On the inverse problem for the Ekman equation

Kurt Frischmuth and Michael Hänler

We consider inverse problems for the partial differential equation

$$\Delta u - 2h^{-1}(\nabla h, \nabla u) = g\,.$$

We assume that the unknown function u – the streamfunction for a class of flow problems – is constant on certain parts of the boundary $\partial\Omega$ while at the remaining parts no boundary condition is imposed.
Instead, values of the gradient of u (velocities) are prescribed at a finite set M of inner points (measuring points).
We look for the minimizer of a functional being a combination of the squared errors at the measuring points and – for regularization – a squared $H^{1/2}$-norm of the (unknown) boundary values.
We present numerical solutions to the problem. Especially, the dependence on the regularization parameter μ is discussed.

Keywords: Elliptic boundary value problems, inverse problems
AMS subject classification: 35R30, 65N30

1 Introduction

In modelling of flows in lakes and estuaries the partial differential equation

$$\Delta u - 2h^{-1}(\nabla h, \nabla u) = g \tag{1}$$

is frequently used, cf. [2, 7, 8, 10]. Here we denote

- Ω a two-dimensional domain,
- h the (given) depth profile,
- u the (unknown) stream function,
- $(\cdot\,,\,\cdot)$ the scalar product in $\mathbb{R}^2$
- $g = k_A(\nabla h, Qw)$ the wind force with
  - k_A an empirical parameter,
  - w the wind vector and
  - Q a clockwise rotation by a right angle.

As a real-life application, the following domain $\Omega \subset \mathbb{R}^2$ and depth profile

$$h : \Omega \longrightarrow \mathbb{R}^{++}$$

are considered:

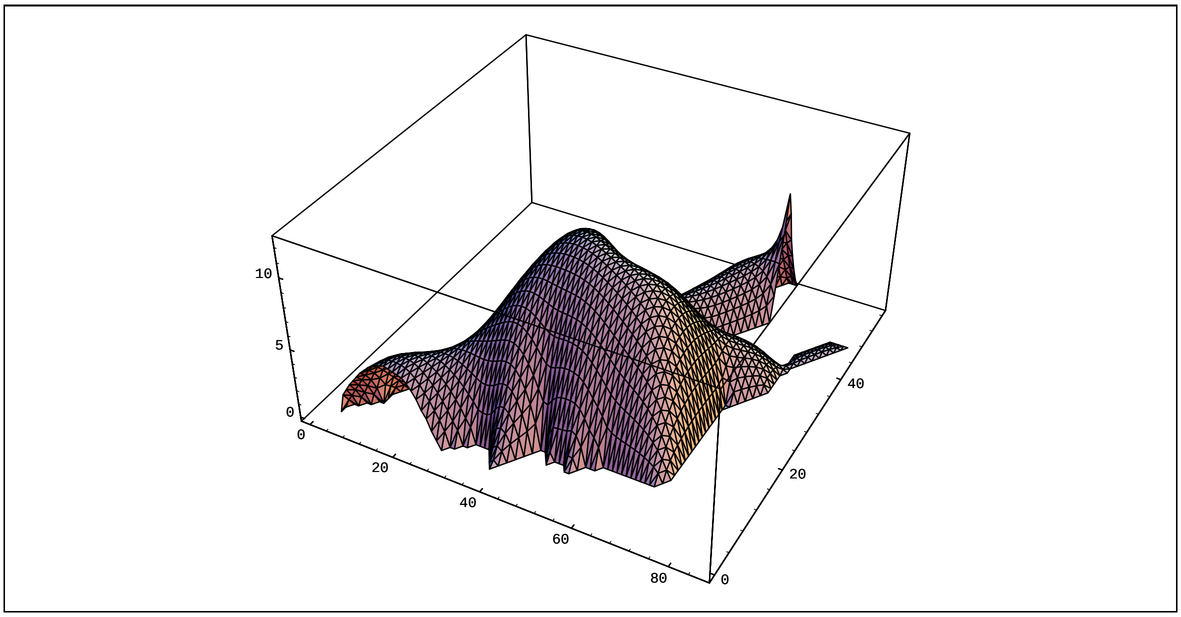

Figure 1: Depth profile h on Ω

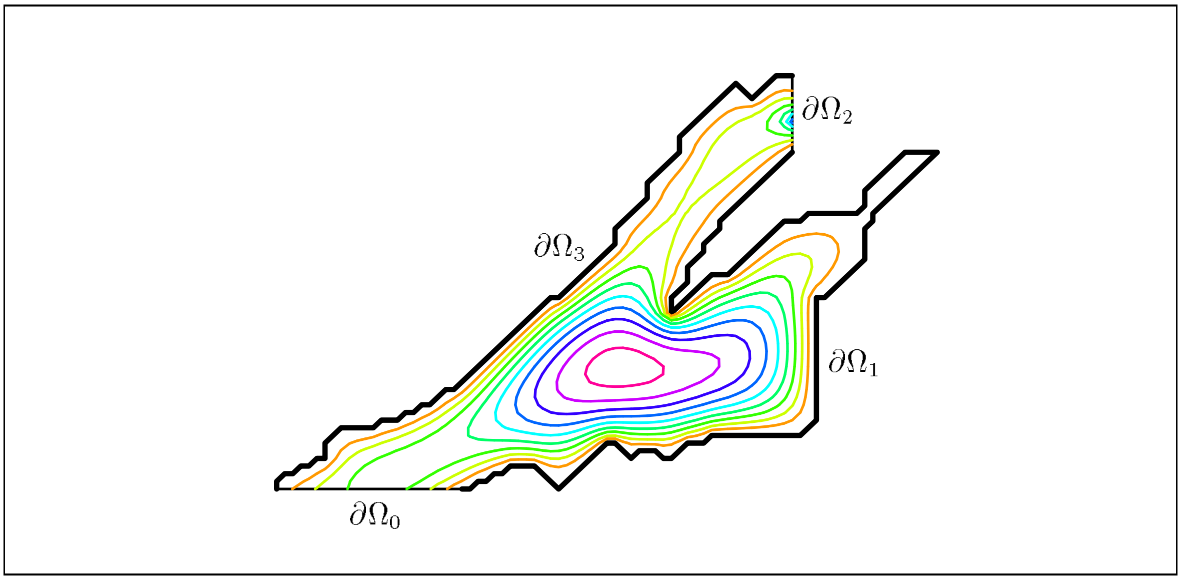

Figure 2: Isolines of h

(Similarities to *Kirrbucht* in the southern Baltic are not accidental.)

For our theoretical considerations we assume throughout this paper that the geometrical data Ω and h are sufficiently smooth. Let in particular Ω be of the Hölder class $\mathcal{C}^{0,1}$ and $h \in C^2(\Omega) \cap C(\overline{\Omega})$. This immediately implies $g \in C^1(\Omega) \subset H^1(\Omega)$. Methods for the

numerical solution of the Dirichlet problem and applications of the model are studied in [1, 10, 4]. For the Dirichlet problem existence and uniqueness of the solution are proven in a weighted Sobolev space [4], asymptotics near the boundary and regularity depend on the behavior of h, cf. [9].
In realistic problems the determination of boundary values is rather difficult. Along parts of the boundary u is constant ("beach"), on other parts ("open sea") there is almost no information on u available. Instead, measurements of velocities $v = Q\,\nabla\, u$ are given on a finite subset $M \subset \Omega$ (positions of probes, cf. [3]).
The aim of the present paper is the numerical approximation of the streamfunction u from this type of mixed data.

2 Formulation of the mixed problem

We assume a decomposition of $\partial\Omega$ into connected arcs $\partial\Omega_0$, $\partial\Omega_1 \ldots \partial\Omega_{2k-1}$ to be given. For the Ekman model, the boundary parts with even index are usually the connected components of $\{x \,:\, h\,(x) = 0\}$.
The streamfunction u is subject to the condition

$$u_{|\partial\Omega_{2l}} = \text{const},\ l = 0 \ldots k-1\,. \tag{2}$$

On the remaining pieces of $\partial\Omega$ no boundary condition is given.
We assume that on a finite set $M \subset \Omega$ values z_x of $\nabla u(x)$ are given (measured and rotated velocities at positions of probes).
Let us consider the functional

$$\phi(u) := \sum_{x\in M} |z_x - \nabla u(x)|^2 + \mu\, \|u_{|\partial\Omega}\|^2\,, \quad (\mu > 0) \tag{3}$$

on the set of all weak solutions of (1), (2) in the weighted Sobolev space

$$H = H^1_h(\Omega) := \{u \in H^1(\Omega) \,:\, h^{-1}\nabla u \in L^2(\Omega)\}\,,$$

where $\|u_{|\partial\Omega}\|$ denotes a suitable semi-norm in $H^{1/2}(\partial\Omega)$. We define

$$a(u,\,v) \;=\; \int_\Omega \frac{(\nabla\, u,\,\nabla\, v)}{h^2}\,dx \tag{4}$$

$$b(v) \;=\; \int_\Omega \frac{g\,v}{h^2}\,dx\,. \tag{5}$$

Then, the equation $a(u,\,v) = b(v)\ \ \forall v \in H^1_0$ is the weak formulation of the divergence form of (1), cf. [4] and (7) of the next section. Without loss of generality we assume $b(v) = 0$. Further, we denote

$$H^{pc} := \{u \in H \,:\, tr(u, \partial\Omega_{2l}) = \text{const},\ l = 0 \ldots k-1\}\,.$$

Remark 1 *It can be shown that $H = H^{pc}$, if there exists an $\alpha \geq 1/2$, s. t. $h(x) \approx$ dist $(x, \partial\Omega_{2l})^\alpha$ as dist $(x, \partial\Omega_{2l}) \to 0$.*

Now, the domain of the functional (3) is $D = \left\{u \in H \,:\, a(u,\,v) = 0 \quad \forall v \in \mathring{H}\right\}$, with $\mathring{H} = H \cap H^1_0$.

Lemma 1 *The domain D is a closed subspace of H. Furthermore, the functional (3) is well defined on D and convex.*

Proof: The bilinear form a is coercive, continuous and symmetric on H (cf. [4]). Hence, D is a closed linear subspace of H. By a classical theorem due to Weyl (cf. [12]) weak solutions of (1) are in $H^3(\Omega')$, for any $\Omega' \subset\subset \Omega$. Every convergent sequence in D is weakly convergent in $H^3(\Omega')$. From Sobolev's imbedding theorem it follows that the first sum in (3) is defined and continous on D. For the second term, we use the semi-norm

$$\|u\|_{1/2}^2 := \int_{\partial\Omega}\int_{\partial\Omega} \frac{(u(x)-u(y))^2}{|x-y|}\,dx\,dy \tag{6}$$

which is well defined on $D \subset H \subset H^1$.
From

$$\phi\left(\frac{u_1+u_2}{2}\right) = \frac{1}{2}\left(\phi(u_1(x)) + \phi(u_2(x)) - \phi_0(u_1(x)-u_2(x))\right)$$

follows the convexity of the functional, where ϕ_0 is the value of ϕ for homogeneous measuring values $z_x = 0\ \forall x \in M$. ■

Remark 2 *In the quotient space $D/\mathbb{R}$ the functional (3) is uniformly convex. Using the properties of the bilinear form $a(\cdot,\cdot)$ it is easy to show that ϕ_0 is a norm on $D/\mathbb{R}$ in the case $\mu > 0$. This immediately implies that there exists a unique minimizer for ϕ in $D/\mathbb{R}$.*
On the other hand, for $\mu = 0$ there is no minimizer of (3) in the general case, cf. [6].

3 Results

We study numerical solutions to the above identification problem. To this end we use a finite difference scheme based on the following equation which is equivalent to (1)

$$\operatorname{div}(h^{-2}\nabla u) = h^{-2} g\,. \tag{7}$$

This formulation leads to a symmetric, positive definite, sparse system of linear equations. For our applications (500 ... 10000 equations) SOR has been applied.
In a first step, we generated a basis of the discretized domain

$$D^{disc} = \left\{ u^{disc} : u^{disc} = \text{ const on } \partial\Omega^{disc} \text{ and } u^{disc} \text{ satisfies the discretized eqn. (7)} \right\}.$$

Next, the discretized functional ϕ^{disc} was defined as a function of the coefficients U_n, $n = 1, \ldots N$, of u^{disc} in that basis

$$\phi^{disc}(u^{disc}) = \|L\,U \; - c\|_{\mathbb{R}^K}^2\,.$$

The first $2\,|M|$ rows of the matrix L correspond to the numerical differentiation at the measuring points $x \in M$, the remaining rows correspond to a discretized version of (6). The vector c consists of the reordered entries of z and zeros,

$$c = \left(z_{x_1 1}, z_{x_1 2}, z_{x_2 1}, z_{x_2 2}, \ldots z_{x_{|M|} 1}, z_{x_{|M|} 2}, 0, 0, 0 \ldots 0\right)^T.$$

For our examples, we had $N = 34$, $|M| = 25$ and $K = 84$. The minimizer of ϕ^{disc} was calculated by solving the overdetermined system of linear equations

$$L\,U = c\,.$$

Numerical experiments have been performed starting with Dirichlet conditions and $g = 0$. The following streamfunction was obtained:

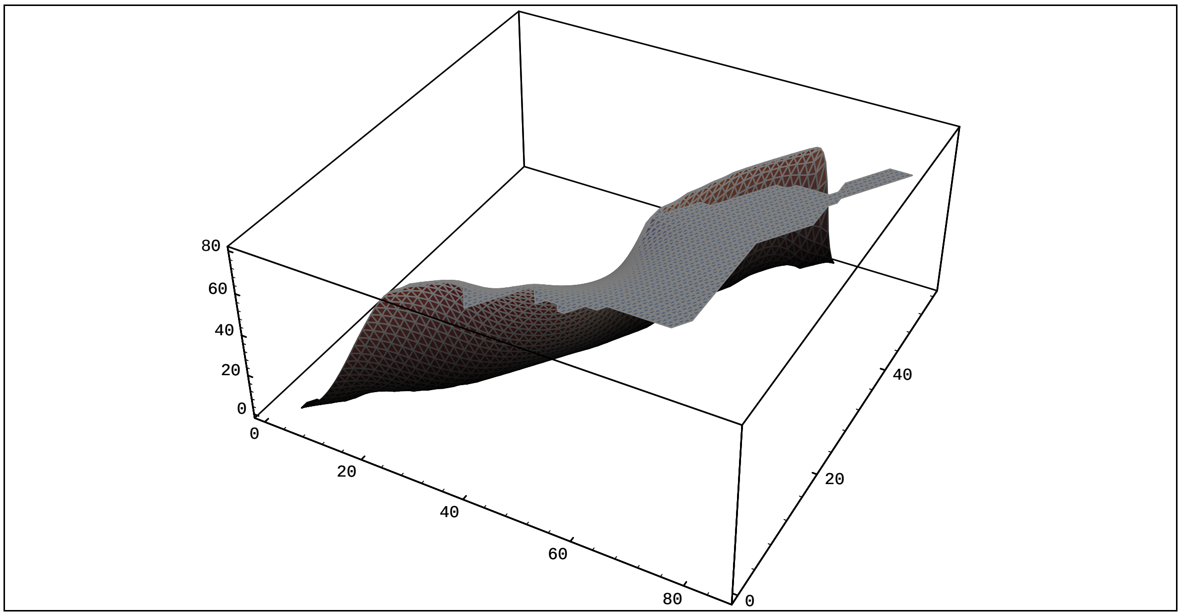

Figure 3: Undisturbed streamfunction u on Ω

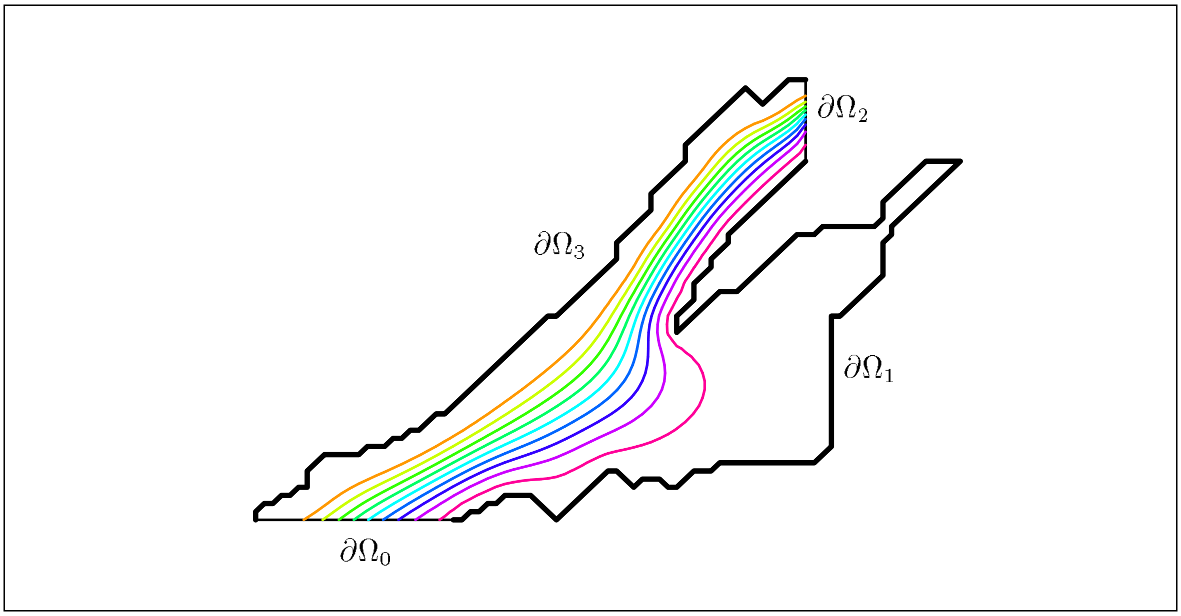

Figure 4: Undisturbed streamfunction u on Ω - streamlines

Next, the numerical derivatives at the measuring points have been calculated and a small random error has been added (vanishing expected value, standard deviation=0, 10^{-4}, 10^{-2}). With these values we formed a vector z and solved the minimization problem for ϕ. The result is compared with the original undisturbed streamfunction. The difference between original and re-calculated streamfunction depends, of course, on the regularization parameter μ. For Figures 5 and 6 we chose the standard deviation to be 10^{-4}.

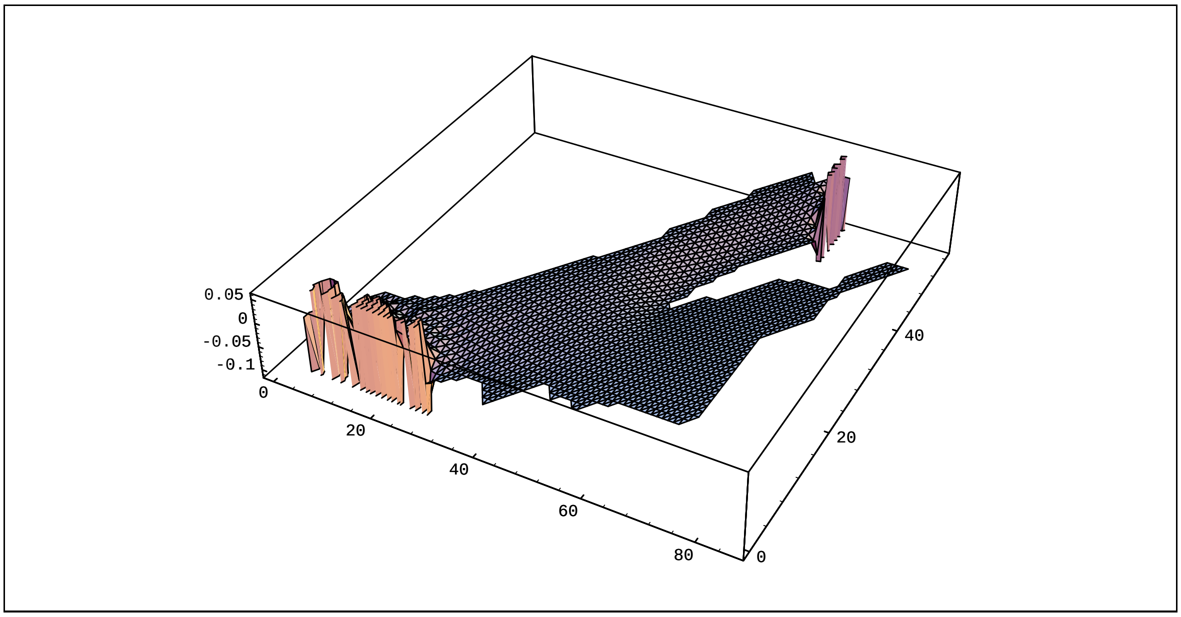

Figure 5: Error for $\mu = 10^{-6}$

Observe that the error is concentrated along the boundary parts with odd indices.

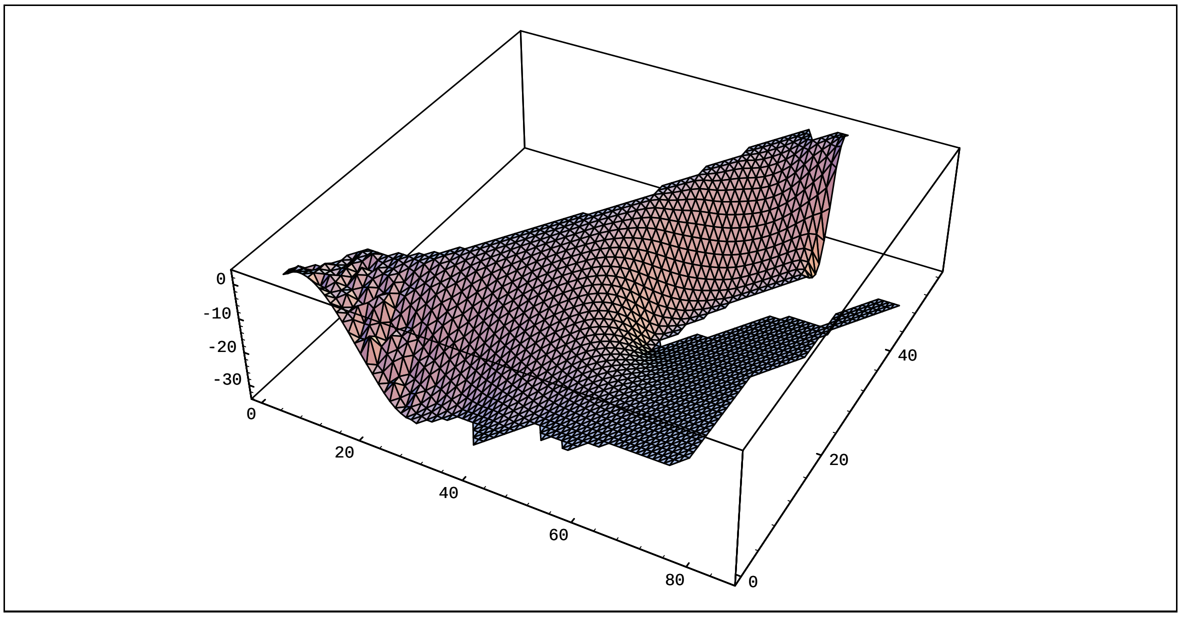

Figure 6: Error for $\mu = 1$

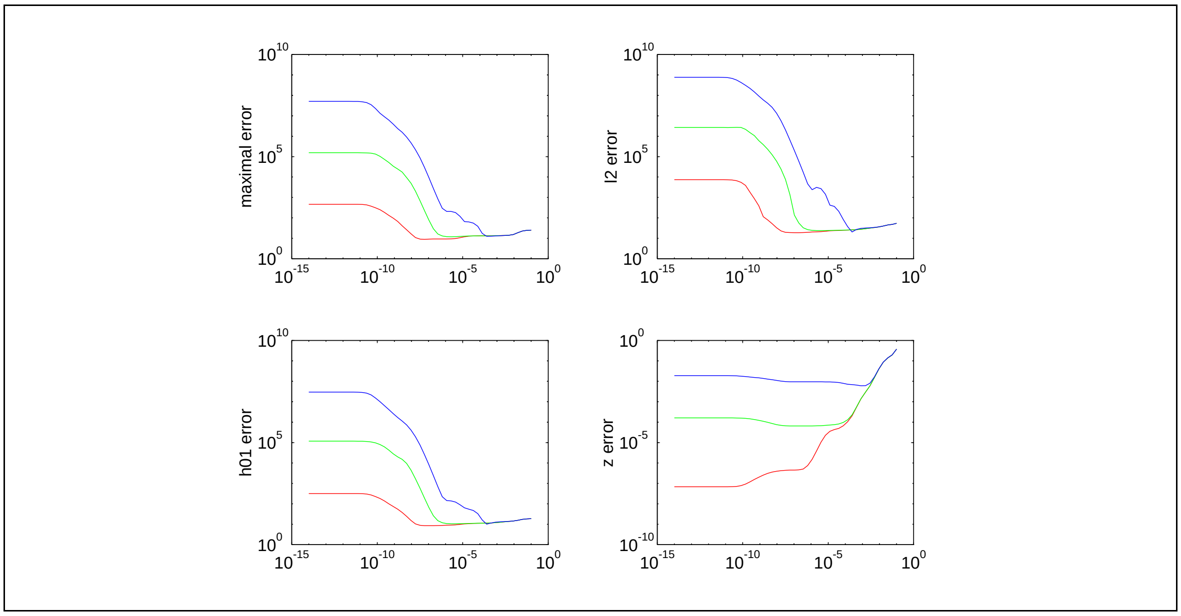

Figure 7: Error vs. μ

A plot of several error norms of this difference versus μ allows us to determine $\mu = 10^{-3}$ as a preferable value. In Figure 7 the three curves correspond to the considered levels of standard deviation.

Remark 3 *Obviously, the performance of the identification procedure depends on the domain Ω, the choice of the set M and the underlying differential equation (cf. [5]). This dependence is especially important for the Ekman equation – due to the depth profile. Hence, an optimization of the choice of M should be considered (cf. [3]).*

4 Conclusion

For "reasonable sets" of measuring points M we are able to reobtain solutions to (1) from disturbed values of their gradients at M with an error comparable to the simulated error of the measurement. The approximation becomes better (by several orders) outside a vicinity of the boundary parts with odd indices.
In our approach the domain of ϕ consists only of weak solutions to (1). Alternatively, one can minimize the sum of (3) and the energy functional of (7) on a broader domain, cf. [11]. In a forthcoming paper we will compare both approaches.

References

[1] Barannik, W. A., and N. J. Seljuk: *Calculation of wind-driven flow of cooling water* (in Russian). Problemy Ochrony Vod, **8** (1977), 49-63.

[2] Felsenbaum, A. I.: *Theoretical foundations and calculation methods for stationary marine flows.* (in Russian). Moskva: Isdat. AN SSSR, 1960.

[3] Frischmuth, K.: *An optimality problem in the identification of the hydrodynamic Ekman-model.* In: Internationale Tagung Mathematische Optimierung – Theorie und Anwendungen, Eisenach, 11.–15. Dezember, 1989.

[4] Frischmuth, K.: *Zur Modellierung und numerischen Lösung von Konfigurations-Fluß-Problemen.* Habilitationsschrift, Universität Rostock, 1990.

[5] Frischmuth, K., and M. Hänler: *Identification of boundary data for the Ekman model.* Proc. of GAMM-SIAM Conference on Inverse Problems in Diffusion Processes, St. Wolfgang, 1994.

[6] Hamann, U.: *Approximation durch Lösungen elliptischer Randwertprobleme auf offenen Mengen II.* Z. Anal. Anw. **10** (1991) 49-61.

[7] Jankowski, A.: *Matematyczne modelowanie pola przepływów w morzu Bałtyckim* (in Polish). Wrocław: Ossolineum, 1988.

[8] Koyalo, M. V., Martuzans, B. Ya., Frischmuth, K., and R. Strauß: *On the calculation of wind driven flows in shallow water* (in Russian). Isv. AN Latp1. SSR, Ser. fizicheskikh i tekhnicheskikh nauk, 1 (1990), 73–77.

[9] Roßmann, J.: *On two classes of weighted Sobolev-Slobodetskiĭ spaces in a dihedral angle.* Warszawa: Banach Center Publications, Volume 27, Polish Academy of Sciences, 1992.

[10] Stoyan, G., Müller, W., and H. Baumert: *Numerische Simulation von wind- und durchflußinduzierten Strömungen in Flachgewässern auf der Basis des Ekman-Modells (Rechenprogramm LAKE).* Acta Hydrophysica, Bd. XXX (1986), 51-67.

[11] Voigtländer, C.: *Verallgemeinerte Spline-Approximation beim Ekman-Modell der Hydrodynamik.* Diplomarbeit, Universität Rostock, 1992.

[12] Wloka, J.: *Partielle Differentialgleichungen.* Stuttgart: Teubner, 1982.

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