introduce $w = u - \hat{u}_0$ and obtain the PDE for w

$$\mathrm{div}\,(h^{-2}\nabla w) = k_A h^{-2}(\nabla h, QW) + \mathrm{div}\,\left(h^{-2}\nabla \hat{u}_0\right) \tag{10}$$

with homogeneous boundary conditions, $w \in H$.
Multiplication by $v \in H$ and integration by parts yields

$$\int_\Omega h^{-2}(\nabla w, \nabla v)\, dx = -\int_\Omega h^{-2}(\ell, \nabla h) v\, dx + \int_\Omega h^{-2}\,(\nabla \hat{u}_0, \nabla v)\; dx\,, \tag{11}$$

where we denote $\ell = k_A Q W$.
We define

$$\begin{aligned} a(w,v) &= \int_\Omega h^{-2}(\nabla w, \nabla v)\, dx\,, \\ b(v) &= -\int_\Omega h^{-2}(\ell, \nabla h) v\, dx + \int_\Omega h^{-2}\,(\nabla \hat{u}_0, \nabla v)\; dx\,. \end{aligned}$$

Hence, the following variational problem is set up:
Find a function $w \in H$ such that for all $v \in H$ there holds

$$a(w,v) = b(v)\,. \tag{12}$$

As a matter of course we can state the following

Lemma 1 *The bilinear form $a(\cdot,\cdot)$ is symmetric, bounded and coercive on $H \times H$, both constants being equal to 1, i.e.*

$$a(w,v) \le \|w\|\|v\| \qquad \textit{and} \qquad a(w,w) \ge \|w\|\,. \tag{13}$$

In fact, the second relation holds as an equality. ∎

Further, we have

Lemma 2 *The terms $\int_\Omega h^{-2}(\ell, \nabla h) v\, dx$ and $\int_\Omega h^{-2}(\nabla \hat{u}_0, \nabla v)\, dx$ both define linear continuous functionals on H.*

Proof: Let $v \in C_o^\infty(\Omega) \subset H$. Then

$$\int_\Omega h^{-2} v \nabla h\, dx = -\int_\Omega v \nabla(h^{-1})\, dx = \int_\Omega h^{-1} \nabla v\, dx\,.$$

Hence, by the Hölder inequality

$$\left| \int_\Omega h^{-2} v \nabla h\, dx \right|^2 \le \int_\Omega |v \nabla (h^{-1})|^2\, dx \int_\Omega 1\, dx = \|v\|^2 \mathrm{mes}\,(\Omega)\,.$$

The first term in the statement of Lemma 2 is, consequently, a well defined linear operator for v in the dense subspace $C_0^\infty(\Omega)$ of H and its norm is bounded by $|\ell|\sqrt{\mathrm{mes}\,(\Omega)} = k_A |W| \sqrt{\mathrm{mes}\,(\Omega)}$. Hence, there exists a unique extension to H which satisfies the same bound.
To $\int_\Omega (\nabla \hat{u}_0, \nabla v) h^{-2}\, dx$ we apply once more Hölder's inequality which yields

$$\int_\Omega h^{-2}\, |(\nabla \hat{u}_0, \nabla v)\, dx| \le \|v\| \|\hat{u}_0\|\,. \qquad ∎$$

Remark 1 *In the last inequality* $\|\hat{u}_0\|$ *is finite due to (9), although* $\hat{u}_0 \in H$ *does not necessarily hold.*

Lemmas 1 and 2 together with the Lax-Milgram Lemma provide a proof of:

Theorem 1 *The variational problem (12) possesses a unique solution* $w \in H$ *with*

$$\|w\| \leq k_A |W| \sqrt{mes\,(\Omega)} + \|u_0\|\,. \qquad \blacksquare \tag{14}$$

Remark 2 *For* $h \leq h_0 < \infty$ *– which is always satisfied, we have* $H \subseteq H^1(\Omega)$*, and we are able to deduce the existence of a solution* $u = \hat{u}_0 + w$ *to equation (1) in the Sobolev space* $H^1(\Omega)$*. Moreover, under reasonable restrictions for* $|\nabla h|$ *and for a suitable class of domains* Ω*, the finiteness of* $\|u\|$ *implies even stronger regularity.*

3 Regularity of the solution

In this section we will assume that the function h has the form

$$h(x_1, x_2) = g(x_1, x_2) \cdot r(x_1, x_2)^{\beta}, \tag{15}$$

where $r = r(x)$ denotes the distance of the point $x = (x_1, x_2)$ to the boundary $\partial\Omega$, β is an arbitrary non-negative real number, and g is a function from $C^{\infty}(\overline{\Omega})$ which satisfies the inequality

$$g(x_1, x_2) \geq g_0 > 0 \qquad \text{in } \Omega.$$

Under this condition the inequality

$$|D^{\alpha}(h^{-1}\,\nabla h)| \leq c_{\alpha} r^{-1-|\alpha|}$$

holds for every multi-index $\alpha = (\alpha_1, \alpha_2)$.
Furthermore, we restrict ourselves to the case that Ω is a domain of polygonal type, i.e., the boundary $\partial\Omega$ of Ω is piecewise smooth and in a neighbourhood of each corner the domain Ω is diffeomorphic to a plane wedge

$$K = \{x = (x_1, x_2) \in R^2 : \ -\phi_0 < \arg(x_1 + i\,x_2) < +\phi_0\}, \qquad 0 < \phi_0 < \pi. \tag{16}$$

We define the space $V^l_{\beta}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm

$$\|u\|_{V^l_{\beta}(\Omega)} = \Big(\int_{\Omega} \sum_{|\alpha \leq l} r^{2(\beta - l + |\alpha|)}\, |D^{\alpha} u|^2\, dx \Big)^{1/2}.$$

It is evident that the norm in H is equivalent to the norm

$$\|u\| = \Big(\int_{\Omega} r^{-2\beta} |\nabla u|^2 dx \Big)^{1/2}. \tag{17}$$

Moreover, from Hardy's inequality it follows that the last norm is equivalent to the norm

$$\|u\| = \Big(\int_{\Omega} (r^{-2\beta - 2}\, |u|^2 + r^{-2\beta}\, |\nabla u|^2) dx \Big)^{1/2}$$

in $C_0^{\infty}(\Omega)$. Consequently, H coincides with the space $V^1_{-\beta}(\Omega)$.

In the following we will prove that the variational solution of the boundary value problem (1), (2) belongs to the space $H^{1+\beta}(\Omega)$. We consider at first the boundary value problem

$$\Delta u - 2\,h^{-1}(\nabla h, \nabla u) = (\ell, \nabla h) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega \tag{18}$$

with the homogeneous Dirichlet condition which corresponds to the variational problem

$$a(u,v) = \int\limits_{\Omega} h^{-2} \cdot (\ell, \nabla h) \cdot v\, dx \qquad \text{for all } v \in H. \tag{19}$$

Obviously, every solution of the variational problem (19) satisfies the equation $\Delta u = f$ in the interior of Ω, where f is an L_2-function on every compact subset of Ω. Hence u belongs to the space $H^2_{loc}(\Omega)$ and satisfies the differential equation $\Delta u - 2h^{-1}(\nabla h, \nabla u) = (\ell, \nabla h)$ in Ω.

Lemma 3 *Let K be the plane wedge and let*

$$L = \sum_{i,j=1}^{2} a_{i,j}(x)\, \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{j=1}^{2} a_j(x) \frac{\partial}{\partial x_j} \tag{20}$$

be an elliptic differential operator in K with coefficients $a_{i,j} \in C^\infty(\overline{K})$, $a_j \in C^\infty(K)$ such that

$$|a_{i,j}(x) - a_{i,j}(0)| < \varepsilon, \qquad |D^\alpha a_j| \le c_\alpha\, r^{-1-|\alpha|} \quad \text{in } \overline{K}$$

for every multi-index $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| \le l-2$, $l \in \{2, 3, \ldots\}$. Here ε is a sufficiently small positive real number.
Furthermore let $u \in V^{l-1}_{-\beta+l-2}(K) \cap H^2_{loc}(K)$ be a solution of the boundary value problem $Lu = f$ in K, $u = 0$ on ∂K, where $f \in V^{l-2}_{-\beta+l-1}(K)$. Then $u \in V^{l}_{-\beta+l-1}(K)$ and

$$\|u\|_{V^l_{-\beta+l-1}(K)} \le c \left(\|f\|_{V^{l-2}_{-\beta+l-1}(K)} + \|u\|_{V^{l-1}_{-\beta+l-2}(K)} \right) \tag{21}$$

with a constant c independent of u.

Proof: We denote by $L_0(x,D)$ the principal part of the operator L. Since $u \in V^{l-1}_{-\beta+l-2}(K)$, we have $Lu - L_0(0,D)u \in V^{l-2}_{-\beta+l-1}(K)$. Consequently, u is a solution of the problem

$$L_0(x,D)u = F \quad \text{in } K, \quad u = 0 \qquad \text{on } \partial K,$$

where $F = f - (L - L_0(x,D))\, u \in V^{l-2}_{-\beta+l-1}(K)$.
Let $\{U_\nu\}$ be a countable covering of K such that $\operatorname{dist}(U, \partial K) = \operatorname{diam} U_\nu =: d_\nu$. Furthermore, let ψ_ν be smooth functions satisfying the conditions

$$\operatorname{supp} \psi_\nu \subset U_\nu\,, \quad \sum_{\nu=0}^{\infty} \psi_\nu = 1 \quad \text{in } K, \quad |D^\alpha \psi_\nu| \le c\, d_\nu^{-|\alpha|}\,. \tag{22}$$

If $x \in U_\nu$, then $x' = x/d_\nu \in d_\nu^{-1} U_\nu =: U'_\nu$, where $\operatorname{diam} U'_\nu = 1$. We define the function u_ν by the equation

$$u_\nu(x') = \psi_\nu(d_\nu x')\, u(d_\nu x') = \psi_\nu(x)\, u(x).$$

Obviously, u_ν is equal to zero on the boundary of U'_ν and the support of u_ν is contained in U'_ν. Using the classical estimates for the solutions of elliptic equations in the domain U'_ν, we get

$$\|\psi_\nu u\|^2_{V^l_{-\beta+l-1}(K)} \le c\, d_\nu^{-2\beta}\, \|u_\nu\|^2_{H^l(U'_\nu)} \le c\, d_\nu^{-2\beta} \left(\|L_0(0,D)\, u_\nu\|^2_{H^{l-2}(K)} + \|u_\nu\|^2_{H^{l-1}(K)} \right) \tag{23}$$

with a constant c independent of u and ν. Since $|a_{i,j}(d_\nu x') - a_{i,j}(0)| < \varepsilon$ and $|D^\alpha_{x'}(a_{i,j}(d_\nu x') - a_{i,j}(0)| \le c\, d_\nu^{|\alpha|}$ for $|\alpha| \ge 1$, from (23) it follows that

$$\begin{aligned}\|\psi_\nu u\|^2_{V^l_{-\beta+l-1}(K)} &\le c\, d_\nu^{-2\beta}\Big(\|L_0(d_\nu x', D_{x'})\, u_\nu\|^2_{H^{l-2}(U'_\nu)} + \|u_\nu\|^2_{H^{l-1}(U'_\nu)}\Big)\\ &\le c\,\Big(\|L_0(x,D)\,(\psi_\nu u)\|^2_{V^{l-2}_{-\beta+l-1}(K)} + \|\psi_\nu u\|^2_{V^{l-1}_{-\beta+l-2}(K)}\Big)\\ &\le c\,\Big(\|\psi_\nu F\|^2_{V^{l-2}_{-\beta+l-1}(K)} + \sum_{\mu\in I_\nu}\|\psi_\mu u\|^2_{V^{l-1}_{-\beta+l-2}(K)}\Big),\end{aligned}$$

where I_ν denotes a set of integer numbers such that $\sum_{\mu\in I_\nu}\psi_\mu = 1$ on U_ν. We can assume that there exists a number N such that every of the sets I_ν consists of not more than N elements and every μ is contained in not more than N of the sets I_ν. Using the equivalence of the norm in $V^l_\beta(K)$ with the norm

$$\|u\| = \Big(\sum_{\nu=0}^{\infty}\|\psi_\nu u\|^2_{V^l_\beta(K)}$$

we get the estimate (21). This proves the lemma. ∎

Since the differential operator in (18) has the form given in the foregoing lemma, as a consequence of this lemma the following theorem holds.

Theorem 2 *Let $u \in H$ be a solution of the variational problem (19). We assume that the function h has the representation (15) with a function $g \in C^\infty(\overline{\Omega})$ satisfying the inequality $|g(x)| \ge g_0 > 0$. Then $u \in V^l_{-\beta+l-1}(\Omega)$ for $l = 1, 2, \ldots$. In particular, we have $u \in H^{1+\beta}(\Omega)$.*

Proof: If $\operatorname{supp} u \cap \partial\Omega = \emptyset$, then the assertion is trivial. Suppose that the support of u is contained in a neighbourhhod of any boundary point. Then with no loss of generality, we may assume that the domain Ω coincides with a half-plane or a plane wedge in this neighbourhood. In both cases we can apply Lemma 3 and obtain $u \in V^l_{-\beta+l-1}(\Omega)$. For functions with arbitrary support this assertion holds by means of a suitable partition of unity on Ω. Using the fact that the space $V^l_{-\beta+l-1}(\Omega)$ is continuously imbedded into $H^{1+\beta}(\Omega)$ for $l \ge 1+\beta$ (cf. Theorem 1 in [16]), we get $u \in H^{1+\beta}(\Omega)$. The proof is complete. ∎

We consider now the boundary value problem (1), (2) with the inhomogeneous Dirichlet condition. Under the condition (15) the assumption (9) can be characterized as follows.

Lemma 4 *Let u_0 be an arbitrary function on $\partial\Omega$. Then there exists a function $\hat{u}_0 \in H^1(\Omega)$ with the properties (9) if and only if*

a) u_0 is a constant in the case $\beta \ge 1/2$,

b) $u_0 \in H^{1/2+\beta}(\partial\Omega)$, if $0 \le \beta < 1/2$.

Proof: Let $W^l_\beta(\Omega)$ be the closure of the set of all function $u \in C^\infty(\overline{\Omega})$ such that

$$\|u\|_{W^l_\beta(\Omega)} = \Big(\|u\|^2_{L_2(\Omega)} + \int_\Omega \sum_{|\alpha|=l} r^{2\beta}\,|D^\alpha u|^2 dx\Big)^{1/2} < \infty \tag{24}$$

with respect to norm (24). In case b) the trace of each functions from $W^1_{-\beta}(\Omega)$ belongs to the space $H^{1/2+\beta}(\partial\Omega)$ (cf. Lemma 1.1, Lemma 1.2 in [15]).
We consider the case $\beta \ge 1/2$. Let $\hat{u}_0$ be a function from $W^1_{-\beta}(\Omega)$ and $\{\hat{u}_n\}_{n\ge1} \subset C^\infty(\overline{\Omega}) \cap W^1_{-\beta}(\Omega)$ a sequence converging to $\hat{u}_0$ in $W^1_{-\beta}(\Omega)$. Since

$$\int_\Omega r^{-2\beta}\,|\nabla \hat{u}_n|^2 dx < \infty,$$

we have $\nabla \hat{u}_n|_{\partial\Omega} = 0$ and therefore, $\hat{u}_n|_{\partial\Omega} = c_n = const$. From Hardy's inequality it follows that

$$\int\limits_\Omega r^{-2\beta-2} \, |\hat{u}_n - c_n|^2 \, dw \le c \int\limits_\Omega r^{-2\beta} \, |\nabla \hat{u}_n|^2 \, dx$$

with a constant c independent of $\hat{u}_n$ and c_n. This implies

$$|c_n| \le c \, \|\hat{u}_n\|_{W^1_{-\beta}(\Omega)} .$$

Analogously, we obtain

$$|c_n - c_m| \le c \, \|\hat{u}_n - \hat{u}_m\|_{W^1_{-\beta}(\Omega)} .$$

Hence the limit of the sequence $\{c_n\}$ exists and coincides with the trace of $\hat{u}_0$ on $\partial\Omega$. ■

Remark 3 In both cases $\beta \ge 1/2$ and $\beta < 1/2$ of Lemma 4 the function u_0 can be extended to a function $\hat{u}_0 \in W^l_{-\beta+l-1}(\Omega)$, where l is an arbitrary integer, $l \ge 1$ (cf. Lemma 1.2 in [15]). For $l \ge \beta + 1$ this function is contained in the space $H^{1+\beta}(\Omega)$ (see Theorem 3 in [16]).

Let u be a solution of the problem (1), (2) and let $\hat{u}_0$ be an extension of u_0, $\hat{u}_0 \in W^l_{-\beta+l-1}(\Omega)$ for $l = 1, 2, \dots$. Then $\Delta \hat{u}_0 - 2h^{-1} (\nabla h, \nabla \hat{u}_0) \in V^{l-2}_{-\beta+l-1}(\Omega)$ for $l = 2, 3, \dots$ and $w = u - \hat{u}_0$ is a solution of the equation

$$\Delta w - 2 \, h^{-1} \, (\nabla h, \nabla w) = (\ell, \nabla h) - \Delta \hat{u}_0 + 2 \, h^{-1} \, (\nabla h, \nabla \hat{u}_0) \tag{25}$$

with the homogeneous Dirichlet condition. Since the right side of (25) belongs to the space $V^{l-2}_{-\beta+l-1}(\Omega)$ for $l = 2, 3, \dots$, we can apply Lemma 3 and obtain the same assertion as in Theorem 2 for the solution u of the boundary value problem (1), (2).

4 Asymptotics of the solution in a neighbourhood of smooth parts of the boundary

Let $u \in H$ be a solution of the boundary value problem (18) and let χ, ψ smooth cut-off functions which are equal to one in a neighbourhood U of any point $x^\circ \in \partial\Omega$ and equal to zero outside of a neighbourhood $U' \supset U$ such that $\chi\psi = \chi$. We suppose again that the function h has the representation (15), where $g \in C^\infty(\Omega)$, $|g(x)| \ge g_0 > 0$. For simplicity, we further assume that $\partial\Omega$ coincides with the x_2-axis in the neighbourhood U'.
The function ψu can be considered as a solution of the equations

$$\Delta(\psi u) - 2h^{-1} \, (\nabla h, \nabla(\psi u)) = F \;\text{ in } \Omega, \quad \psi u = 0 \;\text{ on } \partial\Omega, \tag{26}$$

where $F = \psi(\ell, \nabla h) + 2(\nabla\psi, \nabla u) + u \, \Delta\psi - 2h^{-1} \, (\nabla h, \nabla\psi) \, u$.
We prove at first a regularity assertion for the derivative of ψu in the x_2-direction.

Lemma 5 *If $u \in H = V^1_{-\beta}(\Omega)$ is a solution of (18), then*

$$\frac{\partial(\psi u)}{\partial x_2} \in V^1_{-\beta}(\Omega).$$

Proof: Let v be an arbitrary function in Ω and let $\delta \neq 0$ be an arbitrary real number. Then we define the function v_δ by the equation

$$v_\delta(x_1, x_2) = \delta^{-1} \, ((v(x_1, x_2 + \delta) - v(x_1, x_2)).$$

For sufficiently small δ, the function $(\psi u)_\delta$ satisfies the equations

$$h^{-2}\,\Delta(\psi u)_\delta - 2h^{-3}\,(\nabla h, \nabla(\psi u)_\delta) = h^{-2}\,F_\delta + 2h^{-2}\,((h^{-1}\nabla h)_\delta\,,\,\nabla(\psi u)) \quad \text{in } \Omega, \tag{27}$$

$$(\psi u)_\delta = 0 \qquad \text{on } \partial\Omega. \tag{28}$$

Hence Lemma 1 yields

$$\|(\psi u)_\delta\|_H \le c\,\|h^{-2}F_\delta + 2h^{-2}\,((h^{-1}\nabla h)_\delta\,,\,\nabla(\psi u))\|_{H^*}\,, \tag{29}$$

where H^* denotes the dual space of H. We show that the right side of this inequality can be majorized by the sum of the norm of F in $V^0_{-\delta}(\Omega)$ and the norm of ψu in $V^1_{-\beta}(\Omega)$. For every smooth function v, $\operatorname{supp} v \subset U'$, we have

$$\|v_\delta\|_{V^0_{-\beta}(\Omega)} = \int\limits_\Omega r^{-2\beta}\,|v_\delta|^2\,dx = \int\limits_\Omega x_1^{-2\beta}\,\Big|\int\limits_0^1 \frac{\partial v}{\partial x_2}(x_1, x_2 + t\delta)\,dt\Big|\,dx \le \|v\|^2_{V^1_{-\beta}(\Omega)}\,.$$

Consequently, we obtain

$$\begin{aligned}\Big|\int\limits_\Omega h^{-2}\,F_\delta\,v\,dx\Big| \;&=\; \Big|\int\limits_\Omega F\cdot(h^{-2}\,v)_{-\delta}\,dx\Big| \le c\,\|F\|_{V^0_{-\beta}(\Omega)}\,\Big(\|v\|_{V^0_{-\beta}(\Omega)} + \|v_{-\delta}\|_{V^0_{-\beta}(\Omega)}\Big)\\ &\le\; c\,\|F\|_{V^0_{-\beta}(\Omega)}\;\|v\|_{V^1_{-\beta}(\Omega)}\end{aligned}$$

and therefore, $\|h^{-2}\,F_\delta\|_{H^*} \le c\,\|F\|_{V^0_{-\beta}(\Omega)} < \infty$. Furthermore,

$$\begin{aligned}\|2h^{-2}\,((h^{-1}\nabla h)_\delta\,,\,\nabla(\psi u))\|_{H^*} &\le \|2h^{-2}\,((h^{-1}\nabla h)_\delta\,,\,\nabla(\psi u))\|_{V^0_{\beta+1}(\Omega)}\\ \le\;\; c\Big(\int\limits_\Omega r^{2\beta+2}\,&|r^{-2\beta-1}\,\nabla(\psi u)|^2\,dx\Big)^{1/2} \le c\,\|\psi u\|_{V^1_{-\beta}(\Omega)}\,.\end{aligned}$$

Thus, we get the desired estimate for the right side of (29). The constant in this estimate is independent of δ. By Fatou's lemma we further have

$$\Big\|\frac{\partial(\psi u)}{\partial x_2}\Big\|_{V^1_{-\beta}(\Omega)} = \|\lim_{\delta\to 0}(\psi u)_\delta\|_{V^1_{-\delta}(\Omega)} \le \lim_{\delta\to 0}\,\|(\psi u)_\delta\|_{V^1_{-\beta}(\Omega)}$$

Hence (29) yields the assertion of our lemma. ■

Theorem 3 *Suppose that the function h has the representation*

$$h(x) = g(x_1, x_2)\cdot x_1^\beta$$

in the neighbourhood U' of x°, where $g \in C^\infty(\overline{\Omega})$, $|g(x)| \ge g_0 > 0$. Then the solution $u \in H$ of the boundary value problem (18) admits the decomposition

$$\chi u = \chi\,\ell_1\,g(0, x_2)\,\frac{x_1^{\beta+1}}{\beta+1} + u^{(1)},$$

where $u^{(1)} \in V^1_{-\beta-1}(\Omega)$, if $\beta > 1/2$, and $u^{(1)} \in V^1_{-2\beta-1/2+\varepsilon}(\Omega)$, if $\beta \le 1/2$. Here $\ell_1 = k_A\,W_2$ is the first component of the vector ℓ, and ε is an arbitrary small positive number.

Proof: Rewriting (26) we have

$$\frac{\partial}{\partial x_1}\Big(x_1^{-2\beta}\,\frac{\partial(\psi u)}{\partial x_1}\Big) = x_1^{-2\beta}\,G, \qquad \text{where} \qquad G := F - \frac{\partial^2(\psi u)}{\partial x_2^2} + 2g^{-1}\,(\nabla g, \nabla(\psi u)).$$

Hence

$$\frac{\partial(\psi u)}{\partial x_1} = -x_1^{2\beta} \int\limits_{x_1}^{\infty} \xi^{-2\beta}\, G(\xi, x_2)\, dx + x_1^{2\beta}\, C(x_2). \tag{30}$$

We write $G(x)$ in the form

$$G(x) = \psi(x)\, \ell_1\, \beta\, g(0, x_2)\, x_1^{\beta-1} + G_0(x),$$

where

$$G_0(x) = \psi(x) \left(\ell_1\, \beta\, (g(x_1, x_2) - g(0, x_2))\, x_1^{\beta-1} + (\ell, \nabla g)\, x_1^{\beta} \right)$$
$$+2\, (\nabla\psi, \nabla u) + u\, \Delta\psi - 2\beta\, x_1^{-1} \frac{\partial \psi}{\partial x_1}\, u - 2g^{-1}\, (\nabla g, \nabla\psi)\, u - \frac{\partial^2(\psi u)}{\partial x_2^2} + 2g^{-1}\, \psi\, (\nabla g, \nabla u).$$

Since $\psi u \in V^1_{-\beta}(\Omega)$ and $\partial(\psi u)/\partial x_2 \in V^1_{-\beta}(\Omega)$, the function G_0 belongs to the space $V^0_{-\beta}(\Omega)$. Hence for arbitrary $\varepsilon > 0$ Hardy's inequality yields

$$\int\limits_{\Omega} x_1^{-1+2\max(\varepsilon,\beta-1/2)} \Big| \int\limits_{x_1}^{\infty} \xi^{-2\beta}\, G_0(\xi, x_2)\, d\xi \Big|^2 dx \quad \le \quad c \int\limits_{\Omega} x_1^{1+2\max(\varepsilon,\beta-1/2)} \Big| x_1^{-2\beta}\, G_0(x_1, x_2) \Big|^2 dx$$
$$\le c \int\limits_{\Omega} x_1^{-2\beta}\, |G_0(x_1, x_2)|^2\, dx.$$

Therefore, the function $x_1^{-2\beta}\, G_0(\xi, x_2)$ belongs to the space $V^0_{\kappa}(\Omega)$, where $\kappa = -\beta - 1$, if $\beta > 1/2$, and $\kappa = -2\beta - 1/2 + \varepsilon$, if $\beta \le 1/2$. Let a_1 and a_2 be the following functions:

$$a_1(x_2) = \max\{\xi :\ \psi(\xi, x_2) = 1\}, \qquad a_2(x_2) = \max\{\xi :\ \psi(\xi, x_2) \ne 0\}.$$

Then

$$x_1^{2\beta} \int\limits_{x_1}^{\infty} \beta\, \xi^{\beta-1}\, \psi(\xi, x_2)\, \ell_1\, g(0, x_2)\, d\xi$$
$$= \beta\, \ell_1\, x_1^{2\beta}\, g(0, x_2) \Big(\int\limits_{x_1}^{a_1(x_2)} \xi^{\beta-1}\, d\xi + \int\limits_{a_1(x_2)}^{a_2(x_2)} \psi(\xi, x_2)\, \xi^{\beta-1}\, d\xi \Big)$$
$$= \ell_1\, x_1^{2\beta}\, g(0, x_2)\, (\, -x_1^{\beta} + c_1(x_2)).$$

for all x_2 satisfying the condition $\psi(0, x_2) = 1$. Multiplying (30) by χ we obtain

$$\chi \frac{\partial u}{\partial x_1} = \chi \left(\ell_1\, g_0(0, x_2)\, x_1^{\beta} + c_2(x_2)\, x_1^{2\beta} \right) + R,$$

where $R \in V^0_{\kappa}(\Omega)$. It is evident that $c_2 \in L_2(\partial\Omega \cap U)$, since $\chi\, \partial u/\partial x_1 - R - \ell_1\, g(0, x_2)\, x_1^{\beta} \in V^0_{-\beta}(\Omega)$. This implies $\chi\, c_2(x_2)\, x_1^{2\beta} \in V^0_{-\beta-1}(\Omega)$. Consequently,

$$\chi\, \frac{\partial u}{\partial x_1} = \chi\, \ell_1\, g(0, x_2)\, x_1^{\beta} + R_1\,,$$

where $R_1 \in V^0_{\kappa}(\Omega)$. Using Lemma 5 from this we can conclude that

$$\nabla \left(\chi u - \chi\, \ell_1\, g(0, x_2)\, \frac{x_1^{\beta+1}}{\beta+1} \right) \in V^0_{\kappa}(\Omega).$$

Since $\chi u - \chi\, \ell_1\, g(0, x_2)\, x_1^{\beta+1}/(\beta+1) = 0$ on $\partial\Omega$, Hardy's inequality implies

$$\Big\| \chi u - \chi\, \ell_1\, g(0, x_2)\, \frac{x_1^{\beta+1}}{\beta+1} \Big\|_{V^0_{\kappa-1}(\Omega)} \le c \Big\| \nabla (\chi u - \chi\, \ell_1\, g(0, x_2) \frac{x_1^{\beta+1}}{\beta+1} \Big\|_{V^0_{\kappa}(\Omega)}.$$

Thus, $\chi u - \chi\, \ell_1\, g(0, x_2)\, x_1^{\beta+1}/(\beta+1) \in V^1_{\kappa}(\Omega)$. This proves the theorem. ■

5 Concluding Remarks

The variational formulation (12) was first used for a FEM solution in [9, 7] and [12]. Refined versions have been developped in [3, 6]. Recently, inverse problems have been concidered in [18, 6, 8]. For numerical solutions and applications we refer to [12] and [8]. The Neumann problem and error bounds will be considered in a forthcoming paper.

6 Appendix: The Ekman model

In an attempt to keep the paper self-contained we are going to outline the derivation of the PDE (1) from continuum physics. However, for details compare [5, 17] and the cited papers there. As our starting point we take the general 3-D turbulent Navier-Stokes equations

$$v_t + v\nabla v = G - \nabla p - 2\omega \times v + \operatorname{div}(A\nabla v) \tag{31}$$

in the 3-dimensional domain

$$\Omega^{3d} = \{(x,z) \in R^2 \times R \,:\, x \in \Omega\,,\, \xi(x) > z > -h(x)\}\ . \tag{32}$$

Here we denoted v - the (3 dimensional) velocity field, $G = (0,0,-g)^T$ - the gravitation field, ω - the spin of Earth rotation, p - the pressure, ξ - the (a priori unknown) free surface and $A = \operatorname{diag}(A_H, A_H, A_V)$ the tensor of turbulent impulse transition.
In this section we use the symbols v, ∇ and div regardless of whether they are applied in the 3-dimensional case or in the plane case. In the 3D case z is identified with x_3.
Using Einstein's convention, equation (31) may be re-written as

$$\frac{\partial v_i}{t} + v_j \frac{v_i}{x_j} = G_i - \frac{\partial p}{\partial x_i} - 2\epsilon_{ijk}\omega_j v_k + \frac{\partial}{x_k}\left(A_{kj}\frac{\partial v_i}{\partial x_j}\right)\,,$$

$i = 1,2,3$, with $x_3 = z$, $A_{11} = A_{22} = A_H$, $A_{33} = A_V$, $A_{ij} = 0$ for $i \neq j$, and Riccati's tensor ϵ.
Equation (31) is completed by the incompressibility condition

$$\operatorname{div} v = 0\,. \tag{33}$$

Here and in (31) the density is assumed constant and equal to 1.
As boundary conditions we assume on $\partial\Omega^{3d}$

$$v(x, -h(x)) = 0 \tag{34}$$

(the so called "no slip" condition at the ground), and

$$A_V \frac{\partial v_i}{\partial y}(x, \xi(x)) = \kappa |W| W_i\,, \qquad i = 1,2\,. \tag{35}$$

Here $W = (W_1, W_2)$ is the *wind vector* (usually taken at a height of $z = \xi + 10$ (in meters)), and $k_A > 0$ is an empirical constant describing the shear stress induced by the wind. Equation (35) expresses the continuity of the normal impulse flux through the interface between air and water, cf. [10].
Now, following [5, 11, 17], several assumptions are introduced which lead to a scalar linear elliptic partial differential equation in the 2-dimensional domain Ω. We look for stationary solutions with small accelerations, and further we consider only water bodies with a very small depth to length ratio. Thus we neglect v_t, $v\nabla v$, $\omega \times v$, A_H as being small in comparison with the remaining terms.

Following [5, 17, 11] we also assume A_V to depend only on x and not on z. Hence, we obtain from the third component of (31)

$$p(x,z) = g(\xi - z)\,, \qquad \frac{\partial^2 v_i}{\partial z} = \frac{g}{A_V}\frac{\partial \xi}{x_i}\,, \qquad i = 1,2\,. \tag{36}$$

Now, by integrating (36.2) twice and substituting (34) and (35), we obtain an explicit expression for v_1 and v_2 dependent on $\nabla\xi$ and z.
Integrating once more, and denoting for $x \in \Omega$ the *velocity of mass transport*

$$V(x) = (V_1, V_2)^T\;, \qquad V_i(x) = \int_{-h(x)}^{\xi(x)} v(z)\,dz\,, \qquad i = 1,2 \tag{37}$$

we arrive after elementary calculations at

$$V = \frac{\kappa}{2A_V}(\xi + h)^2|W|W - \frac{g}{3A_V}(\xi + h)^3\nabla\xi\,.$$

Finally, we assume $\xi << h$ and obtain

$$V = \frac{\kappa|W|}{2A_V}h^2 W - \frac{gh^3}{3A_V}\nabla\xi \tag{38}$$

or equivalently

$$\nabla\xi = \frac{3\kappa}{2gh}W - \frac{3A_V}{gh^3}V\,.$$

Note that due to (33) the 2D field V is again divergence free,

$$\operatorname{div} V = 0 \qquad \text{in } \Omega\,. \tag{39}$$

As a constitutive assumption in [5] the empirical formula

$$A_V = A_V^0|W|h \tag{40}$$

was recommended. Assuming this we obtain

$$\nabla\xi = \frac{3\kappa}{2gh}|W| - \frac{k_B|W|}{h^2}V \tag{41}$$

with $k_B = 3A_V^0/g$.
We observe that the velocity of mass transport V is a positive combination of directions of wind and of steepest descent of the surface level. We could have introduced the equation (41) as a phenomenological assumption for wind-driven flow processes instead deriving it from (31)–(35), but this approach explains the form of the coefficient functions.
Now equation (1) turns out to be the integrability condition for the first order PDE (41). In order to make this conspicuous we observe that for sufficiently smooth functions the operator $D := \operatorname{div} Q\nabla$ vanishes. Hence, multiplying (41) by Q and taking the divergence, we obtain

$$???\,. \tag{42}$$

We introduce the streamfunction u by

$$V = Q\nabla u\,. \tag{43}$$

Now, for W constant on Ω, (42) reduces to (1). From (43) we obtain for the boundary values u_0 the interpretation

$$u_0(x) = \int q(s)ds \tag{44}$$

where the integration is carried out along the boundary $\partial\Omega$ and q is the mass flow through the boundary.

Remark 4 *We are not interested here in local velocities $v = v(x,z)$, however they can be derived from (34)–(36) (v_1 and v_2), respectively from (33) (for v_3) by integration over $[-h(x), z]$, cf. [5, 11, 17].*

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