On the Number of Covering Pairs in Specified Lattices

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Abstract

Given a class $\mathcal{F}$ of posets, let $ex(\mathcal{F}, n)$ denote the maximum number of covering pairs achieved by an n-element poset in $\mathcal{F}$. We determine this number for the class of N-free lattices and give crude lower and upper bounds for the classes of distributive and covering lattices. The structural characterization of covering lattices leads us to a class of lattices whose elements have a bounded number of lower or upper covers. This class satisfies a conjecture of Bollobas and Rival, who originally ask for $ex(\mathcal{L}, n)$ where $\mathcal{L}$ denotes the class of all lattices.

Throughout let P denote a partially ordered set, i.e. a non-empty set (which we also call P) together with a reflexive, antisymmetric and transitive binary relation $\leq$ on P. The cardinality of P may be indicated by $|P|$ or the integer n. Sometimes n will be given by a subscript like P_n.

Two elements x and y of P form a *covering pair* if $x < y$ and there is no element $z \in P$ such that $x < z < y$. In a covering pair (x, y) the element x is called a lower cover of y, while y is said to be an upper cover of x. Let C_P be the set of all covering pairs in P and denote its cardinality by $c(P)$. The entire relation $\leq$ is just the reflexive and transitive closure of C_P.

This paper is devoted to the discussion of the following extremal problem. Given a class $\mathcal{F}$ of posets, find the maximum number $ex(\mathcal{F}, n)$ of covering pairs achieved by an n-element poset in $\mathcal{F}$. The study of problem is motivated by similar questions in extremal graph theory. Usually a poset P is represented by its diagram, i.e. the digraph $\vec{H}(P)$ with the vertex set P and the edge set C_P, or its covering graph, i.e. the undirected underlying graph of the diagram. It is a long standing combinatorial problem to determine the maximum number of edges in specified graph classes.

In poset theory, the subject seems to go back to Bollobas and Rival [2]. They asked for the maximum number of covering pairs in lattices. Bollobas and Rival gave the estimate

$$\frac{\sqrt{2}}{4} n^{3/2} - n^{13/10} < ex(\mathcal{L}, n) < 3n^{3/2} \tag{1}$$

for the class $\mathcal{L}$ of lattices whenever n is sufficiently large, and conjectured

$$\lim_{n \to \infty} ex(\mathcal{L}, n) n^{-3/2} = \frac{\sqrt{2}}{4}. \tag{2}$$

This result shall be a starting point of our study. In the following sections we investigate the extremal problem for N-free lattices, distributive lattices and covering lattices. The structural characterization of covering lattices leads us to a subclass $\mathcal{L}_\delta$ of $\mathcal{L}$ that satisfies the Bollobas-Rival-conjecture.

1 Basic Notation

Before proceeding with our study we present some of the terminology to be used in this paper. Throughout we consider only finite graphs and digraphs without loops. V and E denote the vertex and the edge set of the digraph $\vec{G}$, α and β the incidence mappings of $\vec{G}$. When it is important to delineate the sets and incidence mappings we will write $\vec{G} = (V, E, \alpha, \beta)$. Every edge e of $\vec{G}$ corresponds to a pair $(\alpha(e), \beta(e))$ of vertices. Edges that correspond to the same vertex pair are called parallel or multiple. In a digraph without multiple edges the discussed correspondence is one-to-one. In this case we usually associate an edge with its vertex pair and omit the incidence mappings.

The cardinalities of V and E shall be indicated by $|G|$ and $e(G)$, respectively.

Let v be a vertex of $\vec{G}$. If no confusion arises we write sometimes $v \in \vec{G}$ meaning $v \in V$. The sets of predecessors and of successors of v shall be denoted by $D^-(v)$ and $D^+(v)$. Their union forms the neighbourhood of v. The indegree, the outdegree and the (total) degree of v shall be indicated by $d^-(v)$, $d^+(v)$ and $d(v)$. Vertices of indegree 0 are sources, of outdegree 0 are sinks. Let $Min_{\vec{G}}$ and $Max_{\vec{G}}$ denote respectively the sets of sources and sinks. Vertices neither in $Min_{\vec{G}}$ nor in $Max_{\vec{G}}$ are called internal. Let $Int_{\vec{G}}$ be the set of all internal vertices of the digraph. Additionally, put $min_{\vec{G}} := |Min_{\vec{G}}|, max_{\vec{G}} := |Max_{\vec{G}}|$ and $int_{\vec{G}} := |Int_{\vec{G}}|$.

Similar definitions for graphs will be used. The undirected underlying graph of a fixed digraph $\vec{G}$ shall be denoted by G.

Note that for the diagram of a poset P we use the subscript P instead of $\vec{H}(P)$. The minimal (maximal, internal) elements of P are the sources (sinks, internal vertices) of its diagram.

The *line digraph* $L(\vec{G})$ of the digraph $\vec{G}$ is that one whose vertices correspond to the edges of $\vec{G}$ and where an edge goes from u to v iff $\beta(u) = \alpha(v)$ in $\vec{G}$. A digraph $\vec{H}$ is a line digraph there is a digraph $\vec{G}$ such that $\vec{H}$ and $L(\vec{G})$ are isomorphic. Then $\vec{G}$ is said to be a *root digraph* of $\vec{H}$. In general for a fixed digraph $\vec{H}$ there exist more than one root digraph. In order to guarantee uniqueness we assume that all sinks and sources of the root digraph have degree 1.

A poset P whose diagram is a line digraph is said to be a *line poset.* The root digraph of its diagram shall be denoted by $\vec{R}(P)$, the cardinality of the vertex set of $\vec{R}(P)$ by $m(P)$. Obviously $\vec{R}(P)$ is always acyclic.

Let us introduce a special class of digraphs that we shall need in further discussion. A digraph on $\{v_1, \ldots, v_n\}$ is a *walk* if there exists a partition of its edge set into non-empty sets $E_1, \ldots, E_{n-1}$ such that each member e of E_i satisfies $\alpha(e) = v_i$ and $\beta(e) = v_{i+1}$ where $i = 1, \ldots, n-1$. For a walk let μ be the vector of length $n-1$ in which μ_i is the outdegree of v_i and call it the edge vector of the walk. It is clear that two walks are isomorphic iff they have the same edge vector. In this sense, walks are completely determined by their edge vectors.

Next we will recall some usual notations in poset theory. For any posets P and Q, let $P+Q$ be their direct sum, $P \oplus Q$ their ordinal sum and $P \times Q$ their direct product. The n-element chain will be denoted by L_n, the n-element antichain by A_n, the h-dimensional Boolean lattice by $B^{(h)}$. The poset $P_1 \oplus A_{n-2} \oplus P_1$ shall be indicated by M_n, the poset $A_2 \oplus A_2$ by C_4. Further, for any prime p let $\Pi^{(p)}$ be the lattice of subspaces of the finite projective plane over the field of order p. Note that the covering graph of $B^{(h)}$ is the h-dimensional hypercube, while C_4 has the quadrilateral as its covering graph.

The archetypal problem of extremal graph theory is to determine the maximum number of edges in graphs that do not contain some fixed graph H as an induced subgraph. Such graphs are called H-free. Similarly a poset is said to be *Q-free* if its diagram contains no induced subdigraph isomorphic to the diagram of a fixed poset Q. In addition, we call it *strong Q-free* if its diagram contains neither induced nor non-induced subdigraphs isomorphic to $\vec{H}(Q)$. Lattices, for example, are always C_4-free.

2 N-free Lattices

We shall denote by N the poset on four elements w, x, y and z with the covering pairs (w,y), (x,y) and (x,z). The class of N-free posets was introduced by Rival [9] and has been studied extensively in the recent years. Surveys on the subject can be found in Rival [10] and Burosch [3]. According to Heuchenne [7], a poset P is N-free iff it is a line poset.

Our aim is to determine $ex(\mathcal{N}, n)$ for the class $\mathcal{N}$ of N-free lattices. In order to do so we consider posets which are simultanously N-free and C_4-free.

Lemma 2.1 *A poset is N-free and C_4-free iff it is strong N-free.*

Proof. The sufficiency of the given conditions follows from the definition, since the diagram of N is a subdigraph of the diagram of C_4. In order to check their necessity let P be an N-free poset which is C_4-free, too. Assume P is not strong N-free. Then its diagram has a non-induced subdigraph isomorphic to the diagram of N. Denote the vertices of this subdigraph by w, x, y, z such that $(w, y), (x, y)$ and (x, z) are edges of the diagram, i.e. covering pairs of P. The missing edge in the subdigraph induced by w, x, y and z must be (w, z) since P is transitive and its diagram has no multiple edges. Hence, the vertices w, x, y, z induce a subdigraph isomorphic to the diagram of C_4 which gives a contradiction. ■

Theorem 2.2 *Let $\mathcal{N}^*$ be the class of all strong N-free posets. For $n \geq 3$ it holds*

$$ex(\mathcal{N}^*, n) = 2n - 4.$$

We first have to prove the following results.

Lemma 2.3 *An N-free poset P is strong N-free iff each vertex of $\vec{R}(P)$ has indegree 1 or outdegree 1.*

Proof. Take a strong N-free poset P. We have to show that $d^-(v) = 1$ or $d^+(v) = 1$ for all $v \in \vec{R}(P)$. For vertices not in Int_P this is true by convention. Let v be an internal vertex and suppose both, indegree and outdegree are at least 2. There are elements w, x, y, z in P such that $\beta(w) = \beta(x) = v = \alpha(y) = \alpha(z)$. Thus P has the covering pairs $(w, y), (x, y), (w, z)$ and (x, z), i.e. its diagram contains an induced subdigraph isomorphic to $\vec{H}(C_4)$ in contradiction to our assertion. Conversely, let P be an N-free poset whose root digraph satisfies the described condition. Assume P is not strong N-free. Then it is not C_4-free and has four elements w, x, y, z such that $\beta(w) = \beta(x) = \alpha(y) = \alpha(z)$. Denote this vertex by v in $\vec{R}(P)$. Its indegree as well as its outdegree are not less than 2. This gives again a contradiction and concludes the proof. ■

Lemma 2.4 *For an N-free poset P on n elements we have*

$$c(P) \geq 2n - m(P). \tag{3}$$

Equality holds in (3) iff P is strong N-free.

Proof. The number of covering pairs of a line poset can be determined by

$$c(P) = \sum_{v \in \vec{R}(P)} d^-(v)d^+(v).$$

If v is a sink or a source the product $d^-(v)d^+(v)$ equals 0 and, by convention, this is identical to $d(v) - 1$. If v is an internal vertex both, the indegree and the outdegree, are at least 1 and therefore their product is at least $d(v) - 1$. Hence, we obtain

$$\begin{aligned} c(P) &\geq \sum_{v \in \vec{R}(P)} (d(v) - 1) \\ c(P) &\geq 2n - m(P) \end{aligned}$$

which gives the first assertion. Now, the second assertion is an immediate consequence of Lemma 2.3. ■

Lemma 2.4 shows that the number of covering pairs of a strong N-free poset only depends on the cardinality of its ground set and on the number of vertices in the root digraph.

Relation (3) may also be written as $c(P) - n \geq n - m(P)$. The difference between the number of edges and the number of vertices in a digraph is usually called the *cyclomatic number*. Hence, the cyclomatic numbers of the poset diagram $\vec{H}(P)$ and its root digraph $\vec{R}(P)$ are equal for strong N-free posets.

Proof of Theorem 2.2. Let P be a strong N-free poset on $n \geq 3$ elements. Assume $m(P) \leq 3$. $\vec{R}(P)$ has at least one source and a sink, which is different from the source. Hence, in $\vec{R}(P)$ there is at most one internal element. This yields $2n \leq 1+1+(m(P)-1) = m(P)+1 \leq 4$ for the number of edges in $\vec{R}(P)$ and contradicts $n \geq 3$. For $m(P) \geq 4$ we get $c(P) \leq 2n-4$ applying Lemma 2.4. On the other hand, the poset M_n is strong N-free and $c(M_n) = 2n - 4$, which proves the theorem. ■

A poset is said to be *extremal* in a class $\mathcal{F}$ if it achieves, among all posets in this class having the same cardinality, the maximum possible number of covering pairs.

Theorem 2.5 *M_n is the only extremal poset in $\mathcal{N}^*$ on $n \geq 4$ elements.*

Proof. Let P be extremal in $\mathcal{N}^*$ and $n \geq 4$. Hence, $c(P) = 2n - 4$ and $m(P) = 4$. Among the four vertices of $\vec{R}(P)$ there is at least one source and a sink which is different from the source. By convention, both have degree 1. Denote the remaining vertices by u and v. Both are adjacent to at least $n-2$ edges. Since $\vec{R}(P)$ is acyclic, we may assume that all of them start in u and end in v and, thus, $\vec{R}(P)$ is a walk with the edge vector $(1, n-2, 1)$. Its line digraph is just the diagram of M_n, which concludes our proof. ■

The poset M_n $(n \geq 3)$ is not only C_4-free, but also a lattice. Therefore the assertions in Theorem 2.2 and Theorem 2.5 hold for the class $\mathcal{N}$ of N-free lattices, too.

3 Distributive Lattices

Theorem 3.1 *Let $\mathcal{D}$ denote the class of distributive lattices. Then*

$$ex(\mathcal{D}, n) = \frac{1}{2} n \log n + o(n \log n).$$

(The logarithms like all logarithms in this paper have base 2.)

Proof. We will prove the proposition by verifying the inequality

$$\frac{1}{2}(n+1)\{\log(n+1) - 1\} \leq ex(\mathcal{D}, n) \leq \frac{1}{2} n \log n. \tag{4}$$

We start with the upper bound. According to Duffus and Rival [4] the covering graph of a distributive lattice D of length h is a subgraph of the h-dimensional hypercube $H(B^{(h)})$ and its diameter equals h. Hence, all we have to show is that every subgraph G of a finite hypercube satisfies

$$e(G) \leq \frac{1}{2} n \log n \tag{5}$$

where n indicates the number of vertices and $e(G)$ the number of edges of G. We proceed by induction on the dimension h of the hypercube. For $h = 0$ the statement is trivially true. Choose $h \geq 1$ and assume (5) is true for all dimensions less than h. Let G be a subgraph of the hypercube $H(B^{(h)})$. Since the Boolean lattice $B^{(h)}$ is isomorphic to $B^{(h-1)} \times L_2$ its covering graph can be subdivided into two disjoint copies of $H(B^{(h-1)})$. This partition also subdivides G into two subgraphs G' and G". Without loss of generality we may assume that $|G'| \leq |G"|$. Put $n' = |G'|$. For $n' = 0$ the subgraph G is isomorphic to G', i.e. to a subgraph of a $(h-1)$-dimensional hypercube. This implies (5) immediately. If $n' \geq 1$, we obtain

$$\begin{aligned} e(G) &\leq e(G') + e(G") + n' \\ e(G) &\leq \frac{1}{2} n' \log n' + \frac{1}{2}(n-n')\log(n-n') + n'. \end{aligned}$$

The right side of the second inequality is monotone decreasing for $0 < n' \leq \frac{1}{5}n$ and monotone increasing for $\frac{1}{5}n \leq n' \leq \frac{1}{2}n$. For $n' \to +0$ it tends to $\frac{1}{2}n\log n$, while for $n' = \frac{1}{2}n$ it becomes exactly $\frac{1}{2}n\log n$. This proves inequality (5) and gives us the upper bound for $ex(\mathcal{D}, n)$.

Now we continue with the proof of the lower bound. Consider the poset D_n with

$$D_n = \begin{cases} L_1 & \text{if } n = 1 \\ D_{n/2} \times L_2 & \text{if } n \text{ is even} \\ D_{n-1} \oplus L_1 & \text{else} \end{cases}$$

Obviously D_n is a distributive lattice. It remains to show

$$c(D_n) \geq \frac{1}{2}(n+1)\{\log(n+1) - 1\}. \tag{6}$$

We will do this by induction on $r = \lfloor n/2 \rfloor$. Trivially the statement holds for $r = 0$ and $n = 1$. Let $r \geq 1$ be odd and suppose that (6) is true for all integers smaller than r. For $n = 2r+1$ we get

$$\begin{aligned} c(D_n) &= c(D_{2r+1}) = c(D_{2r}) + 1 = 2c(D_r) + r + 1 \\ c(D_n) &\geq (r+1)\{\log(r+1) - 1\} + (r+1) \\ c(D_n) &\geq \frac{1}{2}(n+1)\{\log(n+1) + 1\}. \end{aligned}$$

Additionally, for $n = 2r$ we obtain

$$\begin{aligned} c(D_n) &= c(D_{2r}) = c(D_{2r+1}) - 1 \\ c(D_n) &\geq \frac{1}{2}(n+2)\{\log(n+2) + 1\} - 1, \end{aligned}$$

which is larger than $\frac{1}{2}(n+1)\{\log(n+1) + 1\}$ for $n = 2r \geq 2$.

From $c(D_n) \leq ex(\mathcal{D}, n)$ we obtain the lower bound in (4), which proves the theorem. ■

Remark. For an inductive determination of the exact number of edges of subgraphs of the hypercube in terms of sums, see [5].

Distributive lattices are not neccessary in $\mathcal{N}$. This suggests the question which lattices are both, N-free and distributive.

Lemma 3.2 *Let P be an N-free lattice.*
(i) P is modular iff $\vec{R}(P)$ is a walk whose edge vector μ satisfies

$$\mu_1 = \mu_{m(P)-1} = 1 \text{ and } 1 \in \{\mu_i, \mu_{i+1}\} \text{ for } i = 1, \ldots, m(P) - 2. \tag{7}$$

(ii) Additionally, P is distributive iff

$$\mu_i \leq 2 \text{ for } i = 1, \ldots, m(P) - 1. \tag{8}$$

Proof. Let us recall some definitions. A lattice P, where any two elements have a common lower cover iff they have a common upper cover, is modular. In addition a modular lattice is distributive if it is M_5-free. The latter definition is due to Ward [11].

(i) Consider an N-free lattice P. It is modular iff

$$D^-(v) \cap D^-(w) = \emptyset \Leftrightarrow D^+(v) \cap D^+(w) = \emptyset$$

for any two elements v and w of P. This is equivalent to the relation

$$\alpha(v) = \alpha(w) \Leftrightarrow \beta(v) = \beta(w)$$

in the root digraph $\vec{R}(P)$. Hence, $\vec{R}(P)$ must be a walk and Lemma 2.3 implies (i).

(ii) Now let P be a modular N-free lattice. Its diagram has no induced subdigraph isomorphic to $\vec{H}(M_5)$ iff in $\vec{R}(P)$ we do not find three edges which are pairwise parallel. Since $\vec{R}(P)$ is a walk, this is equivalent to the assertion. ■

The poset M_n (which is extremal in $\mathcal{N}$) is modular but in general not distributive. If we consider our extremal problem in the class of N-free distributive lattices we get the following result.

Theorem 3.3 *Let $\mathcal{D}$ denote the class of distributive lattices. Then*

$$ex(\mathcal{N} \cap \mathcal{D}, n) = \lfloor \tfrac{4}{3}(n-1) \rfloor$$

and a poset P is extremal in $\mathcal{N} \cap \mathcal{D}$ iff it is the line poset of a walk whose edge vector μ can be obtained from a vector μ' of length $2\lfloor (n-1)/3 \rfloor + 1$ with

$$\mu'_i = \begin{cases} 1 & if \quad i \text{ is odd} \\ 2 & if \quad i \text{ is even} \end{cases}$$

by inserting exactly $(n-1)$ mod 3 ones.

Proof. Let P be an N-free distributive lattice. According to Lemma 3.2 its root digraph $\vec{R}(P)$ is a walk. Denote by t_1 respectively t_2 the number of ones and twos in its edge vector μ. By (7) we have

$$m(P) = t_1 + t_2 + 1 \text{ and } n = t_1 + 2t_2.$$

At the same time it holds

$$t_2 \leq t_1 - 1$$

by (7). Hence,

$$n - m(P) + 1 = t_2 \leq \frac{1}{2}m(P) - 1.$$

This implies

$$n \leq \frac{3}{2}m(P) - 2$$

and

$$c(P) = 2n - m(P) \leq \frac{4}{3}(n-1).$$

In order to find also a lower bound for $ex(\mathcal{D} \cap \mathcal{N}, n)$ we consider the walk described in the theorem. It has exactly $t_1 = \frac{1}{3}(n-1-r)+1+r = \frac{1}{3}(n+2+2r)$ ones and $t_2 = \frac{1}{3}(n-1-r)$ twos where r denotes the residue class of $n-1$ modulo 3. Thus, the number of covering pairs in the line poset of the chosen walk is indeed

$$c(P) = t_1 + 3t_2 - 1 = (n-1) + \frac{1}{3}(n-1-r) = \lfloor \tfrac{4}{3}(n-1) \rfloor.$$

Now we look at the extremal posets in $\mathcal{D} \cap \mathcal{N}$. An N-free distributive lattice P has the optimal number of covering pairs iff

$$c(P) = (n-1) + \lfloor \tfrac{1}{3}(n-1) \rfloor.$$

On the other hand, the edge vector of $\vec{R}(P)$ contains $t_2 = (t_1 + 3t_2 - 1) - (t_1 + 2t_2) + 1 = c(P) - n + 1$ twos, and therefore this is equivalent to

$$t_2 = \lfloor \tfrac{1}{3}(n-1) \rfloor.$$

This and Lemma 3.2 imply that the walks described in the theorem are exactly those ones which satisfy this condition. Thus the extremal posets in $\mathcal{D} \cap \mathcal{N}$ are just the claimed ones. ■

4 Lattices with bounded in- or outdegree

For an arbitrary, but fixed integer δ let $\mathcal{L}_\delta$ be the class of all lattices P such that

$$d^-(x) \leq \delta \text{ or } d^+(x) \leq \delta \text{ for every } x \in P. \tag{9}$$

Our aim is to show that this class satisfies the Bollobas-Rival-conjecture (2). Again, we will prove this statement for the slightly larger class ${\mathcal{L}^*}_\delta$ of C_4-free posets satisfying condition (9).

Theorem 4.1 *For every integer δ it holds*

$$ex({\mathcal{L}^*}_\delta, n) = \frac{\sqrt{2}}{4} n^{3/2} + o(n^{3/2}).$$

Before giving the proof of this theorem we need some preparations.

Lemma 4.2 *Let P be a C_4-free poset without internal elements. Then*

$$c(P) \leq \frac{1}{4} n + \frac{1}{4} n (2n-3)^{1/2} \tag{10}$$

for $n \geq 2$.

Proof. For antichains the assertion is trivial. Suppose, P is not an antichain. Behrendt [1] called a three-element subset $\{x, y, z\}$ of a poset P a fork if either $(x, y) \in C_P$ and $(z, y) \in C_P$, or $(y, x) \in C_P$ and $(y, z) \in C_P$ (where x, y, z are taken in a suitable order). Denote by $f_1(P)$ and $f_2(P)$ the number of forks of the first kind and, respectively, of the second kind. This implies

$$f_1(P) = \sum_{x \in P} \binom{d^-(x)}{2} \text{ and } f_2(P) = \sum_{x \in P} \binom{d^+(x)}{2}. \tag{11}$$

Take P without internal elements. For convenience, put $n' = n - max_P$ and $n'' = n - min_P$. Hence, n' counts the non-maximal elements, n'' the non-minimal elements. From (11) and Jensen's inequality we obtain

$$\begin{aligned} f_1(P) &= \sum_{x \in P - Min_P} \binom{d^-(x)}{2} \geq n'' \binom{c(P)/n''}{2} \\ f_2(P) &= \sum_{x \in P - Max_P} \binom{d^+(x)}{2} \geq n' \binom{c(P)/n'}{2}. \end{aligned}$$

In addition, let P be C_4-free. This implies

$$f_1(P) \leq \binom{n'}{2} \text{ and } f_2(P) \leq \binom{n''}{2}.$$

We have

$$\begin{aligned} c^2(P) - n''c(P) &\leq n'n''(n'-1) \\ c^2(P) - n'c(P) &\leq n'n''(n''-1) \\ c^2(P) - \frac{1}{2}(n'+n'')c(P) &\leq \frac{1}{8}(n'+n'')^2(n'+n''-2) \end{aligned}$$

and consequently

$$\begin{aligned} c(P) &\leq \frac{1}{4}(n'+n'') + \frac{1}{4}(n'+n'')(2n'+2n''-3)^{1/2} \\ c(P) &\leq \frac{1}{4}n + \frac{1}{4}n(2n-3)^{1/2} \end{aligned}$$

for any C_4-free poset P_n $(n \geq 2)$ without internal elements. ■

Given a poset P, the *covering split* I_P shall be a poset with minimal elements x' $(x \in P)$ and maximal elements x'' $(x \in P)$ with $x' < y''$ in I_P iff $(x, y) \in C_P$. Obviously I_P has no internal elements and it yields $c(P) = c(I_P)$.

Lemma 4.3 *A poset is C_4-free iff its covering split is C_4-free.*

Proof. The statement is a consequence of the fact, that two elements x, y form a covering pair in P iff x' and y'' form a covering pair in I_P. ■

Proof of Theorem 4.1. We will prove the proposition by checking the inequality

$$\frac{\sqrt{2}}{4}(n^{3/2} - n^{1.276}) < ex(\mathcal{L}^*{}_\delta, n) < \frac{\sqrt{2}}{4} n^{3/2} + (\delta + \frac{1}{4})n \tag{12}$$

for sufficiently large n.

First we check the upper bound of (12). Take a C_4-free poset on $n \geq 6$ elements. Consider the covering split I_P of P.

Let S be the set of all elements in I_P whose degree is not larger than δ. Put $s = |S|$. $P \in \mathcal{L}_\delta$ yields $s \geq n$. Denote by I_P^* the poset induced by $I_P - S$. Then, I_P^* has at least $c(I_P) - \delta s$ covering pairs. Using (10) we obtain

$$\begin{aligned} c(P) &= c(I_P) \leq c(I_P^*) + \delta s \\ c(P) &\leq \begin{cases} \frac{1}{4}(2n - s) + \frac{1}{4}(2n - s)(4n - 2s - 3)^{1/2}\} + \delta s & \text{if } s \leq 2n - 2 \\ 0 & \text{else} \end{cases} \end{aligned}$$

The right side of this inequality is monotone decreasing for $n \leq s \leq 2n$ and consequently

$$\begin{aligned} c(P) &\leq \frac{1}{4}(2n - 3)^{1/2} + (\delta + \frac{1}{4})n \\ c(P) &< \frac{\sqrt{2}}{4} n^{3/2} + (\delta + \frac{1}{4})n. \end{aligned}$$

To verify the lower bound we follow the idea used by Bollobas and Rival [2]. Let p be a prime and $\Pi^{(p)}$ the lattice of subspaces of the finite projective plane over the field of order p. Denote $P_1 \oplus \Pi^{(p)} \oplus P_1$ by $P^{(p)}$. For every $\delta \geq 1$ we have $P^{(p)} \in \mathcal{L}^*{}_\delta$. Further, $P^{(p)}$ has cardinality $n = 2p^2 + 2p + 6$ and $c(P^{(p)}) = (p+3)(p^2 + p + 1) + 2$. Clearly, equality holds in (13):

$$\begin{aligned} (p+3)(p^2+p+1) + 2 &= p^3 + 4p^2 + 4p + 5 \\ &= \frac{1}{2}(p^2 + p + 1)(2p + 1) + \frac{5}{2}(p^2 + p + 3) - 3 \\ &= (\frac{n}{4} - 1)(2n - 11)^{1/2} + \frac{5}{4}n - 3 \end{aligned}$$

Now, for an arbitrary n, let p be the smallest prime such that

$$n^{(p)} = 2p^2 + 2p + 6 \geq n.$$

According to Iwaniec and Pintz [8] consecutive primes q and p satisfy

$$p - q < q^{\alpha}$$

for sufficiently large primes and every $\alpha > \frac{11}{20}$. Thus, if n is large enough, we have

$$n^{(p)} < n + n^{\beta}$$

where $\beta > \frac{\alpha+1}{2} > \frac{31}{40}$.
It is always possible to delete $n^{(p)} - n$ internal elements of $\Pi^{(p)}$ in $P^{(p)}$ such that we obtain a lattice P, which is again in $\mathcal{L}_1$, and thus in $\mathcal{L}^*{}_{\delta}$, too. It holds

$$\begin{aligned}
c(P) &\geq c(P^{(p)}) - (n^{(p)} - n)(p+2) \\
&\geq \frac{1}{2}(n^{(p)} - 4)(p+3) + 2 - (n^{(p)} - n)(p+2) \\
&\geq \frac{1}{2}n^{(p)}(p+2) - (n^{(p)} - n)(p+2) \\
&\geq \frac{1}{2}(n - n^{(p)})(p+2) + \frac{1}{2}n(p+2) \\
&> \frac{1}{2}(n - n^{\beta})(p+2).
\end{aligned}$$

Additionally, $2(p+2)^2 = 2p^2 + 8p + 8 \geq 2p^2 + 2p + 6 \geq n$. Therefore, we have

$$c(P) > \frac{\sqrt{2}}{4}(n^{3/2} - n^{\gamma})$$

where $\gamma > \frac{\alpha+2}{2} > \frac{51}{40}$. This yields the lower bound in (12), and together with the upper bound determined above we obtain the assertion. ∎

The poset $P^{(p)}$ in the proof of Theorem 4.1 is not only C_4-free, but also a lattice. Hence, Theorem 4.1 stays true if we write $\mathcal{L}_{\delta}$ instead of $\mathcal{L}^*{}_{\delta}$.

5 Covering Lattices

Let Q be a poset and C_Q as usual the set of its covering pairs. On C_Q we define a partial order by $(x, x') \leq (y, y')$ iff the pairs are identical or $x' \leq y$ in Q. We call the obtained poset C_Q the *covering poset* of Q. A poset P is a covering poset if there is a poset Q such that P and the covering poset C_Q

are isomorphic. Behrendt [1] proved that a poset P is a covering poset iff it is simultanously N-free and $P_1 \oplus (P_1 + L_t) \oplus P_1$-free for $t = 1, \ldots, n-3$. As a consequence, the poset P is the covering poset of Q if it is the line poset of the diagram of Q.

Lattices which are covering posets, too, shall be called *covering lattices*. The following theorem gives a structural characterization.

Theorem 5.1 *Let Q be a poset without isolated elements, and let P be the covering poset of Q. Then P is a lattice iff the following conditions hold.*
(i) P is C_4-free.
(ii) P has a minimum and a maximum element.
(iii) Q is a lattice.

Proof. First we show that the conditions are necessary. Since (i) and (ii) trivially hold it remains to verify that Q is a lattice. Note that Q has evidently a minimum and a maximum element. Let us suppose that Q is not in $\mathcal{L}$. We assume that there are two incomparable elements u_1 and u_2 in Q without a meet. Let S be the set of elements of Q which are smaller than both, u_1 and u_2. S is not empty and thus there are at least two incomparable elements v_1 and v_2 which are maximal in S. Let x_1 and x_2 be the elements of P such that $\alpha(x_1) = u_1$ and $\alpha(x_2) = u_2$ in the diagram of Q. Since P is a lattice, x_1 and x_2 have a meet z in P. This implies that $\beta(z)$ is greater than both, v_1 and v_2. On the other hand $\beta(z)$ belongs to S and this contradicts the maximality of v_1 and v_2 in S. In the same way, we disprove that there are two incomparable elements without a join. Hence, Q is a lattice.

Now we show the sufficiency of the conditions. Take any posets P and Q that satisfy the assumptions such that P is the covering poset of Q. Suppose, P is not in $\mathcal{L}$. Again, we assume that there are two incomparable elements x_1 and x_2 in P without a meet. Let S be the set of elements of P which are smaller than x_1 and x_2. (ii) yields that S is not empty. Hence, there are two elements y_1 and y_2 which are maximal in S. Choose u_1 and u_2 in Q such that $\alpha(x_1) = u_1$ and $\alpha(x_2) = u_2$ in $\vec{H}(Q)$. By (iii), u_1 and u_2 have a meet w in Q. By (i), the poset P is strong N-free. Now, according to Lemma 2.3, w has exactly one lower cover w'. Denote by z the element of P which corresponds to the covering pair $(w', w) \in C_Q$. z is greater than both, y_1 and y_2. But z belongs to S contradicting the maximality of y_1 and y_2 in S. For the same reasons we do not find two incomparable elements without a join. Hence, our theorem is true. ■

Denote by $\mathcal{L}'_1$ the class of lattices P with

$$d^-(x) = 1 \text{ or } d^+(x) = 1 \text{ for each } x \in P.$$

It is easy to see that $\mathcal{L}'_1$ contains all those lattices from $\mathcal{L}_1$ whose maximum and minimum element have degree 1.

Corollary 5.2 *A poset P is a covering lattice iff it is the covering poset of a lattice from $\mathcal{L}'_1$.*

Proof. The statement follows immediately from Lemma 2.3 and Theorem 5.1. ■

With the help of Theorem 5.1 we are able to give an answer to our extremal problem for covering lattices.

Theorem 5.3 *For the class $\mathcal{C}$ of covering lattices it holds*

$$ex(\mathcal{C}, n) = 2n - 2n^{2/3} + o(n^{2/3}). \tag{13}$$

Proof. Let P_n be any covering lattice. Then there is a lattice $Q \in \mathcal{L}'_1$ such that P is the covering poset of Q. Since the diagram of Q and $\vec{R}(P)$ are isomorphic, we have $c(P) = 2n - |Q| = 2n - m(P)$. By Theorem 4.1 and $\mathcal{L}'_1 \subset \mathcal{L}^*{}_1$, it holds

$$n < \frac{\sqrt{2}}{4}\{m(P)\}^{3/2} + \frac{5}{4}m(P).$$

This implies

$$\begin{aligned} n &< (\{\frac{1}{2}m(P)\}^{1/2} + \frac{5}{6})^3 \qquad (14)\\ 2n^{2/3} - \frac{10}{3}n^{1/3} + \frac{25}{18} &< m(P). \qquad (15)\end{aligned}$$

Hence, we obtain $c(P) = 2n - m(P) < 2n - 2n^{2/3} + \frac{10}{3}n^{1/3}$, which gives an upper bound for $ex(\mathcal{C}, n)$.

Now, for fixed n, let p be the smallest prime such that

$$n^{(p)} = p^3 + 4p^2 + 4p + 5 \geq n.$$

The result of Iwaniec and Pintz on consecutive primes implies

$$(n^{(p)})^{2/3} < n^{2/3} + n^{\beta}$$

whenever n is large enough and $\beta > \frac{\alpha+1}{3} > \frac{31}{60}$. Consider the lattice

$$Q^{(p)} \cong P_1 \oplus \Pi^{(p)} \oplus P_1.$$

If n is sufficiently large it is always possible to delete $n^{(p)} - n$ covering pairs of $Q^{(p)}$ such that the remaining poset Q is in $\mathcal{L}_1'$, too. Thus

$$\begin{aligned} |Q| &= |Q^{(p)}| = 2p^2 + 2p + 6 \\ &\leq 2(p+1)^2 \leq 2(n^{(p)})^{2/3} \\ &\leq 2(n^{2/3} + n^{\beta}) \end{aligned}$$

if we choose n large enough. The covering lattice P of Q has therefore $c(P) \geq 2n - 2n^{2/3} - 2n^{\beta}$ edges, which gives a lower bound for $ex(\mathcal{C}, n)$. Comparing it with the upper bound determined above we get (13). ■

Remark. Note that this time it is not enough to consider the class $\mathcal{C}^*$ of C_4-free covering posets. Here we obtain

$$ex(\mathcal{C}^*, n) = \lfloor 2(n-1) - 2(n-1)^{1/2} \rfloor$$

if $n \geq 8$. For a proof see [6].

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