Iterative Solution of Convex Problems by Fejer-monotone Methods

Dieter Schott

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Fachbereich Elektrotechnik, Hochschule Wismar,
Philipp-Müller-Straße , PF 1210, D-23952 Wismar, Germany

Summary. *Let H be a Hilbert space and M be an implicitely given nonempty, convex and closed subset (solution set of a convex problem). Besides let $Q \supseteq M$ be a further convex and closed subset (feasible domain, constraints). We consider iterative methods*

$$x_{k+1} \in g_k(x_k) \ , \quad x_0 \in Q$$

with point-to-set mappings $g_k : Q \to \mathbb{P}(Q)$, which are Fejer-monotone relative to M (or M-Fejer-monotone). We present conditions ensuring strong convergence of (x_k) to a fixed element x^* in M. Finally we give applications of the results to various types of convex problems.

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1 Introduction

The theory of Fejer-monotone sequences and mappings has a long history . It started already in the sixteeth of our century and was closely connected with the Russian mathematician Eremin ([5],[6]). The monography [7] contains a summarizing description for finite-dimensional spaces with various applications to the iterative solution of convex problems. But up to now you can find papers again and again using Fejer-methods for different problems without relating it to the original basic literature (see e.g. [4],[9],[20]). The great interest in these methods is caused by the wide range of applications. At present they play an important part in computerized tomography and image recovery (see e.g. [21]). Nevertheless there remain open fields for further investigations. My paper [16] supplies a unifying concept for the application of Fejer-methods on the level of Hilbert space assuming somewhat different conditions than in [7].

According to [7:p.49] iterative methods

$$x_{k+1} \in g(x_k), \quad x_0 \in I\!R^n, \quad g : I\!R^n \to \mathbb{P}(I\!R^n), \quad n \in I\!N \tag{1.1}$$

converge to a certain element x^* of a nonempty, convex and closed subset M if g is a closed and (in our terminology a strictly) M-Fejer-monotone mapping. This result can easily be transfered to Hilbert spaces. It suffices to suppose additionally that the bounded iterative sequence (x_k) has strong accumulation values. Then also the strong convergence of (x_k) to $x^* \in M$ follows. But on the one hand it is difficult in general to check this condition, and on the other hand we can not even prove weak convergence if this condition is omitted. We get a modified result by observing the fact that a M-Fejer-monotone sequence (x_k) converges strongly to a certain element in M iff the distance $\rho(x_k, M)$ tends to 0.

The paper [16] contains convergence conditions for iterative methods

$$\begin{array}{l} x_{k+1} = T_k(x_k - \lambda_k t_k), \quad x_0 \in H, \quad \lambda_k \in I\!R, \quad t_k \in H, \\ T_k : H \to H \text{ nonexpansive }, \quad k \in I\!N \end{array} \tag{1.2}$$

based on relations between the sequences (x_k) and (t_k) such that (x_k) is Fejer-monotone relative to a certain subset M for appropriate parameters $\lambda_k \in \Lambda \subseteq I\!R$. Here we prove a slight generalization and strengthening of one result supposing T_k to be M-Fejer-monotone instead of nonexpansive. Then we can formulate conditions which guarantee that iterative methods

$$\begin{array}{l} x_{k+1} \in T(x_k - \Lambda_k t(x_k)), \quad x_0 \in Q \subseteq H, \quad \Lambda_k \subseteq I\!R, \quad k \in I\!N, \\ t : Q \to \mathbb{P}(Q) \ , \quad T : Q \to \mathbb{P}(Q) \end{array} \tag{1.3}$$

converge strongly to a fixed element x^* in M. In [16] only weak convergence was stated. The conditions imply the mappings

$$g_k : Q \to \mathbb{P}(Q) \, , \quad g_k(x) = T(x - \Lambda_k t(x))$$

to be M-Fejer-monotone. So the stationary method (1.1) is replaced by an instationary method (1.3) containing relaxation sets Λ_k, but the mappings g_k are more special than the mapping g. Specialyzing the mappings T and t in (1.3) convergent iterative methods arise for concrete convex problem sets M. In this way we get results which are partly known from the literature but partly also new or more general.

2 Fejer-monotone sequences and mappings

This section contains a short survey of basic definitions, examples and simple consequences (see also [19]).
Let H be a Hilbert space. We consider a sequence (x_k) in H. Its set of (strong) accumulation values is denoted by $A(x_k)$.

Definition 2.1. Let M be a nonempty subset of H. The sequence (x_k) is said to be *M-Fejer-monotone* iff

$$\|x_{k+1} - x\| \leq \|x_k - x\| \quad \forall k \in I\!N\,,\, \forall x \in M.$$

It is said to be *strictly M-Fejer-monotone* iff additionally

$$\|x_{k+1} - x\| < \|x_k - x\| \quad \forall k \in I\!N\,:\, x_k \notin M\,,\, \forall x \in M.$$

It is called *(strictly) Fejer-monotone* iff it is (strictly) M-Fejer-monotone for any subset M. The set

$$C(x_k) = \{x \in H\,:\, \|x_{k+1} - x\| \leq \|x_k - x\| \quad \forall k \in I\!N\}$$

is called the *Fejer-carrier* of (x_k).

Lemma 2.2. *Let M be nonempty and (x_k) be M-Fejer-monotone. Then the following statements hold:*

a) (x_k) *is bounded.*

b) $\|x' - x\| = \inf_k \|x_k - x\| = \lim_k \|x_k - x\| \quad \forall x' \in A(x_k), \forall x \in M.$

c) $A(x_k)$ *ist a singleton in* M *iff* $A(x_k) \cap M \neq \emptyset$.

Lemma 2.3. *Let M be nonempty, convex and closed. Further, let (x_k) be M-Fejer-monotone. Then the following conditions are equivalent:*

a) $A(x_k)$ *is a singleton in* M.

b) $\lim_k \rho(x_k, M) = 0$.

Moreover, under a) or b) the estimates

$$\|P_M x_k - x'\| \leq \rho(x_k, M)\;,\; \|x_k - x'\| \leq 2\rho(x_k, M)$$

hold, where P_M is the metric projector onto M and $x' \in M$ is the limit of (x_k).

Now we consider a nonempty, convex and closed subset Q of H and a point-to-set mapping $g\,:\, Q \to \mathbb{P}(Q)$, where $\mathbb{P}(Q)$ contains all nonempty subsets of Q. As usual mappings (operators) $g\,:\, Q \to Q$ are integrated as imbeddings.

Definition 2.4: Let M be a nonempty subset of Q. Then g is said to be *M-Fejer-monotone* ($g \in \mathbb{F}(M)$) iff

$$\|z - x\| \leq \|y - x\| \quad \forall x \in M\,, \forall y \in Q\,, \forall z \in g(y).$$

It is said to be *strictly M-Fejer-monotone* ($g \in \mathbb{F}_<(M)$) iff additionally

$$\|z - x\| < \|y - x\| \quad \forall x \in M\,, \forall y \in Q \setminus M\,, \forall z \in g(y).$$

It is called *(strictly) Fejer-monotone* iff it is (strictly) M-Fejer-monotone for any M. The set

$$C(g) = \{x \in H \,:\, \|z - x\| \leq \|y - x\| \quad \forall y \in Q\,, \forall z \in g(y)\}$$

is called the *Fejer-carrier* of g.

Remarks 2.5.

1. The above definitions are slight modifications of concepts given in [7], [4]. In [7] a strictly Fejer-monotone mapping is said to be Fejer-monotone. In [4] a strictly Fejer-monotone continuous operator g is called *paracontraction.*

2. For $g : Q \to \mathbb{P}(Q)$ we introduce the fixed point sets
$$F_-(g) = \{x \in Q \,:\, x \in g(x)\}\,,\; F_+(g) = \{x \in Q \,:\, \{x\} = g(x)\} \quad ,$$
which coincide for $g : Q \to Q$ with $F(g) = \{x \in Q \,:\, x = g(x)\}$. The Fejer-carrier $C(g)$ of g is always convex and closed. Besides we get for $g \in \mathbb{F}(M)$ the relations $M \subseteq C(g) \subseteq F_+(g) \subseteq F_-(g)$. If also $F_-(g) \subseteq M$ holds, we have $M = C(g) = F_+(g) = F_-(g)$. Such mappings g are called *regular* ($g \in \mathbb{F}_r(M)$). All mappings $g \in F_<(M)$ are regular, but the reversion is not true.

3. Let the sequence (x_k) be generated by $x_{k+1} \in g(x_k)$. Then (x_k) is (strictly) M-Fejer-monotone if g is (strictly) M-Fejer-monotone.

Examples 2.6.

1. Let $g : Q \to Q$ be a nonexpansive operator with nonempty fixed point set $F(g)$. Since Q is supposed to be nonempty, convex and closed, $F(g) \neq \emptyset$ holds for bounded Q according to the fixed point theorem of BROWDER, GOEDE and KIRK (see e.g. [22: p.115]). Then we have $g \in \mathbb{F}_r(M)$ for $M = F(g)$.
If g is even *strictly nonexpansive*, that means
$$\|g(y) - g(x)\| < \|y - x\| \quad \text{or} \quad g(y) - g(x) = y - x \quad \forall x, y \in Q\,,$$
then $g \in \mathbb{F}_<(M)$ (see [4]). For instance the metric projector P_M onto a convex and closed set $M \subseteq Q$ is strictly nonexpansive. Also contractive operators g are strictly nonexpansive.

2. For $g : Q \to \mathbb{P}(Q)$ we define g_μ by

$$g_\mu(y) = (1-\mu)y + \mu g(y) \ , \quad \text{where} \quad 0 < \mu < 1.$$

Then $g \in \mathbb{F}_r(M)$ implies $g_\mu \in \mathbb{F}_<(M)$.

3. For $g_1, g_2 : Q \to \mathbb{P}(Q)$ let be

$$(g_1 \circ g_2)(y) = \{x \in Q : x \in g_1(z) \, , \, z \in g_2(y)\}.$$

Then $g_1 \in \mathbb{F}(M)$, $g_2 \in \mathbb{F}_<(M)$ leads to $g_1 \circ g_2 \in \mathbb{F}_<(M)$, $g_2 \circ g_1 \in \mathbb{F}_<(M)$. Further, if $g_1, g_2 \in \mathbb{F}_{(<)}(M)$, then $g_1 \cup g_2 \in \mathbb{F}_{(<)}(M)$. Finally, if $g_1 \in \mathbb{F}_{(<)}(M)$ and $g_2(x) \subseteq g_1(x)$ for all $x \in Q$, then $g_2 \in \mathbb{F}_{(<)}(M)$.

3 Iterative methods with Fejer-monotone mappings

We suppose a nonempty, convex and closed subset Q of the Hilbert space H as well as a further Hilbert space H'. Besides let $f : Q \to \mathbb{P}(H')$ be any point-to-set mapping.

Definition 3.1. The mapping f is said to be *closed* iff the graph of f is closed, that is

$$lim_k x_k = x' \in Q \ , \ y_k \in f(x_k) \ , \ lim_k y_k = y' \in H' \Longrightarrow y' \in f(x').$$

Remark 3.2. Upper semi-continuous mappings f are closed (see [1:p.411]). Especially continuous operators $f : Q \to H'$ are closed.

Now we turn to mappings $g : Q \to \mathbb{P}(Q)$.

Method 3.3: $x_0 \in \mathbb{Q}$, $x_{k+1} \in g(x_k)$, $k \in I\!N$.

Theorem 3.4. *Let the following assumptions be fulfilled:*

(A1) $M \neq \emptyset$, $M \subseteq Q$,
(A2) *g strictly M-Fejer-monotone* ,
(A3) *g closed* ,
(A4) *Method* 3.3 *generates a sequence* (x_k) *with* $A(x_k) \neq \emptyset$.

Then (x_k) converges to an element x' in M.

Proof. (A1) is necessary for (A2). As mentioned in Remark 2.5.3 (x_k) is strictly M-Fejer-monotone by (A2). By (A4) the set $A(x_k)$ of accumulation values is nonempty. Since x_k lies in the closed set Q for all k , $A(x_k)$ is a subset of Q. Hence there exists a subsequence $(x_{k'})$ of (x_k) such that

$$\lim_k x_{k'} = x' \ , \ \lim_k x_{k'+1} = x''$$

holds for certain elements x' and x'' in $A(x_k) \subseteq Q$. If x' is in M, then the assertion follows immediately by Lemma 2.2c). But otherwise we get a contradiction. On the one hand it is $x'' \in g(x')$ because of $x_{k'+1} \in g(x_{k'})$ and (A3). Then (A2) implies

$$\|x'' - x\| < \|x' - x\| \quad \forall x \in M.$$

On the other hand Lemma 2.2b) means that we have

$$|x'' - x\| = \|x' - x\| \quad \forall x \in M. \quad \blacksquare$$

Remarks 3.5.

1. If we want to apply Theorem 3.4 to a certain problem we must know by other facts that the solution set is nonempty (see (A1)). Besides we have to construct a suitable closed mapping $g \in \mathbb{F}_<(M)$. This can be a difficult task, since M is unknown. Nevertheless, in many cases the formulation of the problem suggests the choice of g.

2. The sequence (x_k) of Method 3.3 is bounded under the assumptions (A1) , (A2) by Lemma 2.2a). Hence (A4) can be dropped in Theorem 3.4 if H is of finite dimension. So for $H = Q = \mathbb{R}^n$ a theorem of EREMIN in [7:p.49] arises. Furthermore the book [7] contains a lot of applications based on this theorem.

3. The assumption (A4) is troublesome and mostly not satisfied if H is infinite-dimensional. It is necessary for strong convergence. But it is difficult to prove it directly. (A4) is fulfilled under the restrictive condition that (x_k) is contained in a compact set which is realized if the set $\{ y \in Q : \|y - x\| \leq \|x_0 - x\|\}$ is compact for any $x \in M$ or more special if g is a compact operator. Without (A4) the iterative sequence (x_k) is still weakly precompact, but it seems to be unpossible to get at least weak convergence by a slight modification of the theorem.

Conditions different from those given in Theorem 3.4 are known which ensure at least the weak convergence of Method 3.3. Mostly they concern the case of nonexpansive operators g (see e.g. YOULA in [21], [3]). But extensions to Fejer-monotone mappings are obvious. In this paper we are interested in results refering to strong convergence which represent the exception in infinite-dimensional spaces. Hence we look for a result which is more sophisticated but also more satisfactory than Theorem 3.4. Especially we avoid assumption (A4). The starting point is Lemma 2.3. The next two sections supply necessary concepts and some easy consequences.

4 Strongly Fejer-monotone mappings and Fejer-kernels

Here we introduce a stronger version of Fejer-monotone mappings and a related type of mappings called Fejer-kernels. Further we list some simple properties and examples (see [19]). Again we consider a nonempty, convex and closed subset Q of the Hilbert space H as well as a mapping $g : Q \to \mathbb{P}(Q)$.

Definition 4.1. Let M be a nonempty subset of Q and α a positive number. Then g is said to be *strongly* (M, α)*-Fejer-monotone* $(g \in \mathbb{F}_\alpha(M))$ iff $M \supseteq F_-(g)$ and

$$\|y - x\|^2 - \|z - x\|^2 \geq \alpha \|y - z\|^2 \quad \forall x \in M, \forall y \in Q, \forall z \in g(y).$$

It is called *strongly Fejer-monotone* if it is (M, α)-Fejer-monotone for any M and any α.

Remark 4.2. Obviously strongly Fejer-monotone mappings are strictly Fejer-monotone. Especially it is

$$\mathbb{F}_\alpha(M) \subseteq \mathbb{F}_\beta(M) \subseteq \mathbb{F}_<(M) \quad \text{for } \alpha \geq \beta,\ M = F_-(g).$$

Examples 4.3. Let M be a nonempty subset of Q.

1. If M is a convex and closed subset, then the metric projector P_M onto M is in $\mathbb{F}_1(M)$.
2. If $g \in \mathbb{F}_1(M)$ and if we define

   $$g_\lambda(y) = (1 - \lambda)y + \lambda g(y)\ ,\ 0 < \lambda < 2\,,$$

   then $g_\lambda \in \mathbb{F}_\alpha(M)$, where $\alpha = (2 - \lambda)/\lambda$ is strictly monotone decreasing in λ.

Now we consider another type of mappings $t : Q \to \mathbb{P}(H)$ which turn out to be closely related to Fejer-monotone mappings.

Definition 4.4. Let M be a nonempty subset of Q. Then t is said to be a M*-Fejer-kernel* $(t \in \mathbb{K}(M))$ iff

$$\|v - \tfrac{1}{2}(y - x)\| \leq \tfrac{1}{2}\|y - x\| \quad \forall x \in M, \forall y \in Q, \forall v \in t(y).$$

It is called a *Fejer-kernel* iff it is a M-Fejer-kernel for any M. The set

$$K(t) = \{x \in Q : \|v - \tfrac{1}{2}(y - x)\| \leq \tfrac{1}{2}\|y - x\|\ \forall y \in Q,\ \forall v \in t(y)\}$$

is called the *Fejer-kernel-carrier* of t.

Remarks 4.5.

1. The mapping t is a Fejer-kernel iff

$$(v, y-x) \geq (v, v) \quad \forall x \in M,\ \forall y \in Q,\ \forall v \in t(y).$$

This follows immediately if the squared norms in the definition are replaced by corresponding scalar products. Hence, applying Schwarz's inequality, we get for such a mapping

$$\sup\{\|v\| \,:\, v \in t(y)\} \leq \rho(y, M)\ ,$$

where $\rho(y, M)$ denotes the distance between y and M. Thus t is bounded.

2. For $t : Q \to \mathbb{P}(H)$ we introduce the zero sets

$$L_-(t) = \{x \in Q \,:\, 0 \in t(x)\}\ ,\ L_+(t) = \{x \in Q \,:\, \{0\} = t(x)\},$$

which coincide for $t : Q \to H$ with $L(t) = \{x \in Q : t(x) = 0\}$. The carrier $K(t)$ of t is always convex and closed. Besides we have the relations $M \subseteq K(t) \subseteq L_+(t) \subseteq L_-(t)$ for $t \in \mathbb{K}(M)$. If it is also $L_-(t) \subseteq M$, then $M = K(t) = L_+(t) = L_-(t)$ holds. Such Fejer-kernels t are called *regular* ($t \in \mathbb{K}_r(M)$).

Examples 4.6.

1. Let be $g \in \mathbb{F}_1(M)$. Then $t \in \mathbb{K}_r(M)$ for $t(y) = y - g(y)$.

2. Let be $b : Q \to I\!R$ convex, $N(b) = \{x \in Q : b(x) \leq 0\}$ and $\emptyset \neq M \subseteq N(b)$. Then b^+ denotes the positive part and ∂b the subgradient of b. If we define

$$\mu(b, y, v) = \begin{cases} b^+(y)v/\|v\|^2 & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases}\ , \tag{4.1}$$

$$t_b(y) = \{\mu(b, y, v) \,:\, v \in \partial b(y)\}\ , \tag{4.2}$$

then $t_b \in \mathbb{K}(M)$ and for $M = N(b)$ even $t \in \mathbb{K}_r(M)$.

Remark 4.7. We consider two mappings t , g correlated by $g(y) = y - \Lambda t(y)$, $\Lambda \subseteq I\!R$, where

$$\Lambda t(y) = \{z = \lambda v \,:\, \lambda \in \Lambda,\ v \in t(y)\}.$$

Then we get

$$\begin{aligned} &g \in \mathbb{F}(M) \ \text{ for } t \in \mathbb{K}(M) \text{ and } \Lambda \subseteq [0, 2], \\ &g \in \mathbb{F}_r(M) \text{ for } t \in \mathbb{K}_r(M) \text{ and } \Lambda \subseteq (0, 2], \\ &g \in \mathbb{F}_\alpha(M),\ \alpha = (2-\lambda)/\lambda \\ &\qquad\qquad \text{for } t \in \mathbb{K}_r(M),\ \Lambda \subset (0, 2) \text{ and } \lambda = \sup \Lambda < 2. \end{aligned}$$

Further the carrier relation $C(g) = K(t)$ is fulfilled for $\Lambda \subseteq (0, 2]$.

5 Zero-normal mappings

We still need another concept to establish the modified convergence result. Let H and H' be two Hilbert spaces. Besides let be $\emptyset \neq U \subseteq Q \subseteq H,\ t : Q \to \mathbb{P}(H')$. Then $t|U$ denotes the restriction of t to U. Especially it is $t = t|Q$. Moreover we use the notations

$$\begin{aligned}
&L_-^\varepsilon(t|U) = \{x \in U : \exists v \in t(x) : \|v\| \leq \varepsilon\},\ \varepsilon > 0,\\
&L_+^\varepsilon(t|U) = \{x \in U : \forall v \in t(x) : \|v\| \leq \varepsilon\},\ \varepsilon > 0,\\
&0^\varepsilon = \{x \in H : \|x\| \leq \varepsilon\},\ U^\varepsilon = U + 0^\varepsilon.
\end{aligned}$$

Definition 5.1. Let be $\emptyset \neq M \subseteq Q,\ \mathbb{P} \subseteq \mathbb{P}(Q),\ \cup\mathbb{P} = Q$. The mapping t is said to be $(\mathbb{P}, M)$-*zero-normal* iff

$$\forall U \in \mathbb{P} : \forall \delta > 0 : \exists \varepsilon > 0 : L_-^\varepsilon(t|U) \subseteq M^\delta.$$

Obviously $(\mathbb{P}, M)$-zero-normal mappings possess arbitrarily norm-small values v on the system $\mathbb{P}$ only in a corresponding small neighbourhood of M. A sequence interpretation is as follows:

$$\forall U \in \mathbb{P} : \forall (y_k), \{y_k\} \subseteq U : \lim_k t(y_k) = 0 \implies \lim_k \rho(y_k, M) = 0.$$

Remarks 5.2.

1. It is $L_-(t|U) \subseteq L_-^\varepsilon(t|U)$ for all $U \in \mathbb{P}$ and all $\varepsilon > 0$. If t is $(\mathbb{P}, M)$-zero-normal, then we have $L_-(t|U) \subseteq \overline{M}$ and because of $\cup\mathbb{P} = Q$ also $L_-(t) \subseteq \overline{M}$. In the applications the most important case is $L_-(t) = M$ (see Remark 4.5.2). Then we can omit M in the notation.
   For instance, if t is a M-Fejer-kernel $(H' = H)$, then $M \subseteq L_-(t)$ follows by Remark 4.5.2. Hence we obtain $\overline{M} = L_-(t)$ for $(\mathbb{P}, M)$-zero-normal M-Fejer-kernels. Thus we have $L_-(t) = M$ for closed sets M .

2. If we denote

   $$\begin{aligned}
   &\mathbb{P}_1 = \{Q\},\ \mathbb{P}_2 = \{U \subseteq Q : U \text{ bounded}\},\\
   &\mathbb{P}_3(M) = \{\{x_k\} \subseteq Q : (x_k)\ M\text{-Fejer-monotone }\}\ ,
   \end{aligned}$$

   then obviously we get the implications

   $$\begin{aligned}
   t\quad (\mathbb{P}_1, M) - \text{zero-normal} &\implies t\quad (\mathbb{P}_2, M) - \text{zero-normal}\\
   &\implies t\quad (\mathbb{P}_3(M), M) - \text{zero-normal.}
   \end{aligned}$$

   A mapping t with the first property is called simply M-*zero-normal.* The last property is tailored to the iterative methods considered in this paper. Such a mapping t is called M-*Fejer-zero-normal.*

3. To illustrate the different kinds of zero-normality we consider functions
$$t \,:\, I\!R \to I\!R \quad (H = Q = H' = I\!R)$$
and choose $M = L_-(t)$. Let be
$$\begin{array}{lll} t_1(y) = y/2\,, & t_2(y) = y/(1+y^2) & \\ t_3(y|U_1) = 0\,, & t_3(y|U_2) = \min(y, 1-y)\,, & t_3(y|U_3) = 1 \\ t_4(y|U_1) = 0\,, & t_4(y|U_2) = y\,, & t_4(y|U_3) = y-1 \end{array}$$
with $U_1 = (-\infty, 0]$, $U_2 = (0,1)$, $U_3 = [1,\infty)$. Then t_1 is zero-normal, t_2 is $\mathbb{P}_2$-zero-normal but not zero-normal, t_3 is Fejer-zero-normal but not $\mathbb{P}_2$-zero-normal and t_4 is not even Fejer-zero-normal. By the way, all these functions are Fejer-kernels and all with the exception of t_4 are regular.

4. Let be $R(t) = \cup_{x\in Q} t(x) \subseteq H'$. Then the inverse mapping
$$t^- \,:\, R(t) \to \mathbb{P}(Q)$$
is defined by $t^-(v) = \{x \in Q \,:\, v \in t(x)\}$. In this terminology we obtain the relations $t^-(0) = L_-(t)$ and $t^-(0^\varepsilon) = \{t^-(v) \,:\, v \in 0^\varepsilon\} = L_-^\varepsilon(t)$. Consequently the concept of zero-normality is connected with the concept of upper semi-continuity in the following sense (compare [11 p.23]):
$$\begin{aligned} t\ L_-(t)\text{-zero-normal } (\mathbb{P} = \mathbb{P}_1) &\iff t^- \text{ upper semi-continuous in } 0 \\ &\iff \forall \delta > 0 \,:\, \exists \varepsilon > 0 \,:\, t^-(0^\varepsilon) \subseteq t^-(0)^\delta. \end{aligned}$$

5. Evidently t is $(\mathbb{P}, M)$-zero-normal iff the functional $\|t(.)\|$ is $(\mathbb{P}, M)$-zero-normal.

Now we introduce two special classes of zero-normal mappings.

Definition 5.3. Let be $M \subseteq Q$ and $t \,:\, Q \to \mathbb{P}(H')$.

a) t is said to be *degenerated* $(\mathbb{P}, M)$*-zero-normal* iff
$$\forall U \in \mathbb{P} : \inf\{\|v\| \,:\, v \in t(y)\,,\, y \in U\backslash M\} > 0.$$

b) t is said to be *strongly* $(\mathbb{P}, M)$*-zero-normal* iff
$$\forall U \in \mathbb{P} : \exists c > 0 \,:\, \forall y \in U\backslash M \,:\, \forall v \in t(y) \,:\, \|v\| \geq \rho(y, M).$$

Remark 5.4. If P_M is the metric projector onto the convex and closed set M, then $t(y) = y - P_M y$ is strongly $(\mathbb{P}_1, M)$-zero-normal with $c = 1\,(\mathbb{P}_1 = \{Q\})$. Obviously t is $(\mathbb{P}, M)$-zero-normal if t is degenerated $(\mathbb{P}, M)$-zero-normal or strongly $(\mathbb{P}, M)$-zero-normal.

Next we consider a certain composition of zero-normal mappings.

Lemma 5.5. *Let $t_i : Q \to \mathbb{P}(H')$ be $(\mathbb{P}, M_i)$-zero-normal $(i = 1, 2)$. Then the functional $f(.) = \|t_1(.)\|\,\|t_2(.)\|$ is $(\mathbb{P}, M_1 \cup M_2)$-zero-normal.*

Proof. We choose $U \in \mathbb{P}$ arbitrarily and denote $t'_i = t_i|U$ $(i = 1, 2)$. For $\delta > 0$ there exist $\varepsilon_i > 0$ with $L_-^{\varepsilon_i}(t'_i) \subseteq M_i^\delta$ by assumption. Consequently it is

$$\begin{aligned} & L^\varepsilon(\|t'_1(.)\|\,\|t'_2(.)\|) \subseteq L^{\varepsilon_1}(\|t'_1(.)\|)L^{\varepsilon_2}(\|t'_2(.)\|) \\ = \ & L^{\varepsilon_1}(t'_1) \cup L^{\varepsilon_2}(t'_2) \subseteq M_1^\delta \cup M_2^\delta = (M_1 \cup M_2)^\delta \end{aligned}$$

for $\varepsilon = \varepsilon_1\varepsilon_2$. This shows the assertion. ■

Finally we want to characterize $(\mathbb{P}, M)$-zero-normal mappings.

Definition 5.6. Let be $K_\delta(x) = x + 0^\delta$ the δ-ball centred in x and

$$L_*(t) = \{x \in Q \,:\, \forall\delta > 0 \,:\, \forall\varepsilon > 0 \,:\, K_\delta(x) \cap L_-^\varepsilon(t) \neq \emptyset\}$$

the set of *zero-critical elements* of t. The system $\mathbb{P} \subseteq \mathbb{P}(Q)$ is said to be *t-complete* iff $L_*(t)$ coincides with the set

$$\{x \in Q \,:\, \forall\delta > 0 \,:\, \exists U' \in \mathbb{P} \,:\, \forall\varepsilon > 0 \,:\, K_\delta(x) \cap L_-^\varepsilon(t|U') \neq \emptyset\}.$$

Remark 5.7. Evidently $L_-(t)$ is a subset of $L_*(t)$. Furthermore it is easy to see that the systems $\mathbb{P}_1$ and $\mathbb{P}_2$ defined in Remark 5.2.2 are rich enough to be t-complete for all t.

Theorem 5.8. *Considering the conditions*

(a) $L_*(t) \subseteq \overline{M}$,
(b) $\forall U \in \mathbb{P} : \exists\varepsilon' > 0 \,:\, L_-^{\varepsilon'}(t|U)\backslash M$ *precompact,*

the following statements hold:

1. *$(\mathbb{P}, M)$-zero-normal mappings t satisfy* (a) *if $\mathbb{P}$ is t-complete.*
2. *Mappings t satisfying* (a) *and* (b) *are $(\mathbb{P}, M)$-zero-normal.*

Proof.

1. Let t be $(\mathbb{P}, M)$-zero-normal with a t-complete system $\mathbb{P}$. We choose $x \in L_*(t)$ and $\delta > 0$ arbitrarily. Besides we put $\delta = \delta' + \delta''$ with $\delta' > 0$, $\delta'' > 0$. There is $U' \in \mathbb{P}$ such that

   $$\forall\varepsilon > 0 \,:\, K_{\delta'}(x) \cap L_-^\varepsilon(t|U') \neq \emptyset\,,$$

   since $\mathbb{P}$ is t-complete. Besides there exists $\varepsilon'' > 0$ with $L_-^{\varepsilon''}(t|U') \subseteq M^{\delta''}$ considering that t is $(\mathbb{P}, M)$-zero-normal. But then also $K_{\delta'}(x) \cap M^{\delta''} \neq \emptyset$. This means $x \in M^\delta$. As δ is arbitrary, $x \in \cap_\delta M^\delta = \overline{M}$ follows. Hence (a) is fulfilled.

2. We show the assertion indirectly. We suppose that (a) and (b) hold for a mapping t which is not $(\mathbb{P}, M)$-zero-normal. This implies

$$\exists U' \in \mathbb{P} \, : \, \exists \delta' > 0 \, : \, \forall \varepsilon > 0 \, : \, L_-^{\varepsilon}(t|U') \backslash M^{\delta'} \neq \emptyset \; .$$

Now we choose a zero sequence (ε_k) with $\varepsilon_k \leq \varepsilon'$, where $\varepsilon' = \varepsilon'(U')$ is given in (b), and a corresponding sequence (y_k) with

$$y_k \in L_-^{\varepsilon_k}(t|U') \backslash M^{\delta'} \subseteq L_-^{\varepsilon'}(t|U') \backslash M \; .$$

Since $L_-^{\varepsilon'}(t|U') \backslash M$ is precompact by (b), there is a subsequence $(y_{k'})$ of (y_k) converging to an element x. But on the one hand we have $x \notin M^{\delta}$ for any $\delta < \delta'$. On the other hand there is a sequence (v_k) with $v_k \in t(y_k)$, $\lim_k v_k = 0$. Consequently we get $x \in L_*(t)$ and by (a) also $x \in \overline{M}$. This contradiction proves the assertion. ■

Corollary 5.9. *Let condition* (b) *be fulfilled. Besides let* $\mathbb{P}$ *be* t*-complete. Then* t *is* $(\mathbb{P}, M)$*-zero-normal iff condition* (a) *holds.*

Proof. The assertion is an immediate consequence of Theorem 5.8. ■

The following definition leads to a condition which guarantees the coincidence of $L_*(t)$ and $L_-(t)$.

Definition 5.10. The mapping t is said to be *norm lower semi-continuous in* $x \in Q$ iff the mapping $\|t(.)\|$ is lower semi-continuous in x, i.e.

$$\lim_k y_k = x, \; t_k \in t(y_k) \Longrightarrow \exists v \in t(x) \, : \, \|v\| \leq \liminf_k \|t_k\| \; .$$

Moreover t is said to be *norm lower semi-continuous on* $U \subseteq Q$ iff t is norm lower semi-continuous in x for all $x \in U$.

Remark 5.11. It is easy to see that the mapping $t \, : \, Q \rightarrow \mathbb{P}(H')$ is norm lower semi-continuous on Q if the mapping $\|t(.)\|$ is closed (see Definition 3.1). Consequently t is norm lower semi-continuous on Q if t and therefore $\|t(.)\|$ is upper semi-continuous (see Remark 3.2) or if t is closed and compact.

Lemma 5.12. *If* t *is norm lower semi-continuous in* x *and* $x \in L_*(t)$*, then* $x \in L_-(t)$.

Proof. For $x \in L_*(t)$ there are sequences (y_k) and (v_k) satisfying $\lim_k y_k = x$, $v_k \in t(y_k)$, $\lim_k v_k = 0$. If t is norm lower semi-continuous in x, then there exists $v \in t(x)$ with

$$\|v\| \leq \liminf_k \|v_k\| = 0.$$

Hence $v = 0$ and $x \in L_-(t)$. ■

Corollary 5.13. *Let the following conditions be fulfilled:*

(a') $L_-(t) \subseteq \overline{M}$,
(b) $\forall U \in \mathbb{P} : \exists \varepsilon' > 0 : L_-^{\varepsilon'}(t|U)\backslash M$ *precompact ,*
(c) t *is norm lower semi-continuous on* $L_*(t)$.

Then t *is* $(\mathbb{P}, M)$*-zero-normal.*

Proof. Since $L_-(t) \subseteq L_*(t)$ and $L_*(t) \subseteq L_-(t)$ because of (c) and Lemma 5.12, the conditions (a) and (b) of Theorem 5.8 hold. Hence this theorem shows the assertion. ∎

Remark 5.14. By Corollary 5.13 t is M-zero-normal ($\mathbb{P} = \mathbb{P}_1 = \{Q\}$) if t is norm lower semi-continuous on Q , $Q\backslash M$ is precompact and $L_-(t)$ is contained in $\overline{M}$. In the important special case $M = L_-(t)$ the last condition is trivially satisfied.

Remark 5.15. If $\|t(.)\|$ is a nonnegative convex functional with zeros which acts in a finite-dimensional space, then Proposition 13 in [12 : p.191] supplies a condition for t to be M-zero-normal with respect to the zero set M.

Concerning our iterative methods we are especially interested in zero-normal Fejer-kernels. In view of Remark 4.5.1 the images $v \in t(y)$ of Fejer-kernels t tend to 0 if the corresponding originals y tend to the Fejer-kernel. Reversely, zero-normality means that the originals y tend to the Fejer-kernel if there is a choice of images $v \in t(y)$ which tend to 0.

Examples 5.16.

1. Let $M \subseteq Q$ be convex and closed and let P_M be the metric projector onto M. Then the Fejer-kernel t with $t(y) = y - P_M(y)$ is strongly zero-normal (see Examples 4.3.1, 4.6.1). Because of $\|t(y)\| = \rho(y, M)$ we have even equality for a constant $l = 1$.

2. Let us consider convex and closed sets $M_i \, (i = 1, \ldots, m)$ with a nonempty intersection M *(convex feasibility problem).* Then we choose t as

$$t(y) = t_M(y) = y - P_{i(y)}y, \tag{5.1}$$

where P_i is the metric projector onto M_i and the index $i(y)$ fulfils

$$\rho(y, M_{i(y)}) = \max_{1 \leq i \leq m} \rho(y, M_i) \textit{ (remotest set control).} \tag{5.2}$$

Then t is a M-Fejer-kernel since

$$(y - P_i y, \; x - P_i y) \leq 0 \quad \forall i, \forall x \in M$$

and therefore

$$(y - P_{i(y)}y, \; y - x) \geq (y - P_{i(y)}, y - P_{i(y)})$$

(compare Remark 4.5.1). Moreover t is M-zero-normal if one of the following conditions hold:

(a) $M_{i^*} \cap$ int $(\cap_{i \neq i^*} M_i) \neq \emptyset$ for some i^* (int: interior),

(b) M_i halfspaces,

(c) M_i uniformly convex (with the same module),

(d) H finite-dimensional.

In the first two cases t is even strongly M-zero-normal. This shows an analysis of the results in [8].

6 Iterative methods of Fejer-type with relaxation parameters

First we introduce a general iterative method. We consider a Hilbert space H, a subset Q of H, operators $T_k : H \to Q$, elements t_k in H and relaxation parameters $\lambda_k \in I\!R$ $(k \in I\!N)$.

Method 6.1: $x_0 \in Q$, $x_{k+1} = T_k(x_k - \lambda_k t_k)$, $k \in I\!N$.

The assumptions guarantee $\{x_k\} \subseteq Q$. The aim is to state conditions ensuring that this method converges in a Fejer-monotone manner to an element of a certain convex and closed solution set $M \subseteq H$. For this purpose we use the following notations:

$A(x_k)$: set of accumulation values of (x_k),
P_M : metric projector onto M,

$$F(T_k) = \{x \in H \, : \, x = T_k x\}, \tag{6.1}$$
$$C(T_k) = \{x \in H \, : \, \forall y \in H \, : \, \|T_k y - x\| \leq \|y - x\|\}, \tag{6.2}$$
$$Q_k = \{x \in H \, : \, \|t_k - \tfrac{1}{2}(x_k - x)\| \leq \tfrac{1}{2}\|x_k - x\|\}, \tag{6.3}$$
$$X_k = C(T_k) \cap Q_k \, . \tag{6.4}$$

In every case x_k, t_k and T_k relate to Method 6.1. If $C(T_k)$ is nonempty, then T_k is Fejer-monotone with the carrier $C(T_k)$ (see Definition 2.4). The set Q_k in (6.3) can also be written as

$$Q_k = \{x \in H \, : \, (t_k, x_k - x) \geq \|t_k\|^2\} \tag{6.3'}$$

(compare also with Remark 4.5.1). Before stating the convergence theorem we formulate an auxiliary result.

Lemma 6.2. *For $x \in X_k$ and $\lambda_k \geq 0$ the inequality*

$$\|x_{k+1} - x\|^2 \leq \|x_k - x\|^2 - \lambda_k(2 - \lambda_k)\|t_k\|^2$$

holds.

Proof. Considering the procedure of Method 6.1 and the relations in (6.4), (6.1) and (6.3') we get

$$\begin{aligned}\|x_{k+1}-x\|^2 &= \|T_k(x_k-\lambda_k t_k)-x\|^2 \le \|x_k-\lambda_k t_k - x\|^2\\ &= \|x_k-x\|^2 - 2\lambda_k(t_k,\, x_k-x)+\lambda_k^2\|t_k\|^2\\ &\le \|x_k-x\|^2-\lambda_k(2-\lambda_k)\|t_k\|^2.\end{aligned}$$

The first estimate is fulfilled in view of $x \in C(T_k)$, the second estimate in view of $\lambda_k \ge 0$, $x \in Q_k$. ■

Remarks 6.3.

1. If T_k is nonexpansive, then $C(T_k) = F(T_k)$ (see Example 2.6.1). Hence Lemma 1 in [16] is a special case of Lemma 6.2.

2. Let M be a nonempty subset of $\cap_k X_k$. Then $C(T_k) \neq \emptyset$ for all $k \in I\!N$. Consequently the mappings T_k in Method 6.1 are $C(T_k)$-Fejer-monotone and all the more M-Fejer-monotone. Under the additional assumption $\lambda \in [0{,}2]$ Lemma 6.2 shows that Method 6.1 generates a M-Fejer-monotone sequence (x_k). If $t_k \neq 0$ for $x_k \notin M$ and $\lambda_k \in (0{,}2)$, then (x_k) is even strictly M-Fejer-monotone (see Definition 2.1).

Theorem 6.4. *Let the following assumptions be fulfilled for Method* 6.1 :

(B1) M *nonempty, convex and closed,*
(B2) $M \subseteq \cap_k X_k$,
(B3) $\liminf_k \|t_k\| = 0 \Longrightarrow \liminf_k \rho(x_k, M) = 0$,
(B4) $\lambda_k \in [0,2]$ *for all* k, $\sum_k \lambda_k(2-\lambda_k) = \infty$.

Then (x_k) *converges to an element* x' *in* M, *where*

$$\|P_M x_k - x'\| \le \rho(x_k, M)\ ,\ \|x_k - x'\| \le 2\rho(x_k, M).$$

Proof. Using Lemma 6.2 the conditions (B1), (B2) and (B4) imply $\liminf_k \|t_k\| = 0$ (compare Lemma 4 in [16]). Then $\liminf_k \rho(x_k, M) = 0$ follows by (B3). Since (x_k) is M-Fejer-monotone under (B1), (B2) and (B4) (see Remark 6.3.2), the last limit relation is equivalent to $\lim_k \rho(x_k, M) = 0$ (compare Lemma 6 in [16]). Now the application of Lemma 2.3 leads to the assertions. ■

Remarks 6.5.

1. Theorem 6.4 is a generalization of Theorem 4 in [16]. On the one hand T_k is supposed to be M-Fejer-monotone instead of nonexpansive. On the other hand we even get strong convergence and error estimates instead of weak convergence.

2. If we strengthen (B3) and (B4) to

   (B3') $\exists l > 0\ \forall k \in I\!N : \|t_k\| \ge l\rho(x_k, M)$,
   (C4') $\forall k \in I\!N : \lambda_k \in [\varepsilon_1, 2-\varepsilon_2]$, where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_1 + \varepsilon_2 < 2$,

   respectively, then we can show a geometrical convergence rate (see [16] and for an improved version [17], [18]).

7 Iterative methods with Fejer-kernels and relaxation

The results in section 6 are the basis for a convergence theorem with Fejer-monotone mappings which differs from Theorem 3.4 . We avoid the awkward assumption $A(x_k) \neq \emptyset$ and we put the isolated condition $\lim_k \rho(x_k, M) = 0$ in a natural relation to the stationarity of the iterative method. Besides, we replace the conditions in Theorem 6.4 partly by a continuous formulation. So the elements t_k in Method 6.1 fulfil conditions representing the discrete analogon of a zero-normal Fejer-kernel t. The operators $T_k : Q \to Q$ can be substituted by a united mapping $T : Q \to \mathbb{P}(Q)$ (see also Example 2.6.3). But the relaxation parameters are preserved in its discrete form.

Let Q be a convex and closed subset of a Hilbert space H. Let $T, t \in Q \to \mathbb{P}(Q)$ and $\Lambda_k \subseteq \mathbb{R}$ with

$$\underline{\lambda}_k = \inf\{\lambda, 2-\lambda \,:\, \lambda \in \Lambda_k\} \ . \tag{7.1}$$

Method 7.1: $x_0 \in Q$, $x_{k+1} \in T(x_k - \Lambda_k t(x_k))$, $k \in \mathbb{N}$.

Theorem 7.2. *Let the following assumptions be fulfilled for Method* 7.1 *:*

(C1) $\emptyset \neq M = \overline{M} \subseteq Q$,
(C2) T *M-Fejer-monotone* $(T \in \mathbb{F}(M))$,
(C3) t *M-Fejer-kernel* $(t \in \mathbb{K}(M))$,
(C4) t *M-Fejer-zero-normal on a set* $Q' \subseteq Q$ *containing* $\{x_k\}$,
(C5) $\underline{\lambda}_k \in [0,1]$ *for all* k, $\sum_k \underline{\lambda}_k = \infty$.

Then there holds

1) $M = L_+(t) = L_-(t) = K(t)$ *convex* ,
2) $\exists x' \in M \,:\, \lim_k x_k = x'$, *where*

$$\|P_M x_k - x'\| \leq \rho(x_k, M)\,,\ \|x_k - x'\| \leq 2\rho(x_k, M).$$

Proof. By (C1) the set M is nonempty and closed. Since (C3) holds, Remark 4.5.2 supplies $M \subseteq K(t) \subseteq L_+(t) \subseteq L_-(t)$, where $K(t)$ is convex. Because of (C1) and (C4) Remark 5.2.1 shows $L_-(t) \subseteq M = \overline{M}$. Hence 1) is verified. For x_{k+1} in Method 7.1 there are $t_k \in t(x_k)$, $\lambda_k \in \Lambda_k$ and $T_k : Q \to Q$ with $T_k y \in T(y)$ such that Method 6.1, namely

$$x_{k+1} = T_k(x_k - \lambda_k x_k)$$

arises. Now we prove that the assumptions of Theorem 6.4 are satisfied. (B1) follows in view of (C1) and 1). Let x be an arbitrary fixed element of M. Then (C3) means

$$\|v - \tfrac{1}{2}(y-x)\| \leq \tfrac{1}{2}\|y-x\| \quad \forall y \in Q, \forall v \in t(y)$$

using Definition 4.4. We can choose $y = x_k,\ v = t_k$ since $x_k \in Q$. Thus we obtain

$$\|t_k - \tfrac{1}{2}(x_k - x)\| \leq \tfrac{1}{2}\|x_k - x\|.$$

But then we have $x \in Q_k$ by (6.3). Obviously (C2) supplies $T_k \in \mathbb{F}(M)$, where $M \subseteq C(T_k)$ (see (6.2)). Consequently it is also $x \in C(T_k)$. Hence $x \in X_k = Q_k \cap C(T_k)$ (see (6.4)) . This implies $M \subseteq \cap_k X_k$. Therefore (B2) is fulfilled. Evidently it is $\underline{\lambda}_k \le \lambda_k \le 2 - \underline{\lambda}_k$ and $\underline{\lambda}_k \le \min(\lambda_k, 2-\lambda_k)$ for all k. Then we get $0 \le \lambda_k \le 2$ and

$$\sum_k \underline{\lambda}_k \le \sum_k \min(\lambda_k, 2-\lambda_k) \le \sum_k \lambda_k(2-\lambda_k).$$

So (C5) implies (B4).
The iterative sequence (x_k) is M-Fejer-monotone if the conditions (B1), (B2) and (B4) hold (see Remark 6.3.2). Hence the set $U_1 = \{x_k\}$ lies in $\mathbb{P}_3(M)$ (see Remark 5.2.2). Now we assume that $\liminf_k \|t_k\| = 0$. Then there are subsequences (y_k) of (x_k) and (u_k) of (t_k) with $u_k \in t(y_k)$, $\lim_k u_k = 0$. Considering Definition 5.1 and Remark 5.2.2 we get by (C4)

$$\forall U \in \mathbb{P}_3(M) \,:\, \forall \delta > 0 \,:\, \exists \varepsilon > 0 \,:\, L_\varepsilon(t|U) \subseteq M^\delta.$$

Consequently, for an arbitrary $\delta > 0$ there is a number $\varepsilon > 0$ such that $L_\varepsilon(t|U_1) \subseteq M^\delta$. Hence we can find an integer k_0 such that $y_k \in L_\varepsilon(t|U_1)$ and therefore $y_k \in M^\delta$ for all $k \ge k_0$. This means $\rho(y_k, M) \le \delta$ for all $k \ge k_0$. But then we obtain $\lim_k \rho(y_k, M) = 0$ and $\liminf_k \rho(x_k, M) = 0$. Thus (B3) holds. Finally, by Theorem 6.4 the assertions 2) follow. ■

Remarks 7.3.

1. Under the assumptions of Theorem 7.2 the mapping t is a regular Fejer-kernel (see assertion 1) and Remark 4.5.2).

2. Paying attention to assertion 1) the set M in Theorem 7.2 can be replaced by $L_+(t)$ or $K(t)$. Then Method 7.1 supplies approximate solutions of the zero problem $x \in L_+(t)$ or of the variational problem $x \in K(t)$ if the assumptions of Theorem 7.2 hold with the corresponding replacement of M.

3. Instead of (C4) we can use the sharper condition

   (C4') t M-zero-normal (see Remark 5.2.2).

4. Formally more general the assertions of Theorem 7.2 hold also for the method

   $$x_{k+1} \in T_k(x_k - \Lambda_k t(x_k)),\ x_0 \in Q,\ T_k \,:\, Q \to \mathbb{P}(Q),$$

   if (C2) is replaced by

   (C2') $T_k \in \mathbb{F}(M)$ for all k .

   Putting $Ty = \cup_k T_k y$ and observing Example 2.6.3 both results turn out to be equivalent.

5. For $\Lambda_k \subseteq [\varepsilon_1, 2-\varepsilon_2]$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_1 + \varepsilon_2 \le 2$ we get $\underline{\lambda}_k \ge \min(\varepsilon_1, \varepsilon_2) > 0$. Hence condition (C5) is fulfilled.
   For $\Lambda_k = \{\lambda_k\}$ there holds $\underline{\lambda}_k = \min(\lambda_k, 2-\lambda_k)$. In this special case (C5) is equivalent to (B4).

6. Defining the mappings $g_k : Q \to \mathbb{P}(Q)$ by $g_k(y) = T(y - \Lambda_k t(y))$ Method 7.1 can be written in the form

   $$x_{k+1} \in g_k(x_k)\,,\; x_0 \in Q.$$

   Under the assumptions of Theorem 7.2 the mappings g_k are strictly M-Fejer-monotone if only $\underline{\lambda}_k > 0$. Namely, then $y - \Lambda_k t(y)$ is strongly M-Fejer-monotone (see Remark 4.7 and 1)) and consequently strictly M-Fejer-monotone (see Remark 4.2). Finally this property follows for g_k considering (C2) and Example 2.6.3.

7. As a special case of g in 6) we study the mapping $g(y) = y - \lambda t(y)\,,\; \lambda \in (0,2)$. On the one hand g need not be closed if t is M-zero-normal. On the other hand it is easy to see that t is M-zero-normal if g is a strictly M-Fejer-monotone closed mapping and $Q \backslash M$ is precompact.

8. Summarizing 6) and 7) Theorem 7.2 turns out to be partly more special and partly more general than Theorem 3.4. But altogether Theorem 7.2 is more satisfactory for infinite-dimensional spaces.

Using the results in section 5 we can specialize Theorem 7.2 in various ways. We mention here only two possibilities.

Corollary 7.4. *Let the following assumptions be fulfilled for Method* 7.1:

$(D1)$	$\emptyset \neq M \subseteq Q$, $Q' \backslash M$ *precompact for a set* $Q' \subseteq Q$ *containing* $\{x_k\}$,
$(D2)$	$T \in \mathbb{F}(M)$,
$(D3)$	$t \in \mathbb{K}(M)$, $L_-(t) \subseteq M$,
$(D4)$	t *norm lower semi-continuous* ,
$(D5)$	$\underline{\lambda}_k \in [0,1]$ *for all* k, $\sum_k \underline{\lambda}_k = \infty$.

Then the assertions of Theorem 7.2 *hold.*

Proof. The conditions (D2), (D5) coincide with the corresponding conditions (C2), (C5) of Theorem 7.2. By (D3) and the first part of (D1) we get (C1) (see Remark 4.5.2). The first part of (D3) corresponds to (C3). Finally (D4) and the second parts of (D1) and (D3) supply (C4) by Remark 5.14. ■

Remark 7.5. Let be $Q \subseteq I\!R^n$ bounded. Further, let be $t : Q \to Q$ a continuous $L(t)$-Fejer-kernel with $L(t) \neq \emptyset$ and $\lambda \in (0,2)$. Then the iterative method

$$x_0 \in Q\,,\; x_{k+1} = x_k - \lambda t(x_k),\; k \in I\!N$$

converges to an element x' in $L(t)$. This result is an immediate consequence of Corollary 7.4. Namely, if we choose $H = I\!R^n$, $M = L(t)$, $T = I$ (identity) , $\Lambda_k = \{\lambda\}$ for all k, then all assumptions of Corollary 7.4 are fulfilled.
Especially $Q' \backslash M \subseteq I\!R^n$ is as a bounded set always precompact. Further t is norm lower semi-continuous, since t is continuous and therefore upper semi-continuous (see Remark 5.11).

Finally we consider the convex feasibility problem $M = \cap_{i=1}^m M_i$ described in Example 5.2.2.

Method 7.6: $x_0 \in Q,\ x_{k+1} \in x_k - \Lambda_k t_M(x_k),\ k \in I\!N \quad (t_M$ as in (5.1)).

Corollary 7.7. *Let the following assumptions be fulfilled for Method* 7.6:

$(E1)$	$\emptyset \neq M \subseteq Q$,
$(E2)$	*one of the properties* (a) - (d) *in Example* 5.16.2 *concerning* M_i *or* H,
$(E3)$	$\underline{\lambda}_k \in [0,1]$ *for all* k , $\sum_k \underline{\lambda}_k = \infty$.

Then the assertions of Theorem 7.2 *hold with respect to Method* 7.6.

Proof. M is as the intersection of closed sets M_i itself closed. So (E1) implies (C1). Since Method 7.6 has the form of Method 7.1 with $T = I$, (C2) is trivially satisfied. Paying attention to Example 5.16.2 the assumption (C3) and in view of (E2) also the assumption (C4) are fulfilled. The conditions (E3) and (C5) are identical. ■

Remark 7.8. For $\Lambda_k = \{\lambda_k\}$ Method 7.6 has the form

$$x_{k+1} = x_k - \lambda_k(x_k - P_{i(k)}x_k) = (1-\lambda_k)x_k + \lambda_k P_{i(k)}x_k,$$

where $i(k) := i(x_k)$. This is the method of (relaxed) *orthogonal projections* which originates to Bregman (1965), Gubin, Poljak and Raik (1967). After this many authors have investigated variants of this method. Besides Corollary 7.7 supplies a generalization of a result in [8] and of some later results.

8 Iterative methods with Fejer-gradients and relaxation

As before let Q be a convex and closed subset of the Hilbert space H. We consider a mapping $s : Q \to \mathbb{P}(Q)$ and a functional $b : Q \to I\!R$ with the level set

$$N(b) = \{x \in Q : b(x) \leq 0\}$$

and the positive part b^+ $(N(b) = L(b^+))$. We constitute a certain connection between s and b which supplies a mean to construct Fejer-kernels.

Definition 8.1. Let M be a nonempty subset of Q. Then (s,b) is said to be a *M-Fejer-pair* iff

$$(v, y-x) \geq b(y) \quad \forall x \in M, \forall y \in Q\backslash N(b), \forall v \in s(y).$$

Thereby s is called *M-Fejer-gradient* of b and b is called *M-Fejer-potential* of s. Further (s,b) is said to be a *Fejer-pair* if it is a M-Fejer-pair for any $M \neq \emptyset$. Finally the set

$$K(s,b) = \{x \in Q : (v, y-x) \geq b(y)\ \forall y \in Q\backslash N(b), \forall v \in s(y)\}$$

is called the *Fejer-potential-carrier* of (s,b).

Remarks 8.2.

1. The carrier $K(s,b)$ of a Fejer-pair (s,b) is nonempty, convex and closed. Besides we get the relations

   $$M \subseteq K(s,b) \subseteq N(b), \quad L_-(s) \subseteq N(b)$$

   for a M-Fejer-pair (s,b), which is called *regular* if $N(b) \subseteq M$. Then $M = K(s,b) = N(b)$ holds.

2. Eremin denotes in [7] regular M-Fejer-pairs (s,b) *M-separating.* Regular Fejer-pairs with operators s occur also in [20] without defining them explicitely. A discrete version can be found already in [5].

3. Each M-Fejer-pair (s,b) can be related in a natural way to a M-Fejer-kernel t. Let be

   $$\mu(b,y,v) = \begin{cases} b^+(y)v/\|v\|^2 & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases}, \tag{8.1}$$

   $$t_{b,s}(y) = \{\mu(b,y,v) \,:\, v \in s(y)\}. \tag{8.2}$$

   Then (s,b) is a M-Fejer-pair iff $t_{b,s}$ is a M-Fejer-kernel. Moreover (s,b) is regular iff $t_{b,s}$ is regular $(K(s,b) = K(t_{b,s}),\ L_-(t_{b,s}) = N(b))$.

4. Obviously the correspondence between s and b in a Fejer-pair (s,b) is not unique.

Examples 8.3.

1. Let t be a M-Fejer-kernel. Then we can define

   $$b(y) = b(t,y) = \inf\{\|v\|^2 \,:\, v \in t(y)\}.$$

   Since

   $$(v, y-x) \geq \|v\|^2 \geq b(y) \quad \forall x \in M, \forall y \in Q, \forall v \in t(y)$$

   holds, (t,b) is a M-Fejer-pair. Here we have $L(b) = N(b) \supseteq L_-(t)$. Hence $M \supseteq L(b)$ implies $N(b) = L_-(t)$. In this case both t and (t,b) are regular. Observe that t does not coincide in general with $t_{b,t}$ defined in Remark 8.2.3. But, if t is single-valued, then $b(y) = \|t(y)\|^2$ and $t = t_{b,t}$.

2. Let b be convex and continuous. For $M \subseteq N(b)$ the subdifferential $s = \partial b$ is a M-Fejer-gradient of b. If $M = N(b)$, then the Fejer-pair $(b, \partial b)$ is regular. The corresponding Fejer-kernel $t_b = t_{b,\partial b}$ was already mentioned in Example 4.6.2.

3. Let the following relations be fulfilled for some $s : Q \to Q$ and $M \subseteq Q$:

$$\exists \kappa > 1 \, : \, \exists c > 0 \, : \, \exists d > 0 \, : \, \forall x \in M \, : \, \forall y \in Q :$$
$$(s(y), y - x) \geq c\rho(y, M)^{\kappa}, \quad \|s(y)\| \geq d\rho(y, M)^{\kappa - 1}.$$

First $b(y) = c\rho(y, M)^{\kappa}$ is a M-Fejer-potential of s. But an easy calculation shows in view of the second relation that also $b(y) = cd^{\kappa/(1-\kappa)}\|s(y)\|^{\kappa/(1-\kappa)}$ can be chosen as an M-Fejer-potential of s. In both cases the pairs (s, b) are regular (see also [20,p.449]). Lemma 7 in [18] contains a discrete version for arbitrary $\kappa \neq 1$.

Method 8.4: $x_0 \in Q$, $x_{k+1} \in T(x_k - \Lambda_k t_{b,s}(x_k))$, $k \in I\!N$ ($t_{b,s}$ as in (8.2)/(8.1)).

Theorem 8.5. *Let the following assumptions be fulfilled for Method* 8.4:

(F1) $\emptyset \neq M = \overline{M} \subseteq Q$,
(F2) *T M-Fejer-monotone,*
(F3) *(s, b) M-Fejer-pair ,*
(F4) *$r_{b,s}$ M-Fejer-zero-normal on a set $Q' \subseteq Q$ containing $\{x_k\}$, where*

$$r_{b,s}(y) = \{\nu(b, y, v) \, : \, v \in s(y)\},$$
$$\nu(b, y, v) = \begin{cases} b^+(y)/\|v\| & \textit{if } v \neq 0 \\ 0 & \textit{if } v = 0 \end{cases},$$

(F5) $\underline{\lambda}_k \in [0, 1]$ *for all* k, $\sum_k \underline{\lambda}_k = \infty$ ($\underline{\lambda}_k$ *as in* (7.1)).

Then there holds:

1) $M = K(s, b) = N(b)$ *convex* ,
2) $\exists x' \in M \, : \, \lim_k x_k = x'$, *where*
$\|P_M x_k - x'\| \leq \rho(x_k, M)$, $\|x_k - x'\| \leq 2\rho(x_k, M)$.

Proof. We show that the assumptions of Theorem 7.2 are satisfied. The conditions (F1), (F2) and (F5) coincide with the corresponding conditions (C1), (C2) and (C5). By (F3) and Remark 8.2.3 the mapping $t_{b,s}$ is a M-Fejer-kernel. Hence (C3) holds. Obviously we get $r_{b,s}(y) = \{\|v\| \, : \, v \in t_{b,s}(y)\}$. Thus (F4) implies (C4) by Remark 5.2.5. Now the assertions follow from Theorem 7.2 since with $t_{b,s}$ also (s,b) is regular (see Remark 8.2.3). ■

Remarks 8.6.

1. In the assumptions of Theorem 8.5 the set M can be replaced by $N(b)$ (see assertion 1)). Let the functional b be given. Then Method 8.4 supplies approximate solutions of the inequality problem $x \in N(b) \neq \emptyset$ if a corresponding mapping s is known satisfying the conditions in Theorem 8.5. Besides, observe that $x \in N(b)$ is equivalent to $0 = b^+(x) = \min\{b^+(y) \, : \, y \in Q\}$.

2. Instead of (F4) we can use the following more special condition

   (F4') $\quad b^+(y)$ M-Fejer-zero-normal ,

   $$\forall U \in \mathbb{P}_3(M): \; r_M(s,U) = \sup\{\|v\| \, : \, v \in s(y), \; y \in U \backslash M\} < \infty.$$

   Then we show that there is a degenerated $(\mathbb{P}_3(M), M)$-zero-normal mapping $q_s(y)$ with

   $$r_{b,s}(y) = b^+(y) q_s(y).$$

   The first part of (F4') yields $N(b) = L(b^+) \subseteq \overline{M} = M$ considering Remark 5.2.1. (F3) implies $L_-(s) \subseteq N(b)$ and therefore $L_-(s) \subseteq M$ by Remark 8.2.1. Now we fix an arbitrary $U \in \mathbb{P}_3(M)$. Because of $0 < r_M(s,U) < \infty$ we get

   $$r_M^{-1}(s,U) = \inf\{\gamma(v) \, : \, v \in s(y), \, y \in U \backslash M\} > 0,$$

   where

   $$\gamma(v) = \begin{cases} 1/\|v\| & \text{for } v \neq 0 \\ 0 & \text{for } v = 0 \end{cases} .$$

   Obviously we have $\gamma(v) = 1/\|v\|$ for $v \in s(y)$, $y \in U \backslash M$. By Definition 5.3 the mapping

   $$q_s(y) = \{\gamma(v) \, : \, v \in s(y)\}$$

   is degenerated $(\mathbb{P}_3(M), M)$-zero-normal. Considering Remark 5.4 and Lemma 5.5 condition (F4) follows by (F4').

3. If all sets $U \in \mathbb{P}_2$ ($U \subseteq Q$, U bounded) are precompact and b is continuous (and therefore norm lower semi-continuous), then b is $(\mathbb{P}_2, N(b))$-zero-normal. This is a simple consequence of Corollary 5.13.

4. With the preceding remarks Theorem 1 in [20] turns out to be a special version of Theorem 8.5. Observe that there $\mathbb{P}_2$ is used instead of $\mathbb{P}_3(M)$ in (F4') (see Remark 5.2.2). Further specifications are $T = P_Q$, $\Lambda_k = \{\lambda\} \subseteq (0,2)$, s operator, b nonnegative continuous functional. Under the assumptions $H = \mathbb{R}^n$, $Q = \mathbb{R}^n$ or $Q = \mathbb{R}^n_+$, $T = P_Q$, $\Lambda_k = \{\lambda_k\} \subseteq [\varepsilon_1, 2 - \varepsilon_2]$ the paper [5] follows a similar line.

5. If b is convex and continuous, then (F3) is satisfied for $s = \partial b$. This special case is discussed in the following.

Method 8.7: $x_0 \in Q, \; x_{k+1} \in P_Q(x_k - \Lambda_k t_b(x_k)), \; k \in \mathbb{N}$ (t_b as in (4.2)/(4.1)) .

Corollary 8.8. *Let the following conditions be fulfilled:*

(G1) $\emptyset \neq M \subseteq Q$, $Q' \backslash M$ *precompact for a set* $Q' \subseteq Q$ *containing* $\{x_k\}$,
(G2) b *convex and continuous* , $N(b) \subseteq M$,
(G3) $r_M(\partial b) = \sup\{\|v\| \,:\, v \in \partial b(y)\,,\ y \in Q' \backslash M\} < \infty$,
(G4) $\underline{\lambda}_k \in [0,1]$ *for all* k , $\sum_k \underline{\lambda}_k = \infty$ $\quad$ ($\underline{\lambda}_k$ *as in* (7.1)).

Then the assertions of Theorem 8.5 *hold with* $s = \partial b$, *where* (x_k) *is from Method* 8.7.

Proof. The first part of (G2) guarantees the existence of $\partial b \,:\, Q \to \mathbb{P}(Q)$. So we can choose $s = \partial b$ and $T = P_Q$. Then Method 8.4 transverts into Method 8.7. Further the assumptions of Theorem 8.5 hold. Namely, the second part of (G2) implies that M is closed (see Remark 8.2.1). Consequently (F1) is satisfied. Besides, the special mappings T and s fulfil (F2) and (F3), respectively. By (G1) and (G2) the functional b^+ is $(\mathbb{P}_3(M), M)$-zero-normal, that is M-Fejer-zero-normal (see Remark 5.14 and Remark 5.2.2). Considering also (G3) the condition (F4') is satisfied. Remark 8.6.2 shows (F4). Finally (G4) corresponds to (F5). ■

Remarks 8.9.

1. For $\Lambda_k = \{\lambda_k\}$ Method 8.7 is the well-known (projected) *subgradient method* which was investigated e.g. by Poljak [15], Oettli [13],[14], Cegielski [2], Kim, Ahn and Cho [10]. Especially the results of Poljak [15] and Oettli [13] turn out to be special cases of Corollary 8.8. Poljak starts with an equivalent optimization problem mentioned in Remark 8.6.1. But for the parameter range $\Lambda_k = \{\lambda_k\} \subseteq [\varepsilon_1, 2 - \varepsilon_2]$ he establishes only weak convergence of the method. Oettli assumes $H = I\!R^n$, $\Lambda_k = \{1\}$. Observe that in the finite-dimensional case the second part of (G1) and (G3) are fulfilled automatically (for the latter statement see [13]).

2. Corollary 8.8 can be applied to convex inequality systems

   $$f_i(x) \leq 0 \quad (i = 1, \ldots, m)\,,\ x \in Q$$

   with convex and continuous functionals $f_i \,:\, Q \to I\!R$. Then

   $$b(x) = p(f_1^+(x), \ldots, f_m^+(x))\,,$$

   where f_i^+ denotes again the positive part of f_i , is convex and continuous, if $p \,:\, I\!R^m_+ \to I\!R$ is convex, continuous and definite ($p(z) = 0$ iff $z = 0$). This property has each monotone norm p, especially the maximum norm

   $$p(z) = \max\{z_1, \ldots, z_m\},\ z = (z_1, \ldots, z_2)\ \in I\!R^m_+.$$

   Then $\partial f^+_{i(x)} \subseteq \partial b(x)$, where $i(x)$ satisfies

   $$f^+_{i(x)}(x) = \max\{f_1^+(x), \ldots, f_m^+(x)\}.$$

   Special results of this kind can be found in [5] and [14].

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Author: Prof. Dr. Dieter Schott, Neubrandenburger Str. 49a, D-18055 Rostock