Approximation Order of Shift–Invariant Subspaces of $L^2(\mathbb{R})$ Generated by Refinable Function Vectors

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In this paper, finitely generated shift–invariant subspaces $\mathcal{S}(\boldsymbol{\phi})$ of $L^2(\mathbb{R})$ providing controlled approximation order m are considered. Assuming that the vector of generating functions $\boldsymbol{\phi}$ of $\mathcal{S}(\boldsymbol{\phi})$ is refinable, necessary and sufficient conditions for the refinement mask are derived. In particular, if algebraic polynomials can be exactly reproduced in $\mathcal{S}(\boldsymbol{\phi})$, then a factorization of the refinement mask can be given. This result is a natural generalization of the result for the principal shift–invariant subspace of $L^2(\mathbb{R})$, where the refinement mask contains a power of $\left(\frac{1+e^{-iu}}{2}\right)$ if the scaling function is regular.

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1. Introduction

In the last time, a lot of papers have studied the so–called multiresolution analysis of multiplicity r ($r \in \mathbb{N}$, $r \geq 1$), generated by dilates and translates of a finite set of functions ϕ_ν ($\nu = 0, \ldots, r-1$), and the construction of corresponding "multiwavelets" (cf. [9, 11, 12, 14, 20]). In [1], multiwavelets are used for sparse representation of integral operators. Further applications for solving differential equations

by finite element methods seem to be possible, since scaling functions and multiwavelets with very small support can be constructed (cf. [20]). For finite elements short support is crucial. In order to obtain multiwavelets with vanishing moments, the problem remains, how to choose the scaling functions ϕ_ν, such that algebraic polynomials of degree $< m$ ($m \in \mathbb{N}$) can be exactly reproduced in the sample space $\mathcal{S}(\boldsymbol{\phi})$ generated by $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$, or equivalently, such that $\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order m. First ideas to solve this problem can be found in [9] and [23, 24], where examples with two scaling functions ϕ_0, ϕ_1 with very small support are treated, such that the translates $\phi_0(\cdot - l)$, $\phi_1(\cdot - l)$ ($l \in \mathbb{Z}$) form an orthogonal basis of $\mathcal{S}(\boldsymbol{\phi})$ and such that $\mathcal{S}(\boldsymbol{\phi})$ provides approximation order $m = 2$.
In this paper, we consider shift–invariant subspaces of $L^2(\mathbb{R})$ generated by a refinable vector $\boldsymbol{\phi}$ of r ($r \in \mathbb{N}$) functions and providing controlled approximation order m. We will study the consequences for the refinement mask of $\boldsymbol{\phi}$ in some detail.

Let us introduce some notations. Consider the Hilbert space $L^2 = L^2(\mathbb{R})$ of all square integrable functions on $\mathbb{R}$. The Fourier transform of $f \in L^2(\mathbb{R})$ is defined by $\hat{f} := \int_{-\infty}^{\infty} f(x) e^{-ix\cdot} \, \mathrm{d}x$. Further, let $BV(\mathbb{R})$ be the set of all functions which are of bounded variation over $\mathbb{R}$ and normalized by

$$\begin{aligned} f(-\infty) &= f(-\infty + 0) = 0, \qquad f(\infty) = f(\infty - 0) = 0, \\ f(x) &= \frac{1}{2}(f(x+0) + f(x-0)) \quad (-\infty < x < \infty). \end{aligned}$$

If $f \in L^2(\mathbb{R}) \cap BV(\mathbb{R})$, then the Poisson Summation Formula

$$\sum_{l \in \mathbb{Z}} f(l)\, e^{-iul} = \sum_{j \in \mathbb{Z}} \hat{f}(u + 2\pi j) \tag{1.1}$$

is satisfied (cf. [3]). For a measurable function f on $\mathbb{R}$ and $m \in \mathbb{N}$ let

$$\begin{aligned} \|f\|_2 &:= \left(\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x\right)^{1/2}, \\ |f|_{m,2} &:= \|\mathrm{D}^m f\|_2, \qquad \|f\|_{m,2} := \sum_{k=0}^{m} \|\mathrm{D}^k f\|_2 . \end{aligned}$$

Here and in the following, D denotes the differential operator $\mathrm{D} := \mathrm{d}/\,\mathrm{d}\cdot$. Let $W_2^m(\mathbb{R})$ be the usual Sobolev space with the norm $\|\cdot\|_{m,2}$. The l^2–norm of a sequence $\boldsymbol{c} := \{c_l\}_{l\in\mathbb{Z}}$ is defined by $\|\boldsymbol{c}\|_{l^2} := (\sum_{l\in\mathbb{Z}} |c_l|^2)^{1/2}$.
We say that a subspace $\mathcal{S} \subset L^2(\mathbb{R})$ is *shift–invariant* if for each $f \in \mathcal{S}$ all integer translates $f(\cdot - l)$ ($l \in \mathbb{Z}$) are also contained in $\mathcal{S}$. A closed shift–invariant subspace $\mathcal{S} \subset L^2(\mathbb{R})$ generated by one function $\phi \in L^2(\mathbb{R})$

$$\mathcal{S} = \mathcal{S}(\phi) := \mathrm{clos}_{L^2} \, \mathrm{span} \, \{\phi(\cdot - l) : l \in \mathbb{Z}\}$$

is called *principal*. We are especially interested in *finitely generated* shift–invariant closed subspaces $\mathcal{S} \in L^2(\mathbb{R})$ formed by a finite number $r \in \mathbb{N}$ of functions $\phi_\nu \in L^2(\mathbb{R})$ ($\nu = 0, \ldots, r-1$)

$$\mathcal{S} = \mathcal{S}(\boldsymbol{\phi}) := \mathrm{clos}_{L^2} \, \mathrm{span} \, \{\phi_\nu(\cdot - l) : l \in \mathbb{Z};\ \nu = 0, \ldots, r-1\}$$

with $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$. The set $\mathcal{B}(\boldsymbol{\phi}) := \{\phi_\nu(\cdot - l) : l \in \mathbb{Z}, \nu = 0, \ldots, r-1\}$ forms a *Riesz basis* of $\mathcal{S}(\boldsymbol{\phi})$ if there exist constants $0 < A \leq B < \infty$ with

$$A \sum_{\nu=0}^{r-1} \|\boldsymbol{c}_\nu\|_{l^2}^2 \leq \| \sum_{\nu=0}^{r-1} \sum_{l \in \mathbb{Z}} c_{\nu,l}\, \phi_\nu(\cdot - l)\|_{L^2}^2 \leq B \sum_{\nu=0}^{r-1} \|\boldsymbol{c}_\nu\|_{l^2}^2$$

for any sequences $\boldsymbol{c}_\nu = \{\boldsymbol{c}_{\nu,l}\}_{l \in \mathbb{Z}}$ $(\nu = 0, \ldots, r-1)$.
We say that $\mathcal{S}(\boldsymbol{\phi})$ provides the *approximation order* m, if for each $f \in W_2^m(\mathbb{R})$ there are sequences $\boldsymbol{c}_\nu^h = \{c_{\nu,l}^h\}_{l \in \mathbb{Z}}$ $(\nu = 0, \ldots, r-1; h > 0)$ such that for a constant c independent of h we have:

(1) $$\|f - h^{-1/2} \sum_{\nu=0}^{r-1} \sum_{l \in \mathbb{Z}} c_{\nu,l}^h\, \phi_\nu(\cdot/h - l)\|_2 \leq c\, h^m\, |f|_{m,2}.$$

Recently, closed shift–invariant subspaces of $L_2(\mathbb{R})$ providing a specified approximation order were characterized in [2]. In particular, in [2] it was shown that the approximation order of a finitely generated shift–invariant subspace $\mathcal{S}(\boldsymbol{\phi})$ can already be realized by a specifiable principal subspace $\mathcal{S}(f) \subset \mathcal{S}(\boldsymbol{\phi})$, where f can be represented as a finite linear combination

$$f = \sum_{\nu=0}^{r-1} \sum_{l \in \mathbb{Z}} a_{\nu l}\, \phi_\nu(\cdot - l). \qquad (a_{\nu l} \in \mathbb{R}).$$

As known, the approximation order of a principal shift–invariant subspace $\mathcal{S}(\phi)$ can be described in terms of the Fourier transform of ϕ by the so–called Strang–Fix conditions [2, 10, 17, 21, 22].
Following the notations in [15], $\mathcal{S}(\boldsymbol{\phi})$ provides *controlled approximation order* m, if (1) holds, and furthermore the following inequalities are satisfied:
(2) We have

$$\|\boldsymbol{c}_\nu^h\|_{l^2} \leq c\, \|f\|_2 \quad (\nu = 0, \ldots, r-1),$$

where c is independent of h.
(3) There is a constant δ independent of h such that for $l \in \mathbb{Z}$

$$\mathrm{dist}\,(lh, \mathrm{supp}\, f) > \delta \quad \Rightarrow \quad c_{\nu,l}^h = 0 \quad (\nu = 0, \ldots, r-1).$$

In [15], the strong connection of approximation order in a finitely generated subspace $\mathcal{S}(\boldsymbol{\phi})$ and the Strang–Fix conditions for $\boldsymbol{\phi}$ was shown.

A function $\phi \in L^2(\mathbb{R})$ is called *refinable* if ϕ satisfies a *refinement equation* of the form

$$\phi = \sum_{l \in \mathbb{Z}} p_l\, \phi(2 \cdot - l) \qquad (p_l \in \mathbb{R}),$$

or equivalently, if ϕ satisfies the Fourier transformed refinement equation

$$\hat{\phi} = P_\phi(\cdot/2)\, \hat{\phi}(\cdot/2)$$

with the refinement mask

$$P = P_\phi := \frac{1}{2} \sum_{l \in \mathbb{Z}} p_l \, e^{-il\cdot}. \qquad (1.2)$$

Note that P_ϕ is a 2π–periodic function. The following result holds for the principal space:

Theorem 1.1 *Let $\phi \in L^2(\mathbb{R}) \cap BV(\mathbb{R})$ be a refinable function satisfying the decay property $|\phi(x)| = O(|x|^{-m-1-\epsilon})$ $(x \to \infty)$ for $m \in \mathbb{N}$ and $\epsilon > 0$, and let $\mathcal{B}(\phi)$ form a Riesz basis of $\mathcal{S}(\phi)$. Then the closed shift–invariant subspace $\mathcal{S}(\phi)$ provides controlled approximation order m if and only if the refinement mask P_ϕ satisfies the equalities*

$$\begin{aligned} \mathrm{D}^\mu P_\phi(\pi) &= 0 \quad (\mu = 0, \ldots, m-1), & (1.3)\\ P_\phi(0) &= 1. & (1.4) \end{aligned}$$

The assertions (1.3), (1.4) *are equivalent to the following condition:*
The refinement mask P_ϕ can be factorized in the form

$$P_\phi(u) = \left(\frac{1 + e^{-iu}}{2} \right)^m S(u) \qquad (1.5)$$

with $S(0) = 1$, where S is a 2π–periodic $(m-1)$–times continuously differentiable function.

Note that, under the conditions of Theorem 1.1, controlled approximation order m is ensured if and only if algebraic polynomials of degree $< m$ can be exactly reproduced in $\mathcal{S}(\phi)$, i.e., if $\mathcal{S}(\phi)$ provides accuracy m. For corresponding results see also [4]. There is a close connection between accuracy and regularity of ϕ. For $|S(u)| \le 1$, the condition (1.5) is equivalent to the assertion that $\phi \in C^{m-1}(\mathbb{R})$ (cf. [6, 8]).

We want to generalize the result of Theorem 1.1 for finitely generated subspaces $\mathcal{S}(\boldsymbol{\phi})$ of $L^2(\mathbb{R})$. The function vector $\boldsymbol{\phi}$ with elements in $L^2(\mathbb{R})$ is *refinable*, if $\boldsymbol{\phi}$ satisfies a refinement equation of the form

$$\boldsymbol{\phi} = \sum_{l \in \mathbb{Z}} \boldsymbol{P}_l \, \boldsymbol{\phi}(2 \cdot -l) \qquad (\boldsymbol{P}_l \in \mathbb{R}^{r \times r}). \qquad (1.6)$$

By Fourier transform we obtain

$$\hat{\boldsymbol{\phi}} = \boldsymbol{P}_{\boldsymbol{\phi}}(\cdot/2) \, \hat{\boldsymbol{\phi}}(\cdot/2) \qquad (1.7)$$

with $\hat{\boldsymbol{\phi}} := (\hat{\phi}_\nu)_{\nu=0}^{r-1}$ and with the refinement mask

$$\boldsymbol{P} = \boldsymbol{P}_{\boldsymbol{\phi}} := \frac{1}{2} \sum_{l \in \mathbb{Z}} \boldsymbol{P}_l \, e^{-il\cdot}. \qquad (1.8)$$

Note that $\boldsymbol{P}$ is now an $(r \times r)$–matrix of 2π–periodic functions. The purpose of this paper is to find necessary and sufficient conditions for the refinement mask $\boldsymbol{P}$ yielding approximation order m, analogously as for the principal space in Theorem 1.1. The main result in this paper is the following

Theorem 1.2 *Let $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$ be a refinable vector of functions $\phi_\nu \in L^2(\mathbb{R}) \cap BV(\mathbb{R})$ satisfying the decay properties $|\phi_\nu(x)| = O(|x|^{-m-1-\epsilon})$ $(x \to \infty)$ for $m \in \mathbb{N}$, $\nu = 0, \ldots, r-1$ and $\epsilon > 0$. Further, let $\mathcal{B}(\boldsymbol{\phi})$ form a Riesz basis of $\mathcal{S}(\boldsymbol{\phi})$. Then the following assertions are equivalent:*

(a) *The finitely generated subspace $\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order m.*

(b) *Algebraic polynomials of degree $< m$ can be exactly reproduced in $\mathcal{S}(\boldsymbol{\phi})$.*

(c) *The refinement mask $\boldsymbol{P}$ of $\boldsymbol{\phi}$ satisfies the following conditions:*
The elements of $\boldsymbol{P}$ are $(m-1)$–times continuously differentiable functions in $L^2_{2\pi}(\mathbb{R})$, and there are vectors $\boldsymbol{y}_0^k \in \mathbb{R}^r$; $\boldsymbol{y}_0^0 \neq \boldsymbol{0}$ $(k = 0, \ldots, m-1)$ such that for $n = 0, \ldots, m-1$ we have

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^T (2i)^{k-n} (D^{n-k}\boldsymbol{P})(0) &= 2^{-n} (\boldsymbol{y}_0^n)^T, \\
\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^T (2i)^{k-n} (D^{n-k}\boldsymbol{P})(\pi) &= \boldsymbol{0}^T,
\end{aligned}$$

where $\boldsymbol{0}$ denotes the zero vector.

Furthermore, if $\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order m, then there are nonzero vectors $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{m-1}$ such that $\boldsymbol{P}$ factorizes

$$\boldsymbol{P}(u) = \frac{1}{2^m} \boldsymbol{C}_{m-1}(2u) \ldots \boldsymbol{C}_0(2u)\, \boldsymbol{S}(u)\, \boldsymbol{C}_0(u)^{-1} \ldots \boldsymbol{C}_{m-1}(u)^{-1}$$

where the $(r \times r)$–matrices $\boldsymbol{C}_k$ are defined by $\boldsymbol{x}_k$ $(k = 0, \ldots, m-1)$ as in (4.1) – (4.2) and $\boldsymbol{S}(u)$ is an $(r \times r)$–matrix with $(m-1)$–times continuously differentiable entries in $L^2_{2\pi}(\mathbb{R})$. In particular, for the determinant of $\boldsymbol{P}$ it follows

$$\det \boldsymbol{P}(u) = \left(\frac{1 + e^{-iu}}{2^r} \right)^m \det \boldsymbol{S}(u).$$

Note that the assertions of Theorem 1.1 follow from Theorem 1.2 in the special case $r = 1$.

The paper is organized as follows.
In Section 2 we will show that, under mild conditions on the scaling functions ϕ_ν $(\nu = 0, \ldots, r-1)$, the finitely generated subspace $\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order m $(m \in \mathbb{N})$ if and only if algebraic polynomials of degree $< m$ can be exactly reproduced in $\mathcal{S}(\boldsymbol{\phi})$. In Section 3, we introduce the doubly infinite matrix $\boldsymbol{L}$ containing the coefficient matrices $\boldsymbol{P}_l$ which occur in the refinement mask

(1.8). Assuming that algebraic polynomials of degree $< m$ can be reproduced in $\mathcal{S}(\boldsymbol{\phi})$, we derive some consequences for the eigenvalues and eigenvectors of $\boldsymbol{L}$. Similar results can also be found in [23]. Further, we give necessary and sufficient conditions for the refinement mask $\boldsymbol{P}$ of $\boldsymbol{\phi}$ yielding controlled approximation order m in $\mathcal{S}(\boldsymbol{\phi})$ (cf. Theorem 3.2). These conditions use values of the refinement mask and its derivatives at the points $u = 0$ and $u = \pi$, only. Finally, in Section 4, we show that, assuming controlled approximation order m in $\mathcal{S}(\boldsymbol{\phi})$, the refinement mask $\boldsymbol{P}$ can be factorized. As already pointed out in Theorem 1.2, it follows that the determinant of $\boldsymbol{P}(u)$ contains the factor $(1 + e^{-iu})^m$.
During the work at this paper the author has obtained a new preprint [13] which contains the result of Theorem 3.2, but with another proof.

2. Approximation order and accuracy

In this section, we will investigate the connections between approximation order of a finitely generated subspace $\mathcal{S}(\boldsymbol{\phi})$ and reproduction of algebraic polynomials in $\mathcal{S}(\boldsymbol{\phi})$. Let $r \in \mathbb{N}$ be fixed. The following assumptions for the scaling functions $\phi_\nu \in L^2(\mathbb{R})$ $(\nu = 0, \ldots, r-1)$ will often be needed in our further considerations:

(i) For $m \in \mathbb{N}$ and $\epsilon > 0$ we have

$$|\phi_\nu(x)| = \mathrm{O}(|x|^{-m-1-\epsilon}) \quad (x \to \infty).$$

(ii) $\quad \phi_\nu \in BV(\mathbb{R})$.

(iii) The set $\mathcal{B}(\boldsymbol{\phi})$ forms a Riesz basis of $\mathcal{S}(\boldsymbol{\phi})$.

The assumption (i) ensures that the Fourier transforms $\hat{\phi}_\nu$ are $(m-1)$–times continuously differentiable. If the assumption (ii) is also satisfied, then the Poisson Summation Formula (1.1) holds for ϕ_ν. We observe the following

Lemma 2.1 *Let* (i) *and* (iii) *be satisfied for* ϕ_ν $(\nu = 0, \ldots, r-1)$. *Assume that algebraic polynomials of degree* $< m$ *can be exactly reproduced in* $\mathcal{S}(\boldsymbol{\phi})$, *i.e., there are vectors* $\boldsymbol{y}_l^n \in \mathbb{R}^r$ $(l \in \mathbb{Z};\, n = 0, \ldots, m-1)$ *such that the series* $\sum_{l\in\mathbb{Z}} (\boldsymbol{y}_l^n)^T \boldsymbol{\phi}(\cdot - l)$ *are absolutely and uniformly convergent on any compact interval of* $\mathbb{R}$ *and*

$$\sum_{l\in\mathbb{Z}} (\boldsymbol{y}_l^n)^T \boldsymbol{\phi}(x-l) = x^n \qquad (x \in \mathbb{R};\, n = 0, \ldots, m-1). \tag{2.1}$$

Then the vectors $\boldsymbol{y}_l^n$ *can be written in the form*

$$\boldsymbol{y}_l^n = \sum_{k=0}^{n} \binom{n}{k} l^{n-k} \, \boldsymbol{y}_0^k. \tag{2.2}$$

.

Proof. The assertion (2.2) will be proved by induction. For $n = 0$, there are vectors $\boldsymbol{y}_l^0$ $(l \in \mathbb{Z})$ with

$$\sum_{l\in\mathbb{Z}} (\boldsymbol{y}_l^0)^{\mathrm{T}}\, \boldsymbol{\phi}(x-l) = 1.$$

Replacing x by $x+1$, we find

$$\sum_{l\in\mathbb{Z}} (\boldsymbol{y}_l^0)^{\mathrm{T}} \boldsymbol{\phi}(x+1-l) = \sum_{l\in\mathbb{Z}} (\boldsymbol{y}_{l+1}^0)^{\mathrm{T}}\, \boldsymbol{\phi}(x-l) = 1.$$

Thus, by (iii), $\boldsymbol{y}_l^0 = \boldsymbol{y}_{l+1}^0$ $(l \in \mathbb{Z})$, and the assertion follows. Let now

$$\sum_{l\in\mathbb{Z}} (\boldsymbol{y}_l^d)^{\mathrm{T}}\, \boldsymbol{\phi}(x-l) = x^d$$

with $\boldsymbol{y}_l^d$ of the form (2.2) be true for $d = 0, \ldots, n < m-1$. We show that $\boldsymbol{y}_l^{n+1} \in \mathbb{R}^r$ $(l \in \mathbb{Z})$ satisfying

$$\sum_{l\in\mathbb{Z}} (\boldsymbol{y}_l^{n+1})^{\mathrm{T}} \boldsymbol{\phi}(x-l) = x^{n+1} \quad (x \in \mathbb{R}) \tag{2.3}$$

is of the form

$$\boldsymbol{y}_l^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} l^{n+1-k}\, \boldsymbol{y}_0^k.$$

Replacing x by $x+1$ in (2.3) we obtain

$$\begin{aligned}
\sum_{l\in\mathbb{Z}} (\boldsymbol{y}_{l+1}^{n+1})^{\mathrm{T}}\, \boldsymbol{\phi}(x-l) &= (x+1)^{n+1} = \sum_{s=0}^{n+1} \binom{n+1}{s} x^s \\
&= \sum_{s=0}^{n+1} \binom{n+1}{s} \sum_{l\in\mathbb{Z}} (\boldsymbol{y}_l^s)^{\mathrm{T}}\, \boldsymbol{\phi}(x-l).
\end{aligned}$$

Using the induction hypothese and the linear independence of the integer translates of ϕ_ν $(\nu = 0, \ldots, r-1)$ this yields

$$\begin{aligned}
\boldsymbol{y}_{l+1}^{n+1} &= \sum_{s=0}^{n+1} \binom{n+1}{s} \boldsymbol{y}_l^s = \boldsymbol{y}_l^{n+1} + \sum_{s=0}^{n} \binom{n+1}{s} \sum_{k=0}^{s} \binom{s}{k} l^{s-k} \boldsymbol{y}_0^k \\
&= \boldsymbol{y}_0^{n+1} + \sum_{s=0}^{n} \binom{n+1}{s} \sum_{k=0}^{s} \binom{s}{k} \sum_{d=0}^{l} d^{s-k} \boldsymbol{y}_0^k \\
&= \boldsymbol{y}_0^{n+1} + \sum_{k=0}^{n} \binom{n+1}{k} \sum_{s=k}^{n} \binom{n-k+1}{s-k} \sum_{d=0}^{l} d^{s-k} \boldsymbol{y}_0^k \\
&= \boldsymbol{y}_0^{n+1} + \sum_{k=0}^{n} \binom{n+1}{k} \sum_{d=0}^{l} \left(\sum_{s=0}^{n-k} \binom{n-k+1}{s} d^s \right) \boldsymbol{y}_0^k \\
&= \boldsymbol{y}_0^{n+1} + \sum_{k=0}^{n} \binom{n+1}{k} \sum_{d=0}^{l} \left((d+1)^{n+1-k} - d^{n+1-k} \right) \boldsymbol{y}_0^k \\
&= \boldsymbol{y}_0^{n+1} + \sum_{k=0}^{n} \binom{n+1}{k} (l+1)^{n+1-k}\, \boldsymbol{y}_0^k = \sum_{k=0}^{n+1} \binom{n+1}{k} (l+1)^{n+1-k} \boldsymbol{y}_0^k.
\end{aligned}$$

Thus, the assertion follows. ■

Now we can show:

Theorem 2.2 *Assume that the functions $\phi_\nu \in L^2(\mathbb{R})$ $(\nu = 0, \ldots, r-1)$ satisfy the assumptions* (i) – (iii). *Then the following conditions are equivalent:*
(a) *The subspace $\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order m $(m \in \mathbb{N})$.*
(b) *Algebraic polynomials of degree $< m$ can be exactly reproduced in $\mathcal{S}(\boldsymbol{\phi})$.*
(c) *The function vector $\boldsymbol{\phi}$ satisfies the Strang–Fix conditions of order m, i.e., there is a finitely supported sequence of vectors $\{\boldsymbol{a}_l\}_{l\in\mathbb{Z}}$, such that $f := \sum_{l\in\mathbb{Z}} \boldsymbol{a}_l^T \boldsymbol{\phi}(\cdot - l)$ satisfies*

$$\hat{f}(0) \neq 0; \quad \mathrm{D}^n \hat{f}(2\pi l) = 0 \quad (l \in \mathbb{Z} \setminus \{0\}; n = 0, \ldots, m-1).$$

Proof. The equivalence of (a) and (c) is already shown in [15], Theorem 1.1.
1. Let $f \in \mathcal{S}(\boldsymbol{\phi})$ satisfy the Strang–Fix conditions of order m. Then by Corollary 2.3 in [15], algebraic polynomials of degree $< m$ can be exactly reproduced in $\mathcal{S}(f) \subseteq \mathcal{S}(\boldsymbol{\phi})$, i.e, (b) follows from (c).
2. By (i) and (ii), the Poisson Summation Formula can be applied to $\boldsymbol{\phi}$, and we have

$$\begin{aligned}\sum_{l\in\mathbb{Z}} \boldsymbol{\phi}(x-l)\, e^{iul} &= \sum_{j\in\mathbb{Z}} [\boldsymbol{\phi}(\cdot + x)]^\wedge(u + 2\pi j) \\ &= e^{iux} \sum_{j\in\mathbb{Z}} e^{2\pi i j x}\, \hat{\boldsymbol{\phi}}(u + 2\pi j).\end{aligned}$$

Repeated differentiation of this equation by u yields for $\mu = 0, \ldots, m-1$:

$$\sum_{l\in\mathbb{Z}} \boldsymbol{\phi}(x-l)\,(il)^\mu\, e^{iul} = \sum_{k=0}^{\mu} \binom{\mu}{k} (ix)^k\, e^{iux} \sum_{j\in\mathbb{Z}} e^{2\pi i j x} (\mathrm{D}^{\mu-k}\hat{\boldsymbol{\phi}})(u + 2\pi j). \tag{2.4}$$

By (i), the series in (2.4) are absolutely and uniformly convergent for x on any compact interval of $\mathbb{R}$.
We show that (c) follows from (b): Assume that algebraic polynomials of degree $< m$ can be exactly reproduced in $\mathcal{S}(\boldsymbol{\phi})$, i.e., there are vectors $\boldsymbol{y}_l^n$ $(\boldsymbol{y}_0^0 \neq \mathbf{0})$, such that (2.1) with (2.2) is satisfied. Here and in the following, $\mathbf{0}$ denotes the zero vector of length r. Let $f \in \mathcal{S}(\boldsymbol{\phi})$ be defined by

$$f := \sum_{k=0}^{m-1} \boldsymbol{a}_k^{\mathrm{T}}\, \boldsymbol{\phi}(\cdot + k), \tag{2.5}$$

where the coefficient vectors $\boldsymbol{a}_k$ are determined by

$$(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{m-1}) := (\boldsymbol{y}_0^0, \ldots, \boldsymbol{y}_0^{m-1})\, \boldsymbol{V}^{-1} \tag{2.6}$$

with the Vandermonde matrix $\boldsymbol{V} := (k^n)_{k,n=0}^{m-1}$. Hence we have

$$\boldsymbol{y}_0^n = \sum_{k=0}^{m-1} k^n \, \boldsymbol{a}_k \qquad (n = 0, \ldots, m-1). \tag{2.7}$$

By Fourier transform of (2.5) we obtain

$$\hat{f}(u) = \boldsymbol{A}(u)^{\mathrm{T}} \, \hat{\boldsymbol{\phi}}(u)$$

with

$$\boldsymbol{A}(u) := \sum_{k=0}^{m-1} \boldsymbol{a}_k \, e^{iuk}.$$

Observe that by (2.7)

$$(\mathrm{D}^n \boldsymbol{A})(0) = \sum_{k=0}^{m-1} (ik)^n \, \boldsymbol{a}_k = i^n \, \boldsymbol{y}_0^n \quad (n = 0, \ldots, m-1). \tag{2.8}$$

We show by induction, that f satisfies the Strang–Fix conditions of order m. For $n = 0$, we have by (2.1) and (2.2)

$$(\boldsymbol{y}_0^0)^{\mathrm{T}} \sum_{l \in \mathbb{Z}} \boldsymbol{\phi}(x - l) = 1 \quad (x \in \mathbb{R}).$$

Using formula (2.4) for $\mu = 0$ and $u = 0$ this yields

$$(\boldsymbol{y}_0^0)^{\mathrm{T}} \sum_{j \in \mathbb{Z}} e^{2\pi i j x} \, \hat{\boldsymbol{\phi}}(2\pi j) = 1 \quad (x \in \mathbb{R}),$$

and so, by continuity of $\hat{\boldsymbol{\phi}}(u)$ and (2.8),

$$\begin{aligned} \hat{f}(0) &= \boldsymbol{A}(0)^{\mathrm{T}} \, \hat{\boldsymbol{\phi}}(0) = (\boldsymbol{y}_0^0)^{\mathrm{T}} \, \hat{\boldsymbol{\phi}}(0) = 1, \\ \hat{f}(2\pi l) &= \boldsymbol{A}(0)^{\mathrm{T}} \, \hat{\boldsymbol{\phi}}(2\pi l) = (\boldsymbol{y}_0^0)^{\mathrm{T}} \, \hat{\boldsymbol{\phi}}(2\pi l) = 0 \quad (l \in \mathbb{Z} \setminus \{0\}). \end{aligned}$$

Hence, f satisfies the Strang–Fix conditions of order 1.
Now, assume that f satisfies the Strang–Fix conditions of order n $(1 \le n \le m-1)$, i.e., we have

$$\begin{aligned} (\mathrm{D}^\mu \hat{f})(2\pi l) &= \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} (\mathrm{D}^\nu \boldsymbol{A})^{\mathrm{T}}(0) \, (\mathrm{D}^{\mu-\nu} \hat{\boldsymbol{\phi}})(2\pi l) \\ &= \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} i^\nu \, (\boldsymbol{y}_0^\nu)^{\mathrm{T}} \, (\mathrm{D}^{\mu-\nu} \hat{\boldsymbol{\phi}})(2\pi l) = \delta_{0,l} \, \delta_{0,\mu}, \end{aligned} \tag{2.9}$$

where δ denotes the Kronecker symbol. We show that also $(\mathrm{D}^n \hat{f})(2\pi l) = 0$ $(l \in \mathbb{Z})$. By (2.1) and (2.2) we have

$$\sum_{\nu=0}^{n} \binom{n}{\nu} (\boldsymbol{y}_0^\nu)^{\mathrm{T}} \sum_{l \in \mathbb{Z}} l^{n-\nu} \, \boldsymbol{\phi}(x - l) = x^n \qquad (0 \le n \le m-1).$$

Using (2.4) with $\mu = n - \nu$ and $u = 0$ we obtain

$$\sum_{\nu=0}^{n} (-i)^{n-\nu} \binom{n}{\nu} (\boldsymbol{y}_0^\nu)^{\mathrm{T}} \sum_{k=0}^{n-\nu} \binom{n-\nu}{k} (ix)^k \sum_{j\in\mathbb{Z}} e^{2\pi i j x} \, (\mathrm{D}^{n-\nu-k}\hat{\boldsymbol{\phi}})(2\pi j) = x^n.$$

Hence,

$$\sum_{k=0}^{n} \binom{n}{k} x^k \, (-i)^{n-k} \sum_{j\in\mathbb{Z}} e^{2\pi i j x} \sum_{\nu=0}^{n-k} \binom{n-k}{\nu} (\boldsymbol{y}_0^\nu)^{\mathrm{T}} \, i^\nu \, (\mathrm{D}^{n-\nu-k}\hat{\boldsymbol{\phi}})(2\pi j) = x^n.$$

By (2.9), the sum on the left–hand side is zero for $k = 1, \ldots, n-1$. Thus, since f satisfies the Strang–Fix conditions of order 1,

$$(-i)^n \sum_{j\in\mathbb{Z}} e^{2\pi i j x} \left(\sum_{\nu=0}^{n} \binom{n}{\nu} (\boldsymbol{y}_0^\nu)^{\mathrm{T}} \, i^\nu \, (D^{n-\nu}\hat{\boldsymbol{\phi}})(2\pi j) \right) + x^n = x^n,$$

i.e., by (2.9),

$$\sum_{j\in\mathbb{Z}} e^{2\pi i j x} \, (\mathrm{D}^n \hat{f})(2\pi j) = 0.$$

It follows that $\hat{f}$ satisfies the Strang–Fix conditions of order n, so that the proof by induction is complete. ∎

3. Conditions for the refinement mask

In this section, we shall derive necessary and sufficient conditions for the refinement mask $\boldsymbol{P}$ yielding controlled approximation order m in $\mathcal{S}(\boldsymbol{\phi})$. Let now the function vector $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ be refinable in the sense of (1.6). The decay properties (i) imply that the elements of $\boldsymbol{P}(u)$ are $(m-1)$–times continuously differentiable functions in $L^2_{2\pi}(\mathbb{R})$. Following the ideas in [23], we introduce the doubly infinite matrix

$$\boldsymbol{L} := \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \boldsymbol{P}_0 & \boldsymbol{P}_1 & \boldsymbol{P}_2 & \boldsymbol{P}_3 & \boldsymbol{P}_4 & \boldsymbol{P}_5 & \cdots \\ \cdots & \boldsymbol{P}_{-2} & \boldsymbol{P}_{-1} & \boldsymbol{P}_0 & \boldsymbol{P}_1 & \boldsymbol{P}_2 & \boldsymbol{P}_3 & \cdots \\ \cdots & \boldsymbol{P}_{-4} & \boldsymbol{P}_{-3} & \boldsymbol{P}_{-2} & \boldsymbol{P}_{-1} & \boldsymbol{P}_0 & \boldsymbol{P}_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

containing the $(r \times r)$–coefficient matrices $\boldsymbol{P}_l$ occuring in the refinement equation (1.6). Now, with the infinite vector $\boldsymbol{\Phi} := (\ldots, \boldsymbol{\phi}(\cdot + 1)^{\mathrm{T}}, \boldsymbol{\phi}^{\mathrm{T}}, \boldsymbol{\phi}(\cdot - 1)^{\mathrm{T}}, \ldots)^{\mathrm{T}}$ the refinement equation (1.6) can formally be written in vector form

$$\boldsymbol{L}\, \boldsymbol{\Phi}(2\,\cdot) = \boldsymbol{\Phi}. \tag{3.1}$$

First we recall the following Lemma, which can also be found in [23] for compactly supported functions.

Lemma 3.1 *Let the assumptions* (i) *and* (iii) *be satisfied for* ϕ_ν $(\nu = 0, \ldots, r-1)$. *Assume that algebraic polynomials of degree* $< m$ *can be exactly reproduced in* $\mathcal{S}(\boldsymbol{\phi})$, *i.e., there are vectors* $\boldsymbol{y}_l^n \in \mathbb{R}^r$ $(l \in \mathbb{Z};\, n = 0, \ldots, m-1)$ *such that* (2.1) *is satisfied. Then the matrix* $\boldsymbol{L}$ *has the eigenvalues* $1,\, 1/2, \ldots, (1/2)^{m-1}$ *with corresponding left eigenvectors* $\boldsymbol{y}^n := (\ldots, (\boldsymbol{y}_0^n)^T, (\boldsymbol{y}_1^n)^T, \ldots)^T$, *i.e.,*

$$(\boldsymbol{y}^n)^T \boldsymbol{L} = 2^{-n}\,(\boldsymbol{y}^n)^T \qquad (n = 0, \ldots, m-1).$$

Proof. From (2.1) it follows by (3.1) for $n = 0, \ldots, m-1$

$$\begin{aligned} x^n &= (\boldsymbol{y}^n)^{\mathrm{T}}\, \boldsymbol{\Phi}(x) = (\boldsymbol{y}^n)^{\mathrm{T}}\, \boldsymbol{L}\, \boldsymbol{\Phi}(2x) \\ &= 2^{-n}(2x)^n = 2^{-n}\,(\boldsymbol{y}^n)^{\mathrm{T}}\, \boldsymbol{\Phi}(2x) \quad (x \in \mathbb{R}). \end{aligned}$$

Linear independence of the translates $\phi_\nu(\cdot - l)$ $(l \in \mathbb{Z}; \nu = 0, \ldots, r-1)$ gives

$$(\boldsymbol{y}^n)^{\mathrm{T}}\, \boldsymbol{L} = 2^{-n}\,(\boldsymbol{y}^n)^{\mathrm{T}} \quad (n = 0, \ldots, m-1). \quad \blacksquare$$

Now we can prove:

Theorem 3.2 *Let* $r \in \mathbb{N}$, *and let* $\phi_\nu \in L^2(\mathbb{R})$ $(\nu = 0, \ldots, r-1)$ *be functions satisfying the assumptions* (i) – (iii). *Further, let* $\boldsymbol{\phi}$ *be refinable. Then the closed shift–invariant subspace* $\mathcal{S}(\boldsymbol{\phi})$ *provides controlled approximation order* m *if and only if the refinement mask* $\boldsymbol{P}$ *of* $\boldsymbol{\phi}$ *in* (1.8) *satisfies the following conditions:*
There are vectors $\boldsymbol{y}_0^k \in \mathbb{R}^r$; $\boldsymbol{y}_0^0 \neq \boldsymbol{0}$ $(k = 0, \ldots, m-1)$ *such that for* $n = 0, \ldots, m-1$ *we have*

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}}\, (2i)^{k-n}\, (\mathrm{D}^{n-k}\boldsymbol{P})(0) \;=\; 2^{-n}\,(\boldsymbol{y}_0^n)^{\mathrm{T}}, \tag{3.2}$$

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}}\, (2i)^{k-n}\, (\mathrm{D}^{n-k}\boldsymbol{P})(\pi) \;=\; \boldsymbol{0}^{\mathrm{T}}, \tag{3.3}$$

where $\boldsymbol{0}$ *denotes the zero vector.*

Proof. 1. Assume that $\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order m. Since all assumptions for Theorem 2.2 are satisfied, it follows that algebraic polynomials of degree $< m$ can be exactly reproduced in $\mathcal{S}(\boldsymbol{\phi})$.
Thus, applying Lemmata 2.1 and 3.1, there are vectors $\boldsymbol{y}_l^n \in \mathbb{R}^r$ $(l \in \mathbb{Z}; n = 0, \ldots, m-1)$ such that we have with $\boldsymbol{y}^n = (\ldots, (\boldsymbol{y}_0^n)^{\mathrm{T}}, (\boldsymbol{y}_1^n)^{\mathrm{T}}, \ldots)^{\mathrm{T}}$

$$(\boldsymbol{y}^n)^{\mathrm{T}}\, \boldsymbol{L} = 2^{-n}\,(\boldsymbol{y}^n)^{\mathrm{T}} \quad (n = 0, \ldots, m-1), \tag{3.4}$$

where the vectors $\boldsymbol{y}_l^n$ are of the form (2.2). In particular, there is a vector $\boldsymbol{y}_0^0 \in \mathbb{R}^r$ $(\boldsymbol{y}_0^0 \neq \boldsymbol{0})$ such that by Poisson Summation Formula it follows that

$$(\boldsymbol{y}_0^0)^{\mathrm{T}} \sum_{l \in \mathbb{Z}} \boldsymbol{\phi}(x - l) = (\boldsymbol{y}_0^0)^{\mathrm{T}} \sum_{j \in \mathbb{Z}} \hat{\boldsymbol{\phi}}(2\pi j)\, e^{2\pi i j x} = 1,$$

i.e., $(\boldsymbol{y}_0^0)^{\mathrm{T}}\,\hat{\boldsymbol{\phi}}(0) = 1$.
Using the structure of the matrix $\boldsymbol{L}$, (3.4) is equivalent to the equations

$$\begin{aligned} \sum_{l\in\mathbb{Z}} (\boldsymbol{y}_{-l}^n)^{\mathrm{T}}\,\boldsymbol{P}_{2l} &= 2^{-n}\,(\boldsymbol{y}_0^n)^{\mathrm{T}} \quad (n = 0,\ldots,m-1), && (3.5)\\ \sum_{l\in\mathbb{Z}} (\boldsymbol{y}_{-l}^n)^{\mathrm{T}}\,\boldsymbol{P}_{2l+1} &= 2^{-n}\,(\boldsymbol{y}_1^n)^{\mathrm{T}} \quad (n = 0,\ldots,m-1). && (3.6)\end{aligned}$$

Note that by (i) the series in (3.5) – (3.6) are well–defined. Putting (2.2) in (3.5), we find for $n = 0,\ldots,m-1$,

$$\begin{aligned} 2^{-n}\,(\boldsymbol{y}_0^n)^{\mathrm{T}} &= \sum_{l\in\mathbb{Z}}\Big(\sum_{k=0}^{n}\binom{n}{k}(-l)^{n-k}\,(\boldsymbol{y}_0^k)^{\mathrm{T}}\Big)\boldsymbol{P}_{2l}\\ &= \sum_{k=0}^{n}\binom{n}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}}\,(-2)^{k-n}\sum_{l\in\mathbb{Z}}(2l)^{n-k}\,\boldsymbol{P}_{2l}. \qquad (3.7)\end{aligned}$$

Analogously, from (3.6) it follows by (2.2) for $s = 0,\ldots,m-1$,

$$2^{-s}\,(\boldsymbol{y}_1^s)^{\mathrm{T}} = 2^{-s}\sum_{k=0}^{s}\binom{s}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}} = \sum_{k=0}^{s}\binom{s}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}}\,(-2)^{k-s}\sum_{l\in\mathbb{Z}}(2l)^{s-k}\,\boldsymbol{P}_{2l+1}. \qquad (3.8)$$

Multiplying (3.8) with $2^{-n}\binom{n}{s}(-1)^{n-s}2^s$, summation over $s = 0,\ldots,n-1$ yields

$$\begin{aligned} &2^{-n}\sum_{s=0}^{n-1}\binom{n}{s}(-1)^{n-s}\sum_{k=0}^{s}\binom{s}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}} \qquad (3.9)\\ = \;&2^{-n}\sum_{s=0}^{n-1}\binom{n}{s}2^s\,(-1)^{n-s}\left(\sum_{k=0}^{s}\binom{s}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}}\,(-2)^{k-s}\sum_{l\in\mathbb{Z}}(2l)^{s-k}\,\boldsymbol{P}_{2l+1}\right).\end{aligned}$$

Considering the left–hand side of equation (3.9), we find

$$\begin{aligned} &2^{-n}\sum_{s=0}^{n-1}\binom{n}{s}(-1)^{n-s}\sum_{k=0}^{s}\binom{s}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}}\\ = \;&2^{-n}\sum_{k=0}^{n-1}\binom{n}{k}\sum_{s=k}^{n-1}(-1)^{n-s}\binom{n-k}{s-k}(\boldsymbol{y}_0^k)^{\mathrm{T}}\\ = \;&2^{-n}\sum_{k=0}^{n-1}\binom{n}{k}(-1)^{n-k}\sum_{s=0}^{n-1-k}(-1)^s\binom{n-k}{s}(\boldsymbol{y}_0^k)^{\mathrm{T}} = -2^{-n}\sum_{k=0}^{n-1}\binom{n}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}}.\end{aligned}$$

The right–hand side of (3.9) yields

$$\sum_{s=0}^{n-1}\binom{n}{s}(-2)^{s-n}\left(\sum_{k=0}^{s}\binom{s}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}}\,(-2)^{k-s}\sum_{l\in\mathbb{Z}}(2l)^{s-k}\,\boldsymbol{P}_{2l+1}\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \binom{n}{k} \sum_{s=k}^{n-1} (-2)^{k-n} \binom{n-k}{s-k} (\boldsymbol{y}_0^k)^{\mathrm{T}} \sum_{l\in\mathbb{Z}} (2l)^{s-k} \boldsymbol{P}_{2l+1} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}} (-2)^{k-n} \sum_{l\in\mathbb{Z}} \boldsymbol{P}_{2l+1} \sum_{s=0}^{n-1-k} \binom{n-k}{s} (2l)^s \\
&= \sum_{k=0}^{n-1} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}} (-2)^{k-n} \sum_{l\in\mathbb{Z}} \boldsymbol{P}_{2l+1} \Big((2l+1)^{n-k} - (2l)^{n-k} \Big).
\end{aligned}
$$

Hence, by addition of (3.8) with $s = n$ and (3.9) we obtain

$$
\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}} \, (-2)^{k-n} \sum_{l\in\mathbb{Z}} (2l+1)^{n-k} \, \boldsymbol{P}_{2l+1} = 2^{-n} \, (\boldsymbol{y}_0^n)^{\mathrm{T}}. \tag{3.10}
$$

Now, for the sum and the difference of the equations (3.7) and (3.10), respectively, it follows for $n = 0, \ldots, m-1$:

$$
\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} (-2)^{k-n} \, (\boldsymbol{y}_0^k)^{\mathrm{T}} \left(\sum_{l\in\mathbb{Z}} l^{n-k} \, \boldsymbol{P}_l \right) &= 2^{-n+1} \, (\boldsymbol{y}_0^n)^{\mathrm{T}}, && (3.11) \\
\sum_{k=0}^{n} \binom{n}{k} (-2)^{k-n} \, (\boldsymbol{y}_0^k)^{\mathrm{T}} \left(\sum_{l\in\mathbb{Z}} l^{n-k} \, (-1)^l \, \boldsymbol{P}_l \right) &= \boldsymbol{0}^{\mathrm{T}}. && (3.12)
\end{aligned}
$$

By (i), the series in (3.11) – (3.12) are absolutely convergent, and $\boldsymbol{P}$ is $(m-1)$–times continuously differentiable. By

$$
(\mathrm{D}^{n-k} \boldsymbol{P})(u) = \frac{(-i)^{n-k}}{2} \sum_{l\in\mathbb{Z}} \boldsymbol{P}_l \, l^{n-k} \, e^{-iul}
$$

we have

$$
\sum_{l\in\mathbb{Z}} l^{n-k} \, \boldsymbol{P}_l = 2 \, i^{n-k} (\mathrm{D}^{n-k} \boldsymbol{P})(0), \quad \sum_{l\in\mathbb{Z}} (-1)^l \, l^{n-k} \boldsymbol{P}_l = 2 \, i^{n-k} (\mathrm{D}^{n-k} \boldsymbol{P})(\pi).
$$

Hence, (3.11) – (3.12) are equivalent to the conditions (3.2) – (3.3).

2. Assume that $\boldsymbol{P}$ is $(m-1)$–times continuously differentiable with elements in $L^2_{2\pi}(\mathbb{R})$, and that there are vectors $\boldsymbol{y}_0^k \in \mathbb{R}$, $\boldsymbol{y}_0^0 \neq \boldsymbol{0}$ $(k = 0, \ldots, m-1)$, such that the conditions (3.2) and (3.3) are satisfied for $n = 0, \ldots, m-1$. We show that $\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order m.

Let $f \in \mathcal{S}(\boldsymbol{\phi})$ be defined as in (2.5) – (2.6) by

$$
\hat{f}(u) := \boldsymbol{A}(u)^{\mathrm{T}} \, \hat{\boldsymbol{\phi}}(u),
$$

such that the vector of 2π–periodic functions $\boldsymbol{A}(u) := \sum_{k=0}^{m-1} \boldsymbol{a}_k \, e^{iuk} \in C^\infty(\mathbb{R}^r)$ satisfies the conditions

$$
\mathrm{D}^n \boldsymbol{A}(0) = i^n \, \boldsymbol{y}_0^n \qquad (n = 0, \ldots, m-1). \tag{3.13}
$$

We show that f satisfies the Strang–Fix conditions of order m:
For the μ-th derivative of $\hat{f}$ we find by (2.9)

$$(D^\mu \hat{f})(2\pi l) = \sum_{s=0}^{\mu} \binom{\mu}{s} (\mathrm{D}^{\mu-s}\boldsymbol{A})^{\mathrm{T}}(0)\,(\mathrm{D}^s\hat{\boldsymbol{\phi}})(2\pi l) \quad (l \in \mathbb{Z};\ \mu = 0,\ldots,m-1).$$

From (3.13) and the refinement equation (1.7), it follows for $l \in \mathbb{Z}$

$$\begin{aligned}
(D^\mu \hat{f})(2\pi l) &= \sum_{s=0}^{\mu} \binom{\mu}{s} i^{\mu-s}\,(\boldsymbol{y}_0^{\mu-s})^{\mathrm{T}}\,2^{-s} \sum_{d=0}^{s} \binom{s}{d} (\mathrm{D}^{s-d}\boldsymbol{P})(\pi l)\,(\mathrm{D}^d\hat{\boldsymbol{\phi}})(\pi l) \\
&= \sum_{d=0}^{\mu} \binom{\mu}{d} \sum_{s=d}^{\mu} \binom{\mu-d}{s-d} i^{\mu-s}\,(\boldsymbol{y}_0^{\mu-s})^{\mathrm{T}}\,2^{-s}(\mathrm{D}^{s-d}\boldsymbol{P})(\pi l)\,(\mathrm{D}^d\hat{\boldsymbol{\phi}})(\pi l) \\
&= \sum_{d=0}^{\mu} \binom{\mu}{d} \sum_{s=0}^{\mu-d} \binom{\mu-d}{s} i^{\mu-s-d}\,(\boldsymbol{y}_0^{\mu-s-d})^{\mathrm{T}}\,2^{-s-d}(\mathrm{D}^{s}\boldsymbol{P})(\pi l)\,(\mathrm{D}^d\hat{\boldsymbol{\phi}})(\pi l) \\
&= \sum_{d=0}^{\mu} \binom{\mu}{d} \left(\sum_{s=0}^{\mu-d} \binom{\mu-d}{s} i^{s}\,(\boldsymbol{y}_0^{s})^{\mathrm{T}}\,2^{s-\mu}(\mathrm{D}^{\mu-d-s}\boldsymbol{P})(\pi l) \right) (\mathrm{D}^d\hat{\boldsymbol{\phi}})(\pi l).
\end{aligned}$$

Using the relation (3.3) with $n = \mu - d$, we obtain for odd l

$$(D^\mu \hat{f})(2\pi l) = 0 \qquad (\mu = 0,\ldots,m-1).$$

For even l, we find by (3.2) with $n = \mu - d$

$$\begin{aligned}
(D^\mu \hat{f})(2\pi l) &= 2^{-\mu} \sum_{d=0}^{\mu} \binom{\mu}{d} i^{\mu-d}(\boldsymbol{y}_0^{\mu-d})^{\mathrm{T}}\,(\mathrm{D}^d\hat{\boldsymbol{\phi}})(l\pi) \\
&= 2^{-\mu}\,(\mathrm{D}^\mu \hat{f})(l\pi) \quad (\mu = 0,\ldots,m-1).
\end{aligned}$$

Repeating the procedure, we obtain that

$$(\mathrm{D}^\mu \hat{f})(2l\pi) = 0 \qquad (l \in \mathbb{Z}\setminus\{0\};\ \mu = 0,\ldots,m-1).$$

Finally, using the Poisson Summation Formula, we have

$$\hat{f}(0) = (\boldsymbol{y}_0^0)^{\mathrm{T}}\,\hat{\boldsymbol{\phi}}(0) = (\boldsymbol{y}_0^0)^{\mathrm{T}} \sum_{l\in\mathbb{Z}} \boldsymbol{\phi}(\cdot - l) \neq 0,$$

since $\mathcal{B}(\boldsymbol{\phi})$ forms a Riesz basis of $\mathcal{S}(\boldsymbol{\phi})$. Thus, by Theorem 2.2, $\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order m. ■

Remark 3.3 1. In the case $r = 1$, the relations (3.2) and (3.3) can strongly be simplified, and we obtain Theorem 1.1.
2. Let $\boldsymbol{\phi}$ satisfy the assumptions of Theorem 3.2. In order to verify controlled

approximation order $m = 1$, the following conditions must be true: There is a vector $\boldsymbol{y}_0^0 \in \mathbb{R}^r$ ($\boldsymbol{y}_0^0 \neq \boldsymbol{0}$) such that

$$(\boldsymbol{y}_0^0)^{\mathrm{T}}\, \boldsymbol{P}(0) = (\boldsymbol{y}_0^0)^{\mathrm{T}}; \qquad (\boldsymbol{y}_0^0)^{\mathrm{T}}\, \boldsymbol{P}(\pi) = \boldsymbol{0}^{\mathrm{T}}. \tag{3.14}$$

That means, $\boldsymbol{y}_0^0$ is the left eigenvector of $\boldsymbol{P}(0)$ for the eigenvalue 1. At the same time $\boldsymbol{y}_0^0$ is the left eigenvector of $\boldsymbol{P}(\pi)$ for the eigenvalue 0.
3. In order to verify approximation order $m = 2$ we have to show:
a) $\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order 1, i.e., (3.14) is true for $\boldsymbol{y}_0^0 \neq 0$.
b) There is a second vector $\boldsymbol{y}_0^1 \in \mathbb{R}^r$ such that

$$\begin{aligned}(2i)^{-1}\,(\boldsymbol{y}_0^0)^{\mathrm{T}}\,(\mathrm{D}\boldsymbol{P})(0) + (\boldsymbol{y}_0^1)^{\mathrm{T}}\,\boldsymbol{P}(0) &= 2^{-1}\,(\boldsymbol{y}_0^1)^{\mathrm{T}},\\ (2i)^{-1}\,(\boldsymbol{y}_0^0)^{\mathrm{T}}\,(\mathrm{D}\boldsymbol{P})(\pi) + (\boldsymbol{y}_0^1)^{\mathrm{T}}\,\boldsymbol{P}(\pi) &= \boldsymbol{0}^{\mathrm{T}}.\end{aligned}$$

4. We observe that the conditions (3.2) and (3.3) use the values of the refinement mask and its derivatives at the points $u = 0$ and $u = \pi$, only. ♠

Example 3.4 We want to consider the example of quadratic B–splines with double knots. Let N_0^2 and N_1^2 be the B–splines defined by the knots $0, 0, 1, 1$ and $0, 1, 1, 2$, respectively, i.e., we have

$$N_0^2(x) := \begin{cases} 2x(1-x) & x \in [0,1], \\ 0 & \text{otherwise}, \end{cases} \qquad N_1^2(x) := \begin{cases} x^2 & x \in [0,1), \\ (2-x)^2 & x \in [1,2], \\ 0 & \text{otherwise}. \end{cases}$$

Let $\boldsymbol{N} := (N_0^2,\, N_1^2)^{\mathrm{T}}$. With $\mathcal{S}(\boldsymbol{N})$ we denote the closed shift–invariant subspace of $L^2(\mathbb{R})$ generated by N_0^2 and N_1^2, i.e.,

$$\mathcal{S}(\boldsymbol{N}) := \operatorname{clos}_{L^2} \operatorname{span}\{N_0^2(\cdot - k),\, N_1^2(\cdot - k) : \, k \in \mathbb{Z}\}.$$

In [19], it was shown that the Fourier transformed vector

$$\hat{\boldsymbol{N}}(u) = \frac{2}{(iu)^3} \begin{pmatrix} iu - 2 + (2 + iu)e^{-iu} \\ 1 - 2iue^{-iu} - e^{-2iu} \end{pmatrix}$$

satisfies the Fourier transformed refinement equation

$$\hat{\boldsymbol{N}} = \boldsymbol{P}(\cdot/2)\, \hat{\boldsymbol{N}}(\cdot/2)$$

with the refinement mask

$$\boldsymbol{P}(u) := \frac{1}{8} \begin{pmatrix} 2 + 2e^{-iu} & 2 \\ 2e^{-iu} + 2e^{-2iu} & 1 + 4e^{-iu} + e^{-2iu} \end{pmatrix}.$$

It is well–known that $\mathcal{S}(\boldsymbol{N})$ provides the controlled approximation order $m = 3$. In fact, the conditions (3.2) and (3.3) are satisfied with $\boldsymbol{y}_0^0 := (1,1)^{\mathrm{T}}$, $\boldsymbol{y}_0^1 := (1/2,1)^{\mathrm{T}}$, $\boldsymbol{y}_0^2 := (0,\, 1)^{\mathrm{T}}$. We find for $n = 0,\, 1,\, 2$

$$
\begin{aligned}
\frac{1}{8}(1,\,1)\begin{pmatrix} 4 & 2 \\ 4 & 6 \end{pmatrix} &= (1,\,1),\\
\frac{1}{8}(1,\,1)\begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} &= (0,\,0),\\
-\frac{i}{16}(1,\,1)\begin{pmatrix} -2i & 0 \\ -6i & -6i \end{pmatrix} + \frac{1}{8}(1/2,\,1)\begin{pmatrix} 4 & 2 \\ 4 & 6 \end{pmatrix} &= \frac{1}{2}(1/2,\,1),\\
-\frac{i}{16}(1,\,1)\begin{pmatrix} 2i & 0 \\ -2i & 2i \end{pmatrix} + \frac{1}{8}(1/2,\,1)\begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} &= (0,\,0),\\
-\frac{1}{32}(1,\,1)\begin{pmatrix} -2 & 0 \\ -10 & -8 \end{pmatrix} - \frac{i}{8}(1/2,\,1)\begin{pmatrix} -2i & 0 \\ -6i & -6i \end{pmatrix} &+ \frac{1}{8}(0,\,1)\begin{pmatrix} 4 & 2 \\ 4 & 6 \end{pmatrix}\\
&= \frac{1}{4}(0,\,1),\\
-\frac{1}{32}(1,\,1)\begin{pmatrix} 2 & 0 \\ -6 & 0 \end{pmatrix} - \frac{i}{8}(1/2,\,1)\begin{pmatrix} 2i & 0 \\ -2i & 2i \end{pmatrix} &+ \frac{1}{8}(0,\,1)\begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix}\\
&= (0,\,0). \quad \spadesuit
\end{aligned}
$$

4. Factorization of the refinement mask

As known, if the principal space $\mathcal{S}(\phi)$, generated by one refinable function ϕ, provides controlled approximation order m, then the refinement mask of ϕ factorizes

$$P(u) = \left(\frac{1+e^{-iu}}{2}\right)^m S(u)$$

(cf. Theorem 1.1). In this section, we shall find a matrix factorization of the refinement mask in the case of a finite set of refinable functions.

Let $r \in \mathbb{N}$ be fixed. First, let us define the $(r \times r)$–matrix $\boldsymbol{C} := (C_{j,k})_{j,k=0}^{r-1}$ by the vector $\boldsymbol{y} = (y_0, \ldots, y_{r-1})^{\mathrm{T}} \in \mathbb{R}^r$, $\boldsymbol{y} \neq \boldsymbol{0}$. Let $j_0 := \min\{j;\, y_j \neq 0\}$ and $j_1 := \max\{j; y_j \neq 0\}$. Further, for all j with $y_j \neq 0$ let $d_j := \min\{k :\; k > j,\, y_j \neq 0\}$. For $j_0 < j_1$, the entries of $\boldsymbol{C}$ are defined for $j, k = 0, \ldots, r-1$ by

$$C_{j,k}(u) := \begin{cases} y_j^{-1} & y_j \neq 0 \quad \text{and} \quad j = k, \\ 1 & y_j = 0 \quad \text{and} \quad j = k, \\ -y_j^{-1} & y_j \neq 0 \quad \text{and} \quad d_j = k, \\ -e^{-iu}/y_{j_1} & j = j_1 \quad \text{and} \quad k = j_0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

For $j_0 = j_1$, $\boldsymbol{C}$ is a diagonal matrix of the form

$$\boldsymbol{C}(u) = \operatorname{diag}(\underbrace{1, \ldots, 1}_{j_0}, (1 - e^{-iu})/y_{j_0}, \underbrace{1, \ldots, 1}_{r-1-j_0}). \tag{4.2}$$

In particular, for $r = 1$ we have $\boldsymbol{C}(u) := (1 - e^{-iu})/y_0$. Note that $\boldsymbol{C}$ is chosen such that

$$\boldsymbol{y}^{\mathrm{T}} \boldsymbol{C}(0) = \boldsymbol{0}^{\mathrm{T}}. \tag{4.3}$$

We obtain:

Theorem 4.1 *Let $\phi_\nu \in L^2(\mathbb{R})$ $(\nu = 0, \ldots, r-1)$ be functions satisfying the assumptions* (i) – (iii)*. Further, let $\boldsymbol{\phi}$ be refinable. Then the following conditions are equivalent:*
(a) *$\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order* 1*, i.e., there is a vector $\boldsymbol{y} = (y_0, \ldots, y_{r-1})^T \in \mathbb{R}^r$ $(\boldsymbol{y} \neq \boldsymbol{0})$ with*

$$\boldsymbol{y}^T \sum_{l \in \mathbb{Z}} \boldsymbol{\phi}(\cdot - l) = 1.$$

(b) *The refinement mask $\boldsymbol{P}$ of $\boldsymbol{\phi}$ satisfies*

$$\boldsymbol{y}^T \boldsymbol{P}(0) = \boldsymbol{y}^T, \quad \boldsymbol{y}^T \boldsymbol{P}(\pi) = \boldsymbol{0}^T \tag{4.4}$$

with $\boldsymbol{y}$ as in (a)*.*
(c) *The refinement mask $\boldsymbol{P}$ of $\boldsymbol{\phi}$ is of the form*

$$\boldsymbol{P}(u) = \frac{1}{2} \boldsymbol{C}(2u) \, \tilde{\boldsymbol{P}}(u) \, \boldsymbol{C}(u)^{-1},$$

where $\boldsymbol{C}(u)$ is defined by (4.1) – (4.2) *with $\boldsymbol{y}$ as in* (a)*, and where $\tilde{\boldsymbol{P}}(u)$ is an $(r \times r)$–matrix satisfying $\tilde{\boldsymbol{P}}(0)\boldsymbol{e} = \boldsymbol{e}$ with $\boldsymbol{e} := (\operatorname{sign}(|y_0|), \ldots, \operatorname{sign}(|y_{r-1}|)^{\mathrm{T}}$.*

Proof. 1. Without loss of generality we suppose that $\boldsymbol{y} \neq \boldsymbol{0}$ is of the form

$$\boldsymbol{y} = (y_0, \ldots, y_{l-1}, 0 \ldots, 0)^{\mathrm{T}}$$

with $1 \leq l \leq r$ and $y_\nu \neq 0$ for $\nu = 0, \ldots, l-1$. Introducing the direct sum of quadratic matrices $\boldsymbol{A} \oplus \boldsymbol{B} := \operatorname{diag}\,(\boldsymbol{A}, \boldsymbol{B})$, the matrix $\boldsymbol{C}(u)$ reads $\boldsymbol{C}(u) = \boldsymbol{C}_l(u) \oplus \boldsymbol{I}_{r-l}$, where $\boldsymbol{I}_{r-l}$ is the $(r-l) \times (r-l)$–unit matrix and

$$\begin{aligned}
\boldsymbol{C}_l(u) &:= \begin{pmatrix} y_0^{-1} & -y_0^{-1} & 0 & \ldots & 0 \\ 0 & y_1^{-1} & -y_1^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & y_{l-2}^{-1} & -y_{l-2}^{-1} \\ -e^{-iu}/y_{l-1} & 0 & \ldots & 0 & y_{l-1}^{-1} \end{pmatrix} \qquad (l > 1), \\
\boldsymbol{C}_1(u) &:= \frac{1 - e^{-iu}}{y_0}.
\end{aligned}$$

It can be easily observed that $\boldsymbol{C}(u)$ is invertible for $u \neq 0$, and we have $\boldsymbol{C}(u)^{-1} = \boldsymbol{C}_l(u)^{-1} \oplus \boldsymbol{I}_{r-l}$ with

$$\boldsymbol{C}_l(u)^{-1} = \frac{1}{1-z}\,\boldsymbol{E}_l(u) := \frac{1}{1-z}\begin{pmatrix} y_0 & y_1 & y_2 & \dots & y_{l-1} \\ y_0 z & y_1 & y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & y_{l-1} \\ y_0 z & y_1 z & \ddots & y_{l-2} & y_{l-1} \\ y_0 z & y_1 z & \dots & y_{l-2} z & y_{l-1} \end{pmatrix} \qquad (z := e^{-iu}).$$

We introduce $\boldsymbol{E}(u) := (1 - e^{-iu})\,\boldsymbol{C}(u)^{-1} = \boldsymbol{E}_l(u) \oplus (1 - e^{-iu})\boldsymbol{I}_{r-l}$.
2. The equivalence of (a) and (b) was already shown in Theorem 3.2. Assume that (4.4) holds. We show that $\tilde{\boldsymbol{P}}(u) := 2\,\boldsymbol{C}(2u)^{-1}\,\boldsymbol{P}(u)\,\boldsymbol{C}(u)$ satisfies the assertion $\tilde{\boldsymbol{P}}(0)\boldsymbol{e} = \boldsymbol{e}$ with $\boldsymbol{e} = (\underbrace{1,\dots,1}_{l},\underbrace{0\dots,0}_{r-l})^{\mathrm{T}}$. Using the rule of L'Hospital, we obtain

$$\begin{aligned} \tilde{\boldsymbol{P}}(0) &= \lim_{u\to 0}\frac{2}{(1-e^{-2iu})}\,\boldsymbol{E}(2u)\,\boldsymbol{P}(u)\,\boldsymbol{C}(u) \\ &= \frac{1}{i}\,(2(\mathrm{D}\boldsymbol{E})(0)\,\boldsymbol{P}(0)\,\boldsymbol{C}(0) + \boldsymbol{E}(0)\,(\mathrm{D}\boldsymbol{P})(0)\,\boldsymbol{C}(0) + \boldsymbol{E}(0)\,\boldsymbol{P}(0)\,(\mathrm{D}\boldsymbol{C})(0))\,. \end{aligned}$$

Observe that $\boldsymbol{C}(0)\boldsymbol{e} = \boldsymbol{0}$ and $(\mathrm{D}\boldsymbol{C})(0)\boldsymbol{e} = (i/y_{l-1})\,\boldsymbol{e}_l$ with the l-th unit vector $\boldsymbol{e}_l := (\underbrace{0,\dots,0}_{l-1},1,0,\dots,0)^{\mathrm{T}}$. Hence,

$$\tilde{\boldsymbol{P}}(0)\,\boldsymbol{e} = \frac{1}{y_{l-1}}\boldsymbol{E}(0)\,\boldsymbol{P}(0)\,\boldsymbol{e}_l.$$

By the assumption (4.4), we have $\boldsymbol{E}(0)\,\boldsymbol{P}(0) = \boldsymbol{E}(0)$. Thus,

$$\tilde{\boldsymbol{P}}(0)\,\boldsymbol{e} = \frac{1}{y_{l-1}}\boldsymbol{E}(0)\,\boldsymbol{e}_l = \boldsymbol{e}.$$

3. Let the refinement mask $\boldsymbol{P}(u)$ be of the form

$$\boldsymbol{P}(u) = \frac{1}{2}\,\boldsymbol{C}(2u)\,\tilde{\boldsymbol{P}}(u)\,\boldsymbol{C}(u)^{-1} = \frac{1}{2(1-e^{-iu})}\boldsymbol{C}(2u)\,\tilde{\boldsymbol{P}}(u)\,\boldsymbol{E}(u)$$

with $\boldsymbol{C}(u)$ and $\boldsymbol{E}(u)$ defined by $\boldsymbol{y} \neq \boldsymbol{0}$ as in part 1 of the proof, and with $\tilde{\boldsymbol{P}}(0)\,\boldsymbol{e} = \boldsymbol{e}$. We show that (4.4) is satisfied for $\boldsymbol{P}(u)$.
Since $\boldsymbol{C}(\pi)$ is regular, the second assertion in (4.4) immediately follows from (4.3). Again, using the rule of L'Hospital, we find

$$\boldsymbol{P}(0) = \frac{1}{2i}\left(2(\mathrm{D}\boldsymbol{C})(0)\,\tilde{\boldsymbol{P}}(0)\,\boldsymbol{E}(0) + \boldsymbol{C}(0)\,(\mathrm{D}\tilde{\boldsymbol{P}})(0)\,\boldsymbol{E}(0) + \boldsymbol{C}(0)\,\tilde{\boldsymbol{P}}(0)\,(\mathrm{D}\boldsymbol{E})(0)\right).$$

By (4.3), $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{P}(0)$ simplifies to

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{P}(0) = \frac{1}{i}\boldsymbol{y}^{\mathrm{T}}\,(\mathrm{D}\boldsymbol{C})(0)\,\tilde{\boldsymbol{P}}(0)\,\boldsymbol{E}(0).$$

Observing that $\boldsymbol{y}^{\mathrm{T}}(\mathrm{D}\boldsymbol{C})(0) = i\boldsymbol{e}_1^{\mathrm{T}}$ with $\boldsymbol{e}_1 := (1,0,\ldots,0)^{\mathrm{T}}$, and $\tilde{\boldsymbol{P}}(0)\,\boldsymbol{E}(0) = \boldsymbol{E}(0)$, we have

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{P}(0) = \boldsymbol{e}_1^{\mathrm{T}}\boldsymbol{E}(0) = \boldsymbol{y}^{\mathrm{T}}. \qquad \blacksquare$$

Moreover, we obtain the following

Theorem 4.2 *Let $m \geq 1$ be fixed. Let $\boldsymbol{P}$ be an $(r\times r)$–matrix with $(m-1)$–times continuously differentiable entries in $L^2_{2\pi}(\mathbb{R})$, and assume that $\boldsymbol{P}$ satisfies (3.2) and (3.3) for $n = 0,\ldots,m-1$ with vectors $\boldsymbol{y}_0^0,\ldots,\boldsymbol{y}_0^{m-1}$ ($\boldsymbol{y}_0^0 \neq \boldsymbol{0}$). Then the matrix $\tilde{\boldsymbol{P}}$,*

$$\tilde{\boldsymbol{P}}(u) := 2\,\boldsymbol{C}(2u)^{-1}\,\boldsymbol{P}(u)\,\boldsymbol{C}(u) \tag{4.5}$$

with $\boldsymbol{C}(u)$ defined as in (4.1) – (4.2) by $\boldsymbol{y}_0^0$, satisfies the conditions (3.2) and (3.3) for $n = 0,\ldots,m-2$ with vectors $\tilde{\boldsymbol{y}}_0^0,\ldots,\tilde{\boldsymbol{y}}_0^{m-2}$ given by

$$(\tilde{\boldsymbol{y}}_0^k)^T := \frac{1}{k+1}\sum_{\nu=0}^{k+1}\binom{k+1}{\nu}\, i^{\nu-k-1}\,(\boldsymbol{y}_0^\nu)^T\,(D^{k+1-\nu}\boldsymbol{C})(0) \quad (k = 0,\ldots,m-2). \tag{4.6}$$

In particular, we have $\tilde{\boldsymbol{y}}_0^0 \neq \boldsymbol{0}$.

Proof. 1. From $2\,\boldsymbol{P}(u)\,\boldsymbol{C}(u) = \boldsymbol{C}(2u)\,\tilde{\boldsymbol{P}}(u)$ it follows by $(n-k)$–fold differentiation for $0 \leq n-k \leq m-1$

$$2\sum_{l=0}^{n-k}\binom{n-k}{l}(\mathrm{D}^{n-k-l}\boldsymbol{P})(u)(\mathrm{D}^l\boldsymbol{C})(u) = \sum_{l=0}^{n-k}\binom{n-k}{l}2^{n-k-l}(\mathrm{D}^{n-k-l}\boldsymbol{C})(2u)(\mathrm{D}^l\tilde{\boldsymbol{P}})(u),$$

i.e.,

$$(\mathrm{D}^{n-k}\boldsymbol{P})(u)\,\boldsymbol{C}(u) = \boldsymbol{S}_1(u) - \boldsymbol{S}_2(u) \tag{4.7}$$

with

$$\begin{aligned}
\boldsymbol{S}_1(u) &:= \frac{1}{2}\sum_{l=0}^{n-k}\binom{n-k}{l}\,2^{n-k-l}(\mathrm{D}^{n-k-l}\boldsymbol{C})(2u)\,(\mathrm{D}^l\tilde{\boldsymbol{P}})(u),\\
\boldsymbol{S}_2(u) &:= \sum_{l=1}^{n-k}\binom{n-k}{l}\,(\mathrm{D}^{n-k-l}\boldsymbol{P})(u)\,(\mathrm{D}^l\boldsymbol{C})(u).
\end{aligned}$$

2. First, we observe that by (3.3) for $u = \pi$ and $1 \leq n \leq m-1$

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}}\,(2i)^{k-n}\boldsymbol{S}_2(\pi)\\
=\;& \sum_{l=1}^{n}\binom{n}{l}\left(\sum_{k=0}^{n-l}\binom{n-l}{k}\,(\boldsymbol{y}_0^k)^{\mathrm{T}}(2i)^{k-n+l}\,(\mathrm{D}^{n-l-k}\boldsymbol{P})(\pi)\right)(2i)^{-l}\,(\mathrm{D}^l\boldsymbol{C})(\pi) \qquad (4.8)\\
=\;& \sum_{l=1}^{n}\binom{n}{l}\,\boldsymbol{0}^{\mathrm{T}}\,(2i)^{-l}\,(\mathrm{D}^l\boldsymbol{C})(\pi) = \boldsymbol{0}^{\mathrm{T}}.
\end{aligned}$$

Analogously, for $u = 0$, from (3.2) it follows for $1 \le n \le m-1$ by the definition (4.6) of $\tilde{\boldsymbol{y}}_0^{n-1}$

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}}\,(2i)^{k-n} \boldsymbol{S}_2(0) \\
= \;& \sum_{l=1}^{n} \binom{n}{l} \left(\sum_{k=0}^{n-l} \binom{n-l}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}} (2i)^{k-n+l}\,(\mathrm{D}^{n-l-k}\boldsymbol{P})(0) \right) (2i)^{-l}\,(\mathrm{D}^l \boldsymbol{C})(0) \quad (4.9) \\
= \;& \sum_{l=1}^{n} \binom{n}{l} 2^{-n+l}\,(\boldsymbol{y}_0^{n-l})^{\mathrm{T}}\,(2i)^{-l}\,(\mathrm{D}^l \boldsymbol{C})(0) \\
= \;& 2^{-n} \sum_{l=0}^{n-1} \binom{n}{l} i^{-n+l}\,(\boldsymbol{y}_0^l)^{\mathrm{T}}\,(\mathrm{D}^{n-l}\boldsymbol{C})(0) = 2^{-n} n (\tilde{\boldsymbol{y}}_0^{n-1})^{\mathrm{T}} - 2^{-n} (\boldsymbol{y}_0^n)^{\mathrm{T}} \boldsymbol{C}(0).
\end{aligned}$$

3. We will show the following. If $\boldsymbol{P}$ satisfies (3.2) – (3.3) for a certain n ($1 \le n \le m-1$) with $\boldsymbol{y}_0^k$ ($k = 0, \dots, n$) then $\tilde{\boldsymbol{P}}$ satisfies (3.2) – (3.3) for $n-1$ with $\tilde{\boldsymbol{y}}_0^k$ ($k = 0, \dots, n-1$).
We multiply (3.3) with $\boldsymbol{C}(\pi)$ and replace $(\mathrm{D}^{n-k}\boldsymbol{P})(\pi)\,\boldsymbol{C}(\pi)$ ($k = 0, \dots, n$) by the right–hand side of (4.7) for $u = \pi$. Then, by (4.8), we have for $1 \le n \le m-1$

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}} (2i)^{k-n}\,(\boldsymbol{S}_1(\pi) - \boldsymbol{S}_2(\pi)) \\
= \;& \sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}} (2i)^{k-n}\,\frac{1}{2} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^{n-k-l}\,(\mathrm{D}^{n-k-l}\boldsymbol{C})(0)\,(\mathrm{D}^l \tilde{\boldsymbol{P}})(\pi) = 0.
\end{aligned}$$

Hence, by the definition of $\tilde{\boldsymbol{y}}_0^{n-l-1}$

$$\begin{aligned}
&\frac{1}{2} \sum_{l=0}^{n} \binom{n}{l} \left(\sum_{k=0}^{n-l} \binom{n-l}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}} i^{k-n+l}\,(\mathrm{D}^{n-k-l}\boldsymbol{C})(0) \right) (2i)^{-l}\,(\mathrm{D}^l \tilde{\boldsymbol{P}})(\pi) \\
= \;& \frac{n}{2} \sum_{l=0}^{n-1} \binom{n-1}{l} (2i)^{-l}\,\tilde{\boldsymbol{y}}_0^{n-l-1}\,(\mathrm{D}^l \tilde{\boldsymbol{P}})(\pi) = 0.
\end{aligned}$$

The substitution $l' = n-1-l$ yields

$$\sum_{l'=0}^{n-1} \binom{n-1}{l'} (2i)^{-n+l'+1}\,(\tilde{\boldsymbol{y}}_0^{l'})^{\mathrm{T}}\,(\mathrm{D}^{n-l'-1}\tilde{\boldsymbol{P}})(\pi) = 0,$$

that is, $\tilde{\boldsymbol{P}}$ satisfies (3.3) for $0 \le n \le m-2$ with the vectors $\tilde{\boldsymbol{y}}_0^k$ ($k = 0, \dots, m-2$).
4. We multiply (3.2) with $\boldsymbol{C}(0)$ and replace $(\mathrm{D}^{n-k}\boldsymbol{P})(0)\,\boldsymbol{C}(0)$ ($k = 0, \dots, n$) by the right–hand side of (4.7) for $u = 0$. Then, by (4.9), we obtain for $1 \le n \le m-1$

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}}\,(2i)^{k-n}\,\boldsymbol{S}_1(0)$$

$$
\begin{aligned}
&= 2^{-n}\,(\boldsymbol{y}_0^n)^{\mathrm{T}}\,\boldsymbol{C}(0) + \sum_{k=0}^{n}\binom{n}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}}\,(2i)^{k-n}\,\boldsymbol{S}_2(0)\\
&= 2^{-n}(\boldsymbol{y}_0^n)^{\mathrm{T}}\boldsymbol{C}(0) + 2^{-n}n(\tilde{\boldsymbol{y}}_0^{n-1})^{\mathrm{T}} - 2^{-n}\,(\boldsymbol{y}_0^n)^{\mathrm{T}}\boldsymbol{C}(0) = n2^{-n}(\tilde{\boldsymbol{y}}_0^{n-1})^{\mathrm{T}}.
\end{aligned}
$$

Hence, by the definitions of $\boldsymbol{S}_1(0)$ and $\tilde{\boldsymbol{y}}_0^{n-l-1}$ it follows that

$$
\begin{aligned}
&2\sum_{k=0}^{n}\binom{n}{k}(\boldsymbol{y}_0^k)^{\mathrm{T}}\,(2i)^{k-n}\,\boldsymbol{S}_1(0)\\
&= \sum_{l=0}^{n}\binom{n}{l}\left(\sum_{k=0}^{n-l}\binom{n-l}{k}\,(\boldsymbol{y}_0^k)^{\mathrm{T}}i^{k-n+l}\,(\mathrm{D}^{n-k-l}\boldsymbol{C})(0)\right)(2i)^{-l}\,(\mathrm{D}^l\tilde{\boldsymbol{P}})(0)\\
&= n\sum_{l=0}^{n-1}\binom{n-1}{l}(2i)^{-l}\,(\tilde{\boldsymbol{y}}_0^{n-l-1})^{\mathrm{T}}\,(\mathrm{D}^l\tilde{\boldsymbol{P}})(0)\\
&= n\sum_{l=0}^{n-1}\binom{n-1}{l}(2i)^{-n+1+l}\,(\tilde{\boldsymbol{y}}_0^l)^{\mathrm{T}}\,(\mathrm{D}^{n-1-l}\tilde{\boldsymbol{P}})(0) = n2^{-n+1}(\tilde{\boldsymbol{y}}_0^{n-1})^{\mathrm{T}},
\end{aligned}
$$

that is, $\tilde{\boldsymbol{P}}$ satisfies (3.2) for $0 \le n \le m-2$ with the vectors $\tilde{\boldsymbol{y}}_0^k$ $(k = 0, \ldots, m-2)$.

5. Finally, we consider

$$(\tilde{\boldsymbol{y}}_0^0)^{\mathrm{T}} := (-i)(\boldsymbol{y}_0^0)^{\mathrm{T}}\,(\mathrm{D}\boldsymbol{C})(0) + \boldsymbol{y}_0^{1\,T}\,\boldsymbol{C}(0).$$

Computing this vector, we easily observe that the sum of the components of $\tilde{\boldsymbol{y}}_0^0$ equals 1. Consequently, $\tilde{\boldsymbol{y}}_0^0 \ne \boldsymbol{0}$. ■

With the help of Theorem 4.2 we find the following factorization of refinement matrices:

Corollary 4.3 *Let $m \in \mathbb{N}$ be fixed. Let $\boldsymbol{P}$ be an $(r \times r)$–matrix with $(m-1)$–times continuously differentiable entries in $L^2_{2\pi}(\mathbb{R})$, and assume that $\boldsymbol{P}$ satisfies (3.2) – (3.3) for $n = 0, \ldots, m-1$ with vectors $\boldsymbol{y}_0^0, \ldots, \boldsymbol{y}_0^{m-1}$ $(\boldsymbol{y}_0^0 \ne \boldsymbol{0})$. Then there are vectors $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{m-1}$, such that $\boldsymbol{P}$ factorizes*

$$\boldsymbol{P}(u) = \frac{1}{2^m}\boldsymbol{C}_{m-1}(2u)\;\ldots\;\boldsymbol{C}_0(2u)\,\boldsymbol{S}(u)\,\boldsymbol{C}_0(u)^{-1}\;\ldots\;\boldsymbol{C}_{m-1}(u)^{-1},$$

where the matrices $\boldsymbol{C}_k$ are defined by $\boldsymbol{x}_k$ $(k = 0, \ldots, m-1)$ as in (4.1) – (4.2) and $\boldsymbol{S}(u)$ is an $(r \times r)$–matrix with $(m-1)$–times continuously differentiable entries in $L^2_{2\pi}(\mathbb{R})$. In particular, for the determinant of $\boldsymbol{P}$ it follows

$$\det\,\boldsymbol{P}(u) = \left(\frac{1+e^{-iu}}{2^r}\right)^m\,\det\,\boldsymbol{S}(u). \tag{4.10}$$

Proof. The first assertion can easily be observed by repeated application of Theorem 4.2. The formula (4.10) follows from

$$\det\,(\boldsymbol{C}_k(2u)\,\boldsymbol{C}_k(u)^{-1}) = \frac{1-e^{-2iu}}{1-e^{-iu}} = 1 + e^{-iu} \quad (k = 0, \ldots, m-1). \qquad ■$$

Remark 4.4 The vectors $\boldsymbol{x}_n$ $(n = 0, \ldots, m-1)$ in Corollary 4.3 can be computed by repeated application of (4.6). In particular, we obtain $\boldsymbol{x}_0^{\mathrm{T}} := (\boldsymbol{y}_0^0)^{\mathrm{T}}$, $\boldsymbol{x}_1^{\mathrm{T}} := (\tilde{\boldsymbol{y}}_0^0)^{\mathrm{T}} = (-i)(\boldsymbol{y}_0^0)^{\mathrm{T}}(\mathrm{D}\boldsymbol{C})(0) + (\boldsymbol{y}_0^1)^{\mathrm{T}}\,\boldsymbol{C}(0)$, where $\boldsymbol{C}$ is defined by $\boldsymbol{x}_0 = \boldsymbol{y}_0^0$. ♠

Example 4.5 1. Let us again consider the refinement mask of the vector of quadratic B–splines with double knots

$$\boldsymbol{P}(u) = \frac{1}{8}\begin{pmatrix} 2+2e^{-iu} & 2 \\ 2e^{-iu}+2e^{-2iu} & 1+4e^{-iu}+e^{-2iu} \end{pmatrix}$$

(cf. Example 3.4). By Theorem 4.2, we obtain $\tilde{\boldsymbol{P}}$ by (4.5) with $\boldsymbol{C}(u)$ defined by $\boldsymbol{y}_0^0 = (1,1)^{\mathrm{T}}$:

$$\begin{aligned} \tilde{\boldsymbol{P}}(u) &= \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -z^2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2+2z & 2 \\ 2z+2z^2 & 1+4z+z^2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -z & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 2+z & 1 \\ z & 1+2z \end{pmatrix} \qquad (z := e^{-iu}). \end{aligned}$$

In particular, we have $\tilde{\boldsymbol{P}}(0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\tilde{\boldsymbol{y}}_0^0$ we obtain with $\boldsymbol{y}_0^1 = (1/2,\, 1)^{\mathrm{T}}$ (cf. Example 3.4),

$$(\tilde{\boldsymbol{y}}_0^0)^{\mathrm{T}} = (1,\,1)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (1/2,\,1)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = (1/2,\,1/2).$$

Repeating this procedure, we finally obtain with $z := e^{-iu}$ the factorization

$$\begin{aligned} \boldsymbol{P}(u) &= \frac{1}{8}\begin{pmatrix} 1 & -1 \\ -z^2 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & -1/2 \\ -z^2/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1-z^2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad\times\begin{pmatrix} 1 & 0 \\ 0 & 1-z \end{pmatrix}^{-1}\begin{pmatrix} 1/2 & -1/2 \\ -z/2 & 1/2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & -1 \\ -z & 1 \end{pmatrix}^{-1}. \end{aligned}$$

For the determinant of $\boldsymbol{P}(u)$ we have

$$\det\,\boldsymbol{P}(u) = \frac{1}{32}\,(1+e^{-iu})^3.$$

2. Now we consider the example of two scaling functions ϕ_0 and ϕ_1 treated in [9] and [23]. The refinement mask of $\boldsymbol{\phi} := (\phi_0,\,\phi_1)^{\mathrm{T}}$ is given by

$$\boldsymbol{P}(u) := \frac{1}{20}\begin{pmatrix} 6+6z & 8\sqrt{2} \\ (-1+9z+9z^2-z^3)/\sqrt{2} & -3+10z-3z^2 \end{pmatrix} \qquad (z = e^{-iu}).$$

The translates $\phi_0(\cdot - l)$ and $\phi_1(\cdot - l)$ $(l \in \mathbb{Z})$ form an orthogonal basis of $\mathcal{S}(\boldsymbol{\phi})$. We observe that supp $\phi_0 = [0,\,1]$ and supp $\phi_1 = [0,\,2]$. Further, the symmetry relations $\phi_0 = \phi_0(1-\cdot)$ and $\phi_1 = \phi_1(2-\cdot)$ are satisfied. The refinement mask $\boldsymbol{P}$ satisfies the conditions (3.2) – (3.3) with $\boldsymbol{y}_0^0 := (\sqrt{2},\, 1)^{\mathrm{T}}$ and $\boldsymbol{y}_0^1 := (\sqrt{2}/2,\, 1)^{\mathrm{T}}$. With $\boldsymbol{x}_0 := (\sqrt{2},\, 1)^{\mathrm{T}}$ and $\boldsymbol{x}_1 := (1/2,\, 1/2)^{\mathrm{T}}$ we find the factorization

$$\begin{aligned}\boldsymbol{P}(u) \;&=\; \frac{1}{40}\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ -z^2 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & -1/2 \\ -z^2/2 & 1/2 \end{pmatrix}\begin{pmatrix} 10 & 0 \\ -1+20z-z^2 & -4-4z \end{pmatrix} \\ &\quad\times \begin{pmatrix} 1/2 & -1/2 \\ -z/2 & 1/2 \end{pmatrix}^{-1}\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ -z & 1 \end{pmatrix}^{-1}.\end{aligned}$$

A further factorization is not possible. For the determinant of $\boldsymbol{P}$ it follows

$$\det\, \boldsymbol{P}(u) = -\frac{1}{40}(1+e^{-iu})^3.$$

In fact, it was shown in [9, 23] that $\mathcal{S}(\boldsymbol{\phi})$ provides controlled approximation order $m = 2$. ♠

Remark 4.6 1. In the following, let $m \in \mathbb{N}_0$ and $1 \le r \le m$ be given integers. We consider equidistant knots of multiplicity r

$$x_l = x_l^r := \lfloor l/r \rfloor \quad (l \in \mathbb{Z}),$$

where $\lfloor x \rfloor$ means the integer part of $x \in \mathbb{R}$. Let $N_k^{m,r} \in C^{m-r-1}(\mathbb{R})$ $(1 \le r \le m;\, k \in \mathbb{Z})$ denote the normalized B-splines of order m and defect r with the knots $x_k, \ldots, x_{k+m}$. We introduce the spline vector $\boldsymbol{N}_m^r := (N_\nu^{m,r})_{\nu=0}^{r-1}$. In [19], it was shown that the refinement mask $\boldsymbol{P}_m^r(u)$ of $\boldsymbol{N}_m^r$ can be factorized in the form

$$\boldsymbol{P}_m^r(u) = \frac{1}{2^m}\, \boldsymbol{A}_{m-1}(z^2)\, \ldots\, \boldsymbol{A}_0(z^2)\, \boldsymbol{P}_{-1}^r\, \boldsymbol{A}_0(z)^{-1}\, \ldots\, \boldsymbol{A}_{m-1}(z)^{-1}$$

with $\boldsymbol{P}_{-1}^r := \mathrm{diag}\,(2^{r-1}, \ldots, 2^0)$ and matrices $\boldsymbol{A}_\nu$ $(\nu = 0, \ldots, r-1)$ defined by the vector of knots $\boldsymbol{x}_\nu := (x_\nu, \ldots, x_{\nu+r-1})^{\mathrm{T}}$ in a similar manner as $\boldsymbol{C}$ in (4.1) – (4.2). In particular, we have

$$\det\, \boldsymbol{P}_m^r(u) = 2^{-r(m-1)+r(r-3)/2}\,(1+e^{-iu})^m$$

(cf. [19]). Since the approximation order of the spline space generated by $\boldsymbol{N}_m^r$ is m, it follows that the B-splines with multiple knots are optimal in the sense that for a fixed small support $[0, \lfloor (m-1)/r \rfloor + 1]$ of $\boldsymbol{\phi}$ the best possible approximation order m is achieved.
2. For the construction of scaling functions providing a given approximation order, also a reversion of Corollary 4.3 would be useful. This problem will be delt with in a forthcomming paper.

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